Note Title 5/14/2013

2.I.1. Exercise: Let d be a non-negative integer and let

$$V = \mathbb{C}_d[x] = \{\text{all polynomials in } x \text{ of degree } \leq d\}.$$

Let $z_1, \ldots, z_n \in \mathbb{C}$ be distinct complex numbers and define

$$\langle f, g \rangle = \sum_{k=1}^{n} f(z_i)\overline{g(z_i)}, \quad f, g \in V.$$

Prove that $\langle -,- \rangle$ is an inner product on V if and only if $n \geq d+1$.

2.II.1. Exercise: Suppose $V = V_{\mathbb{C}}$ and $\langle -,- \rangle$ is a complex-valued inner product on V. Show that then

$$(x, y) = \mathrm{Re}\langle x, y \rangle$$

is a real-valued inner product on V, where the scalars are restricted to the real field.

2.II.2. Exercise: Let V be a normed vector space with norm $\|\cdot\|$. Prove that the norm $\|\cdot\|$ comes from an inner product if and only if

$$(\star) \quad \|x+y\|^2 + \|x-y\|^2 = 2\left[\|x\|^2 + \|y\|^2\right] \text{ for all } x, y \in V$$

What is the geometric meaning of this identity?

(Hint: you could start by reconstructing $\mathrm{Re}\langle x, y \rangle$ from the expressions in $(\star)$ using the cosine law; then proceed to tweak those for the complex case)

2.I.3. Exercise: Let $V = \mathcal{C}(a,b)$, the space of continuous functions from $[a,b]$ to $\mathbb{C}$.

(1) Prove that $\langle f, g \rangle = \int_a^b f(t)\overline{g(t)}\,dt$ is an inner product.

(2) Verify that, for every positive integer n, the set

$$F_n := \{1, \cos x, \cos 2x, \dots, \cos nx, \sin x, \sin 2x, \dots, \sin nx\}$$

is an orthogonal set in $\mathcal{C}(0, 2\pi)$.

2.I.4. Verify the last statement, that is: prove that for every $v \in V$ and every orthogonal set $(v_1, \dots, v_d)$ one has

$$\|v\|^2 \geq \sum_{i=1}^{d} \left| \frac{\langle v, v_i \rangle}{\|v_i\|} \right|^2$$

with equality iff $v \in \mathrm{Span}(v_1, \dots, v_d)$.

Bessel's Inequality

2.I.5. Riesz's Representation theorem, baby version: Let V be a finite-dimensional inner product space. Prove that every $f \in V^*$ has some $v_f \in V$ such that

$$f(u) = \langle u, v_f \rangle$$

holds for all $u \in V$. Also answer the following questions:

(1) Is v_f uniquely determined by f?

(2) Is the mapping $f \mapsto v_f$ linear? $\mathbb{R}$-linear?

2.III.1. <u>Exercise</u>: Prove the following properties of orthogonal complements

(1) $S^{\perp}$ is a subspace of V;

(2) $S \subseteq S^{\perp\perp}$;

(3) $S \cap S^{\perp} \subseteq \{0\}$.

2.III.2. <u>Exercise</u>: Prove the following, for W, U subspaces of a FDIPS V:

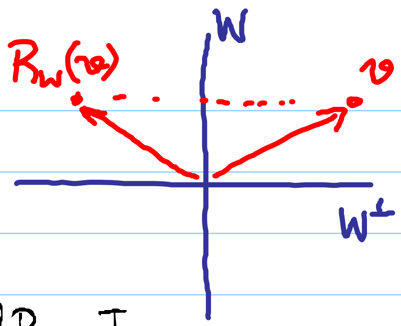
$$W^{\perp\perp} = W, \quad (U+W)^{\perp} = U^{\perp} \cap W^{\perp}, \quad (U \cap W)^{\perp} = U^{\perp} + W^{\perp}.$$

2.III.3. <u>Exercise</u>: Let V be a finite-dimensional inner prod. space and U, W are subspaces. Then (1) $(U+W)^{\perp} = U^{\perp} \cap W^{\perp}$, (2) $(U \cap W)^{\perp} = U^{\perp} + W^{\perp}$.

2.III.4. <u>Exercise</u>: Let W be a subspace of a fin. dim'l inner prod. spc. V and let $v \in V$.

