__Proof:__ first assume f is given in the form ⊛. Then

$$f^* = \sum_{\lambda\in\mathbb{C}} \bar{\lambda} P_\lambda^* = \sum_{\lambda\in\mathbb{C}} \bar{\lambda} P_\lambda,$$

and since $P_\lambda P_\mu = P_\mu P_\lambda$ for all $\lambda,\mu\in\mathbb{C}$ we conclude that f is normal.

Moreover, for any $x\in E$ write:

$$x = I(x) = \sum_{\lambda\in\mathbb{C}} P_\lambda(x)$$

and note that

$$x\in E_\mu(f) \iff f(x) = \mu x$$

$$\iff \sum_{\lambda\in\mathbb{C}} \lambda P_\lambda(x) = \mu \sum_{\lambda\in\mathbb{C}} P_\lambda(x)$$

$$\iff \sum_{\lambda\in\mathbb{C}} (\lambda-\mu) P_\lambda(x) = 0$$

a sum of orthogonal vectors is zero $\iff$ all the summands are zero

$$\iff \forall_{\lambda\in\mathbb{C}}\ \lambda-\mu = 0 \text{ or } P_\lambda(x) = 0$$

$$\iff P_\lambda(x) = 0 \text{ for all } \lambda\neq\mu$$

$$\iff x\in \operatorname{Im}(P_\mu)$$

Thus, the P_λ are uniquely determined as the orthogonal projections to the eigenspaces of f.

For the converse, let f be normal and let P_λ be the orthogonal projections to the eigenspaces of f. By the preceding theorem, these eigenspaces are orthogonal — hence $P_\lambda P_\mu = 0$ for $\lambda\neq\mu$ — and they sum up to E — hence $\sum_{\lambda\in\mathbb{C}} P_\lambda = I$. The formula ⊛ comes from

$$f(x) = f\Big(\sum_\lambda P_\lambda x\Big) = \sum_\lambda f(P_\lambda x) = \sum_\lambda \lambda P_\lambda(x).$$

Finally, letting $q_\lambda(t)$ be a polynomial satisfying $q_\lambda(\lambda)=1$ and $q_\lambda(\mu)=0$ for all $\mu \in \mathrm{Spec}(f)$, $\mu\neq\lambda$ (Lagrange interpolant) we obtain

$$q_\lambda(f) = q_\lambda\Big(\sum_{\mu\in\mathbb{C}} \mu P_\mu\Big) = q_\lambda\Big(\sum_{\mu\in\mathrm{Spec}(f)} \mu P_\mu\Big)$$

$$= \sum_{\mu\in\mathrm{Spec}(f)} q_\lambda(\mu)\, P_\mu = P_\lambda,$$

the P_μ commute and $P_\mu P_{\mu'} = \delta_{\mu\mu'} P_\mu$

and we are done. $\boxed{}$

Other consequences:

① f is normal iff f^* is a polynomial in f:

- simply find $p(t)\in\mathbb{K}[t]$ with $p(\lambda)=\bar\lambda$ for all $\lambda\in\mathrm{Spec}(f) \Rightarrow f^*=p(f)$.

② f is self adjoint $\Longleftrightarrow$ f is normal with only real eigenvalues

Note: we have this right now only for $f\in\mathcal{L}(E)$, E unitary.

③ Analytic functions of f can be defined in $\mathcal{L}(E)$, e.g.:

$$f = \sum_{\lambda\in\mathrm{Spec}(f)} \lambda P_\lambda \Longrightarrow e^f \overset{\text{def.}}{=} \sum_{\lambda\in\mathrm{Spec}(f)} e^\lambda P_\lambda$$

with all the properties

will satisfy $e^{f+g} = e^f e^g$ when f & g are commuting normal operators.(*)

(*) over

(*) **Theorem:** Suppose $\mathcal{F} \subseteq \mathcal{L}(E)$ is a collection of commuting orthogonally/unitarily diagonalizable operators, that is:

1. every $f \in \mathcal{F}$ has an ONB α_f of E s.t. $[f]_{\alpha_f} = \text{diag.}$;
2. every $f, g \in \mathcal{F}$ satisfy $fg = gf$.

Then there exists an ONB α of E such that $[f]_\alpha$ is diagonal for all $f \in \mathcal{F}$.

3.II.1. **Exercise:** Prove this theorem using induction on $\dim E$, after proving a lemma: There exists a common eigenvector for $\mathcal{F}$ in E, that is there exists $0 \neq x \in E$ such that $f(x) \in \text{Span}(x)$ for every $f \in \mathcal{F}$.

