HENCEFORTH ALL SPACES ARE FDIPS

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[Until Further Notice, Unless otherwise stated]

IV The adjoint operator

Consider a linear operator $A : V_{\mathbb{F}} \to W_{\mathbb{F}}$ and let $A^T : W^* \to V^*$ be its transpose, or dual. We recall:

$$\forall_{f \in W^*} \forall_{v \in V} \quad \langle A^T f \,|\, v \rangle \overset{def}{=} \langle f \,|\, Av \rangle$$

Recalling that V & W have inner products, let us consider a vector $w \in W$ and the functional $f \in W^*$ defined as:

$$\langle f | y \rangle \overset{def}{=} \langle y, w \rangle$$

For every $v \in V$ we have:

$$\langle Av, w \rangle = \langle f \,|\, Av \rangle = \langle A^T f \,|\, v \rangle$$

Now, $A^T f \in V^*$ and since V is finite-dimensional we know there exists a unique vector $u \in V$ such that

$$\langle A^T f \,|\, v \rangle = \langle v, u \rangle$$

holds identically for all $v \in V$. More simply put:

Theorem & Defn: (Existence & Uniqueness of adjoints)
Let $A: V \to W$ be a linear operator between FDIPS. Then there exists one and only one linear operator $A^*: W \to V$ satisfying the identity
$$\langle Av, w \rangle = \langle v, A^* w \rangle$$
for all $v \in V$ and $w \in W$. Moreover, the following properties hold whenever relevant:

(1) $(A+B)^* = A^* + B^*$

(2) $(AB)^* = B^* A^*$

(3) $(\lambda A)^* = \overline{\lambda} \cdot A^*$ for all $\lambda \in \mathbb{R}$

(4) $Id_V^* = Id_V$

$\blacksquare$

Some nomenclature for the case when $A: V \to V$:

* $A = A^*$ (Self-adjoint; $\mathbb{R}$ – symmetric, $\mathbb{C}$ – Hermitian)
* $AA^* = I$ ($\mathbb{R}$ – Orthogonal, $\mathbb{C}$ – Unitary)
* $AA^* = A^*A$ (Normal)
* $A^* = -A$ (Skew; $\mathbb{R}$ – skew-symmetric, $\mathbb{C}$ – skew-Hermitian)

Theorem (Representation of the adjoint): Let $\mathcal{E} = (v_1, \ldots, v_d)$ be an orthonormal basis of the FDIPS V and sps $A: V \to V$. Then
$$[A^*]_\mathcal{E} = \overline{[A]_\mathcal{E}}^T := [A]_\mathcal{E}^*$$
(conjugate & transposed)

Proof: Recall the identity $[A]_\mathcal{E}[v]_\mathcal{E} = [Av]_\mathcal{E}$. For each i we have:

$$[Av_i]_\mathcal{E} = [A]_\mathcal{E}[v_i]_\mathcal{E} = [A]_\mathcal{E} e_i = i\text{-th column of } A.$$

At the same time, $\langle v^j | Av_i \rangle = j$-th coordinate of $[Av_i]_\mathcal{E}$

Thus, letting $M = [A]_\mathcal{E}$ we obtain $M_{ji} = \langle v^j | Av_i \rangle$. So much is true always.

But now, invoking the orthonormality of $\mathcal{E}$:

$$M_{ji} = \langle v^j | Av_i \rangle = \langle Av_i , v_j \rangle = \langle v_i , A^* v_j \rangle$$
$$= \overline{\langle A^* v_j , v_i \rangle} = \overline{\langle v^i | A^* v_j \rangle} = \overline{([A^*]_\mathcal{E})_{ij}}. \quad \blacksquare$$

Given this result, it seems sensible to define classes of matrices as follows:

* $A \in M_n(\mathbb{R})$,
  - $A^T A = I \ (\Leftrightarrow AA^T = I)$ ⟶ Orthogonal matrix
  - $A^T = A$ ⟶ Symmetric matrix
  - $A^T = -A$ ⟶ Skew-Symmetric
  - $A^T A = AA^T$ ⟶ Real-normal

* $A \in M_n(\mathbb{C})$,
  - $A^* A = I \ (\Leftrightarrow AA^* = I)$ ⟶ Unitary matrix
  - $A^* = A$ ⟶ Hermitian
  - $A^* = -A$ ⟶ Skew-Hermitian
  - $A^* A = AA^*$ ⟶ $\mathbb{C}$-normal

Remark: By the uniqueness of the adjoint, given an operator $A: V \to V$ it suffices to find/guess an operator $B: V \to V$ such that the identity
$$\langle Av, w \rangle = \langle v, Bw \rangle$$
holds over V. However, more is true:

Lemma: Let V be a unitary space and $A: V \to V$. Then
$$\langle Av, v \rangle = 0 \text{ for all } v \in V \text{ implies } A = 0.$$

Note in particular that, as a direct application one has:

Corollary: Let V be a unitary space and $A, B: V \to V$ linear ops. Then $B = A^*$ iff $\langle Av, v \rangle = \langle v, Bv \rangle$ for all $v \in V$.

$$\Updownarrow$$

$$\langle Av - B^* v, v \rangle = 0 \quad \text{explains it all...}$$

At the same time observe that the lemma is FALSE if V is a Euclidean (Real FDIPS) space: pick $V = \mathbb{R}^2$ with the standard inner product and let
$$Av = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v.$$

Proof of the lemma: let $v, w \in V$. Our goal is to show $\langle Av, w \rangle = 0$.

$$0 = \langle A(v+w), v+w \rangle = \underbrace{\langle Av, v \rangle + \langle Aw, w \rangle}_{0} + \langle Av, w \rangle + \langle Aw, v \rangle$$

$$\Longrightarrow \langle Av, w \rangle = -\langle Aw, v \rangle \quad (*)$$

$$0 = \langle A(v+iw), v+iw \rangle = \underbrace{\langle Av, v \rangle + \langle A(iw), iw \rangle}_{0} + \langle A(iw), v \rangle + \langle Av, iw \rangle$$

$$\Rightarrow 0 = i\langle Aw, v \rangle - \bar{i}\langle Av, w \rangle \quad (**)$$

$$\langle Aw, v \rangle \underset{(**)}{=} \langle Av, w \rangle \underset{(*)}{=} -\langle Aw, v \rangle$$

$\blacksquare$

Orthogonal Projections:

Let V be a FDIPS and $W < V$. We consider $P = P_W : V \to V$ and observe that, for all $v \in V$:

$$v = Pv + (I-P)v \quad \text{with} \quad Pv \in W, \; (I-P)v \in W^\perp$$

$$\Rightarrow \|v\|^2 = \|Pv\|^2 + \|(I-P)v\|^2$$

$$\Rightarrow \|Pv\| \leq \|v\|, \; \|(I-P)v\| \leq \|v\|.$$

Let us try to consider P^* now. For any $u, v \in V$:

$$\langle Pu, v\rangle = \langle Pu, Pv + (I-P)v\rangle = \langle Pu, Pv\rangle + \underbrace{\langle Pu, (I-P)v\rangle}_{=0}$$

$$\langle u, Pv\rangle = \langle Pu + (I-P)u, Pv\rangle = \langle Pu, Pv\rangle + \underbrace{\langle (I-P)u, Pv\rangle}_{=0}$$

Thus we must have $\langle Pu, v\rangle = \langle u, Pv\rangle$ for all $u, v \in V$, and hence $P^* = P$, by uniqueness of the adjoint.

Theorem: Let V be a FDIPS and let $P : V \to V$ be a projection. The following are equivalent:

(1) $\operatorname{Im} P \perp \operatorname{Ker} P$ (P is an orthogonal projection)
(2) P is self-adjoint
(3) $\|Pv\| \leq \|v\|$ for all $v \in V$

Proof: We have just shown (1) $\Rightarrow$ (2). Here is an alternative proof:

Pick an orthonormal basis $\alpha_1 = (v_1, \dots, v_d)$ for W and $\alpha_2 = (v_{d+1}, \dots, v_n)$ for $W^\perp$ and observe that $\alpha = \alpha_1 \cup \alpha_2$ is an orthonormal basis of V.

By construction, $[P]_\alpha = \left[\begin{array}{c|c} I_d & 0 \\ \hline 0 & 0 \end{array}\right]$ which is real and symmetric and $P^* = P$.

To verify (2) $\Rightarrow$ (3), assume $P^* = P$ and observe:

Whenever $\|Pv\| = 0$ we have $\|Pv\| = 0 \le \|v\|$ hence we may assume

$\|Pv\| \neq 0$: $\|Pv\|^2 = \langle Pv, Pv \rangle = \langle v, P^* P v \rangle = \langle v, P^2 v \rangle = \langle v, Pv \rangle$