Show that $\|v - P_W(v)\| \leq \|v - w\|$ for all $w \in W$ [$P_W(v)$ is the <u>best</u> <u>approximation</u> of v in W]

2.III.5. <u>Exercises</u> Let $U, W < V$ as before. Find a necessary and sufficient condition for P_U and P_W to commute ($P_U P_W = P_W P_U$), and prove it.



2.IV.1. <u>Exercise:</u> Let $W < V$ be a subspace, where V is a FDIPS. Define the <u>reflection in W</u> to be
$$R_W = I - 2P_{W^\perp} = 2P_W - I$$
Prove:

(1) R_W is both self-adjoint and unitary

(2) If $T: V \to V$ is both self-adjoint and unitary, then it is a reflection.

2.IV.2. <u>Exercise:</u> Let $Q: V \to V$ be a linear map. Prove that the following are equivalent:

(1) $\langle Qv, Qw\rangle = \langle v, w\rangle$ for all $v, w \in V$

(2) $\|Qv\| = \|v\|$ for all $v \in V$

(3) $Q^*Q = QQ^* = I$

A $Q: V \to V$ satisfying (any one of) these is called a <u>linear isometry</u>

2.IV.3. <u>Exercise</u> (Complexification) Let V be a Euclidean space. We construct a unitary space $\mathbb{C}\otimes V$ called the <u>complexification</u> of V as follows:

$\mathbb{C}\otimes V = V \times V$, using the notation $\left\{\begin{array}{l}(v,w) \overset{\text{df}}{=} v + iw \\ (v,0) = \mathbb{C}\otimes v\end{array}\right.$, we define addition and scalar multiplication:

- $(v_1 + iw_1) + (v_2 + iw_2) = (v_1 + v_2) + i(w_1 + w_2)$
- $z \cdot (v + iw) = (\mathrm{Re}(z)\, v - \mathrm{Im}(z) w) + i(\mathrm{Re}(z) w + \mathrm{Im}(z) v)$

and an inner product:

- $\langle v_1 + iw_1, v_2 + iw_2\rangle = \langle v_1, v_2\rangle + \langle w_1, w_2\rangle + i\langle w_1, v_2\rangle - i\langle v_1, w_2\rangle$

2.IV.3. (cntd.) ① Prove that $\mathbb{C}\otimes V$ is a unitary space.

② Prove that if $(v_1, \ldots, v_n)$ is an orthonormal basis for V then $\mathbb{C}\otimes\alpha$ is an orthonormal basis of $\mathbb{C}\otimes V$, too.

Now, if $T: V \to W$ is a linear map between Euclidean spaces, we construct

$$\mathbb{C}\otimes T : \mathbb{C}\otimes V \longrightarrow \mathbb{C}\otimes W$$

by setting

$$\mathbb{C}\otimes T(v + iw) = T(v) + i\,T(w).$$

③ Letting $\alpha = (v_1, \ldots, v_n)$, $\beta = (w_1, \ldots, w_m)$ be bases of V & W, respectively, prove that:

$$\left[\mathbb{C}\otimes T\right]^{\mathbb{C}\otimes\alpha}_{\mathbb{C}\otimes\beta} = \left[T\right]^{\alpha}_{\beta} \quad \text{where:} \left\{ \begin{array}{c} \alpha = (v_1, \ldots, v_n) \\ \Downarrow \text{def.} \\ \mathbb{C}\otimes\alpha = (\mathbb{C}\otimes v_1, \ldots, \mathbb{C}\otimes v_n) \end{array} \right.$$

④ Show that $(\mathbb{C}\otimes T)^* = \mathbb{C}\otimes(T^*)$.

In view of the above, we $\boxed{\text{MAY ALWAYS ASSUME } T: V \to W \text{ IS AN OPERATOR BETWEEN UNITARY SPACES.}}$

2.IV.4. <u>Exercise</u>: Let $V = \mathcal{C}(0,1)$, the space of continuous real-valued functions in $[0,1]$.
Let $W < V$ be the subspace of polynomial functions of degree ≤ 3.
Let $u(t) = \cos(2\pi t)$ and $v(t) = \sin(2\pi t)$. Find the best approximations of u & of v in W with respect to the norm

$$\|f\| \stackrel{\text{def.}}{=} \int_0^1 |f(t)|^2 dt.$$