(III) Before we move on to applications, let us investigate the real case a little bit. I will leave much of the work to you, though.

We first recall from the exercises that any operator f on a Euclidean FDIPS extends as a $\mathbb{C}$-linear operator $\mathbb{C}\otimes f$ on a unitary FDIPS $\mathbb{C}\otimes E$ in such a way that for any basis α of E one has

$$[\mathbb{C}\otimes f]_{\mathbb{C}\otimes\alpha} = [f]_\alpha, \quad \text{and} \quad \mathbb{C}\otimes f^* = (\mathbb{C}\otimes f)^*.$$

Thus effectively this means that if we think of $f \in \mathcal{L}(E)$ in terms of its representation as a real-valued matrix acting on $\mathbb{R}^n$ via left multiplication, then f is simply the restriction of that same matrix acting on $\mathbb{C}^n$ in the same way. Moreover, if α is orthogonal, then the Gram matrix is I, and the inner product on $\mathbb{C}^n$ corresponding to the one on $\mathbb{C}\otimes E$ is the standard one.

The remarks above enable arguments such as the following one:

<u>Lemma</u>: Let E be a FDIPS (<u>Real or Complex</u>), and $f \in \mathcal{L}(E)$. Then

1. f symmetric or Hermitian $\Longrightarrow \operatorname{Spec}(f) \subseteq \mathbb{R}$
2. f is skew-Symmetric/Hermitian $\Longrightarrow \operatorname{Spec}(f) \subseteq i\mathbb{R}$,
3. f is orthogonal/unitary $\Longrightarrow \operatorname{Spec}(f) \subseteq S^1$ (unit circle in $\mathbb{C}$)

Moreover, the converse implications hold whenever f is normal.

<u>Proof</u>: First, note the ease with which the "Moreover" part is demonstrated: if f is normal on E is unitary, then the claim follows from spectral decomposition; if E is Euclidean then $\mathbb{C}\otimes f$ has the same spectrum as f and is normal b/c f is $\Rightarrow$ $\mathbb{C}\otimes f$ satisfies the converse statements. But then so does f.

Now, for 1 consider $\langle fx, x\rangle$ for $x \in E$, $x \neq 0$, $fx = \lambda x$, $\lambda \in \mathbb{C}$:

$$\lambda \|x\|^2 = \langle fx, x\rangle = \langle x, f^* x\rangle = \langle x, fx\rangle = \langle x, \lambda x\rangle = \bar{\lambda}\|x\|^2$$

$$\Longrightarrow (\lambda - \bar{\lambda})\|x\|^2 = 0 \underset{x\neq 0}{\Longrightarrow} \lambda - \bar{\lambda} = 0.$$

Note, if E is Euclidean, then $E_\lambda(f) = 0$ <u>despite</u> $\chi_f(\lambda) = 0$!

Nevertheless, the eigenvector x <u>does</u> exist in $\mathbb{C}\otimes E$, which facilitates the <u>same</u> argument. $\boxed{\equiv}$

3.III.1 <u>Exercise</u>: Finish the proof of 2 and 3.

IV Applications of the Spectral Theorem:

We return to Gram matrices by recalling, for a basis $\alpha=(v_1,\dots,v_n)$ of E

$$(\star)\quad \langle x,y\rangle \equiv [y]_\alpha^* G_\alpha [x]_\alpha \quad ; \quad (G_\alpha)_{ij} = \langle v_i, v_j\rangle .$$

We observe an alternative way of reading $(\star)$: Think of $[x]_\alpha \mapsto G_\alpha [x_\alpha]$ as a linear operator on E. This motivates the following definitions:

Defn: An operator $f\in\mathcal{L}(E)$ is positive definite if f is self-adjoint and $\langle fx,x\rangle>0$ whenever $0\neq x\in E$.

NOTATION $f>0$

Ex: if E is unitary, the requirement that $\langle fx,x\rangle>0$ suffices:

$$\langle fx,x\rangle \geq 0 \Rightarrow \langle fx,x\rangle \in \mathbb{R} \text{ for all } x\in E$$

$$\Rightarrow \langle fx,x\rangle = \langle x, f^*x\rangle = \langle f^*x,x\rangle \text{ for all } x\in E$$

(a lemma we had)

$$\Rightarrow f=f^*.$$

3.IV.1. **Exercise:** find an Euclidean example where $\langle fx,x\rangle>0$ for all $x\in E, x\neq 0$ but f is not self-adjoint.

Defn: A self-adjoint matrix A is positive-definite if $x^*Ax>0$ whenever $0\neq x\in \mathbb{K}^n$. (Notation $A>0$)

Clearly, a matrix M is pos. definite iff it is the Gram matrix of an inner product on $\mathbb{K}^n$ with respect to the standard basis $\mathcal{E}=(e_1,\dots,e_n)$.
If $\alpha=(v_1,\dots,v_n)$ is another basis of $\mathbb{K}^n$ then

$$G_\alpha = \left[{}^\alpha_\mathcal{E}\right]^* G_\mathcal{E} \left[{}^\alpha_\mathcal{E}\right] = \left[{}^\alpha_\mathcal{E}\right]^* M \left[{}^\alpha_\mathcal{E}\right]$$

and thus

$$[x]_\alpha^* G_\alpha [x]_\alpha = \left(\left[\begin{smallmatrix}\alpha\\ \varepsilon\end{smallmatrix}\right][x]_\alpha\right)^* M \left(\left[\begin{smallmatrix}\alpha\\ \varepsilon\end{smallmatrix}\right][x]_\alpha\right)$$

Since $\left[\begin{smallmatrix}\alpha\\ \varepsilon\end{smallmatrix}\right]$ is invertible, we have $x \neq 0$ iff $\left[\begin{smallmatrix}\alpha\\ \varepsilon\end{smallmatrix}\right][x]_\alpha \neq 0$, and conclude that G_α is pos. definite as well.

Viewing G_α as an operator on $\mathbb{K}^n$, on the other hand, allows us to apply the spectral theorem and conclude:

<u>Thm</u>: The following are equivalent for a matrix $M \in M_n(\mathbb{K})$:

(1) M is positive definite,

(2) M is a Gram matrix of an inner product on an n-dim'l inner prod. space,

(3) M is a Gram matrix of an inner product on $\mathbb{K}^n$

(4) M is self-adjoint with strictly positive eigenvalues.

Criterion for inner products!

(5) M is self-adjoint with all positive diagonal minors. ☺

$\boxtimes$

<u>Exercise</u>: Find and read a proof of (5) being equivalent to M being positive definite, e.g. in Hoffman & Kunze's textbook on Linear Algebra.

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<u>Defn</u>: An operator $f \in \mathcal{L}(E)$ on a FDIPS E is <u>positive-semidefinite</u> if

(1) $f = f^*$, and (2) $\langle fx, x\rangle \geq 0$ for all $x \in E$.

NOTATION: $f \geq 0$

MAIN EXAMPLE OF POS. SEMIDEFINITE OP: $f \in \mathcal{L}(E) \Rightarrow f^*f \geq 0$

Indeed, $\langle f^*f x, x\rangle = \langle fx, fx\rangle = \|fx\|^2 \geq 0$;

Also, $(f^*f)^* = f^* f^{**} = f^*f$.

We want to go a bit deeper into the analysis of such operators.

Since $g = f^* f$ is self-adjoint, apply the spectral decomposition:

$$g = \sum_{\lambda \geq 0} \lambda P_\lambda \quad , \quad \underset{i}{P_\lambda P_\mu = \delta_{\lambda\mu} P_\lambda} \, , \quad \underset{ii}{P_\lambda^* = P_\lambda} \, , \quad \underset{iii}{\sum P_\lambda = I} \, , \; \lambda \geq 0$$

$$\Rightarrow \forall_{x \in E} \quad \|f x\|^2 = \langle gx, x \rangle = \left\langle \sum_{\lambda \geq 0} \lambda P_\lambda x, \sum_{\mu \geq 0} P_\mu x \right\rangle$$

$$= \sum_{\lambda, \mu \geq 0} \lambda \langle P_\lambda x, P_\mu x \rangle = \sum_{\lambda \geq 0} \lambda \|P_\lambda x\|^2$$

$P_\lambda P_\mu = 0$
$\Downarrow$
$\operatorname{Im} P_\mu \subseteq \operatorname{Ker} P_\lambda$
$\Downarrow$ orth. proj.
$\operatorname{Im} P_\mu \perp \operatorname{Im} P_\lambda$

$$\leq \max_{\lambda \in \operatorname{Spec}(g)} \lambda \cdot \sum_\lambda \|P_\lambda x\|^2$$

$$= \left(\max_{\lambda \in \operatorname{Spec}(g)} \lambda \right) \cdot \left\| \sum_\lambda P_\lambda x \right\|^2$$

$$= \left(\max_{\lambda \in \operatorname{Spec}(g)} \lambda \right) \|x\|^2$$

Thus, we have proved that $\sup_{x \neq 0} \frac{\|f(x)\|}{\|x\|} \leq \max_{\lambda \in \operatorname{Spec}(g)} \sqrt{\lambda}$ (Remember $\lambda \geq 0$?)

and moreover, if $\sigma_1(f) := \max_{\lambda \in \operatorname{Spec}(g)} \sqrt{\lambda}$, we have that the above bound is realized by choosing $x \in E_g(\sigma_1(f))$. We have proved:

<u>Theorem</u>: for any $f \in \mathcal{L}(E)$,

$$\|f\| \overset{\text{def}}{=} \max_{x \neq 0} \frac{\|f(x)\|}{\|x\|} = \max_{\|x\| = 1} \|f(x)\| = \sigma_1(f)$$

In particular, $\operatorname{Spec}(f)$ is contained in the closed disc of radius $\sigma_1(f)$ about the origin in $\mathbb{C}$.

3.IV.3. <u>Exercise</u>: Prove that $\|\cdot\|$ as defined above is a norm on $\mathcal{L}(E)$ satisfying $\|fg\| \leq \|f\|\cdot\|g\|$ for all $f, g \in \mathcal{L}(E)$.

A nice & quick application of this result is the existence of e^A for any matrix $A \in \mathbb{C}^n$:

$$e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!} := \lim_{K\to\infty} \sum_{n=0}^{K} \frac{1}{n!} A^n \quad (\star)$$

We observe:

$$\left\| \sum_{n=0}^{K} \frac{1}{n!} A^n \right\| \leq \sum_{n=0}^{K} \frac{1}{n!} \|f\|^n \leq e^{\|f\|} \quad \text{here } f \in \mathcal{L}(\mathbb{C}^n),\ f(x) = Ax$$

Thus the series $(\star)$ converges absolutely $\Rightarrow$ converges, by completeness of $\mathcal{L}(\mathbb{C}^n) \cong \mathbb{C}^{n\times n}$.

(if you're bothered by this "black box", you should be!)

$\Rightarrow e^A$ is well-defined. ▨

3.IV.4. <u>Exercise</u>: Suppose $f \in \mathcal{L}(E)$ has $\|f\| < 1$. Prove:

$$(I - f)^{-1} = \sum_{n=0}^{\infty} f^n$$

Note the special rôle of Unitary/Orthogonal operators:

<u>Defn</u>: Let E, F be FDIPS and suppose $g \in \mathcal{L}(E,F)$. TFAE:

(1) $g^* g = Id_E$;

(2) for all $x, y \in E$, $\langle gx, gy\rangle = \langle x, y\rangle$;

(3) for all $x \in E$, $\|gx\| = \|x\|$.

<u>Proof</u>: If (1) holds then $\forall x, y \in E$ $\langle gx, gy\rangle = \langle g^*gx, y\rangle = \langle x, y\rangle$.

If (2) holds then (3) holds by setting $y := x$.
If (3) holds, we recall that $h = g^* g$ is a positive semidefinite op. (why?) with real non-negative spectrum and an ONB of eigenvectors in E.

If $\lambda \in \mathrm{Spec}(h)$, $x \in E_\lambda(h)$, $x \neq 0$ then

$$\|x\|^2 = \|gx\|^2 = \langle hx, x \rangle = \lambda \|x\|^2 \Rightarrow \lambda = 1.$$

By the spectral decomposition theorem, $h = \mathrm{Id}_E$. $\blacksquare$

Note: An isometry $g : E \to E$ of a FDIPS is a function satisfying the identity:

$$\|gx - gy\| = \|x - y\|.$$

It can be shown that any isometry of E is the composition of a linear isometry with a translation.

The bottom line is that we would like to think of isometries as the mathematical model for rigid motions. This is the context in which the following result becomes really striking (not to mention useful...):

Theorem (Polar Decomposition): Let E be a FDIPS. Then every invertible $f \in \mathcal{L}(E)$ has a unique expression in the form $f = up$ where p is positive definite and u is a linear isometry.

Proof: We start by forming $g = f^* f$ which is then positive definite, as:

$$0 = \langle gx, x \rangle = \|f(x)\| \iff x \in \mathrm{Ker}\, f \iff x = 0 \text{ since } f \text{ is invertible.}$$

Next, writing $g = \sum_{\lambda \in \mathrm{Spec}(g)} \lambda P_\lambda$ (spec decomp.) we set $p = \sqrt{g} := \sum_{\lambda \in \mathrm{Spec}(g)} \sqrt{\lambda}\, P_\lambda$

so that p is normal with only pos. eigenvalues $\Rightarrow$ p is pos. definite.

Now consider $u \in \mathcal{L}(E)$ constructed as $u := f p^{-1}$

$$u^* u = (f p^{-1})^* (f p^{-1}) = (p^{-1})^* f^* f p^{-1} = (p^*)^{-1} g p^{-1} = p^{-2} g = I.$$

Thus, $f = up$ with $p > 0$ and u - a linear isometry, as required.

To verify uniqueness, suppose $f = up = vq$ where u, v are isometries and p, q are positive definite. We look at

$$p = u^* f \quad (u^* = u^{-1} \text{ after all}), \quad q = v^* f$$

Then $p^2 = p^* p = f^* u u^* f = f^* f$, $q^2 = q^* q = f^* v v^* f = f^* f$. Thus the proof is finished given the next lemma! ∎

Lemma: Let $p \in \mathcal{L}(E)$ be positive definite. Then p has a unique positive-definite square root.

Proof: We've shown existence already: writing $p = \sum_{\lambda > 0} \lambda P_\lambda$, $\sqrt{p} \overset{\text{def.}}{=} \sum_{\lambda > 0} \lambda^{1/2} P_\lambda$ is pos. definite by construction, and $(\sqrt{p})^2 = p$.

Now suppose $q \in \mathcal{L}(E)$ has $q > 0$ and $q^2 = p$. Then $qp = qq^2 = q^2 q = pq$. This implies $E_\lambda(p)$ is q-invariant for all $\lambda > 0$. Then for each λ, the restriction $q_\lambda := q|_{E_\lambda(p)}$ is a positive-definite operator with $q_\lambda^2 = \lambda I$.

Writing down the spectral decomposition of q_λ as an operator on $E_\lambda(p)$

$$q_\lambda = \sum_{\mu > 0} \mu Q_\mu \Rightarrow \sum_{\mu > 0} \mu^2 Q_\mu = \lambda^2 I = \sum_{\mu > 0} \lambda^2 Q_\mu$$

By uniqueness of spectral decomposition (of $\lambda^2 I$), $\operatorname{Spec}(q_\lambda) = \{\sqrt{\lambda}\}$ and $q_\lambda = \sqrt{\lambda} I_{E_\lambda(p)}$, proving uniqueness. ∎

Now consider what we have achieved, in the language of matrices:

Let $A \in M_n(\mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then there exist matrices P and U in $M_n(\mathbb{K})$ such that

$$(1)\ P > 0, \quad (2)\ U^*U = UU^* = I, \quad (3)\ A = UP$$

Now, let $\alpha = (v_1, \dots, v_n)$ be an ONB of $\mathbb{K}^n$ with respect to the standard inner prod. such that $[P]_\alpha = \mathrm{diag}(\sigma_1, \dots, \sigma_n) =: \Sigma$ and write $V := [v_1 | \dots | v_n] = [{}^\alpha_\varepsilon]$

Then V is unitary/orthogonal and we have $P = V \Sigma V^*$

$$\Longrightarrow A = \underbrace{(UV)}_{\text{unitary}} \underbrace{\Sigma}_{\text{pos. diag.}} \underbrace{V^*}_{\text{unitary}}$$

Defn: A decomposition of a matrix as a product as above is called a <u>singular value decomposition</u>. It is common to arrange the singular values $\sigma_1, \dots, \sigma_n$ in a non-increasing order.

<u>Theorem</u>: The singular values of a matrix A are independent of the particular decomposition.

<u>Pf</u>: Indeed, the singular values are the eigenvalues of A^*A:

If $A = U \Sigma V$ with U, V unitary and $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$, $\sigma_i > 0$

then $A^*A = V^* \Sigma^* U^* U \Sigma V = V^* \Sigma^* \Sigma V$

$\Longrightarrow \mathrm{Spec}(A^*A) = \{|\sigma_1|^2, \dots, |\sigma_n|^2\} \Rightarrow \sigma_1, \dots, \sigma_n$ are determined by A alone.

$\blacksquare$