[Thus $\langle v, Pv \rangle \geq 0$ for all v]

$$\overset{C\text{-}S}{\leq} \|v\| \, \|Pv\|$$

$$\Rightarrow \|Pv\| \leq \|v\| .$$

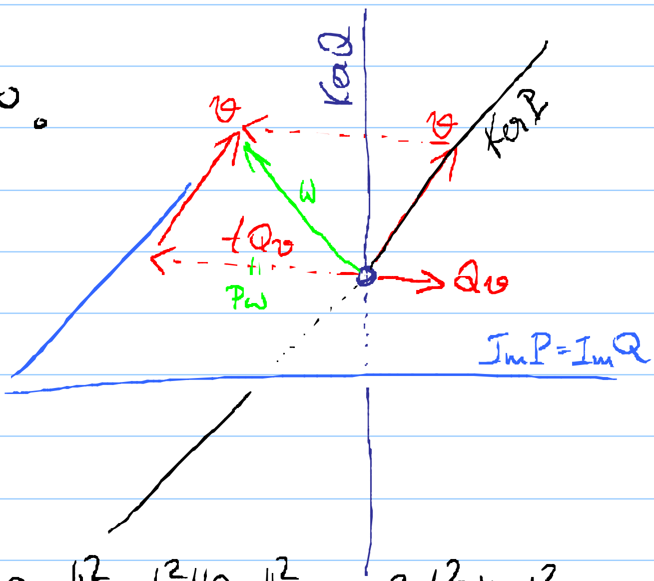
To prove (3) $\Rightarrow$ (1), let Q denote the orthogonal projection onto $W = \mathrm{Im}\,P$, and we have $Q^2 = Q$, $Q = Q^*$ and $\|Qv\| \leq \|v\|$ for all $v \in V$ from the preceding arguments.

Take $v \in \mathrm{Ker}\,P$ and assume $Qv \neq 0$.
Write

$$0 < \varepsilon \overset{\text{def}}{=} \frac{\|Qv\|}{\|v\|}$$

and recall

$$\|Qv\|^2 = \langle v, Qv \rangle .$$



Now consider $w = v - t\,Qv$

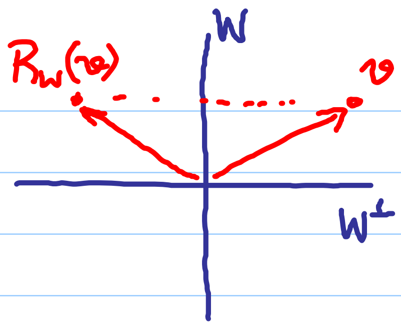
$$\|Pw\|^2 = \|Pv - tPQv\|^2 = \|0 - tQv\|^2 = t^2 \|Qv\|^2 = \varepsilon^2 t^2 \|v\|^2$$

$$\|w\|^2 = \|v\|^2 + t^2 \|Qv\|^2 - 2t\,\mathrm{Re}\langle v, Qv \rangle = \|v\|^2 \left[1 + \varepsilon^2 t^2 - 2t\varepsilon^2 \right]$$

Since $\|Pw\| \leq \|w\|$ for all w, for all $t \in \mathbb{R}$ we have $1 - 2t\varepsilon^2 \geq 0$, which is impossible!

∎

Reflections:



2.IV.1. Exercise: Let $W < V$ be a subspace, where V is a FDIPS. Define the reflection in W to be

$$R_W = I - 2P_{W^\perp} = 2P_W - I$$

Prove:

1. R_W is both self-adjoint and unitary
2. If $T: V \to V$ is both self-adjoint and unitary, then it is a reflection.

2.IV.2. Exercise: Let $Q: V \to V$ be a linear map. Prove that the following are equivalent:

1. $\langle Qv, Qw \rangle = \langle v, w \rangle$ for all $v, w \in V$
2. $\|Qv\| = \|v\|$ for all $v \in V$
3. $Q^*Q = QQ^* = I$

A $Q: V \to V$ satisfying (any one of) these is called a linear isometry

2.IV.3. Exercise (Complexification) Let V be a Euclidean space. We construct a unitary space $\mathbb{C} \otimes V$ called the complexification of V as follows:

$\mathbb{C} \otimes V = V \times V$, using the notation $\left\{\begin{array}{l}(v,w) \overset{\text{df}}{=} v + iw \\ (v,0) = \mathbb{C} \otimes v\end{array}\right.$, we define addition and scalar multiplication:

- $(v_1 + iw_1) + (v_2 + iw_2) = (v_1 + v_2) + i(w_1 + w_2)$
- $z \cdot (v + iw) = (\mathrm{Re}(z)\, v - \mathrm{Im}(z) w) + i(\mathrm{Re}(z) w + \mathrm{Im}(z) v)$

and an inner product:

- $\langle v_1 + iw_1, v_2 + iw_2 \rangle = \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle + i\langle w_1, v_2 \rangle - i\langle v_1, w_2 \rangle$

2.IV.3. (cntd.) (1) Prove that $\mathbb{C}\otimes V$ is a unitary space.

(2) Prove that if $(v_1,\ldots,v_n)$ is an orthonormal basis for V then $\mathbb{C}\otimes\alpha$ is an orthonormal basis of $\mathbb{C}\otimes V$, too.

Now, if $T: V\to W$ is a linear map between Euclidean spaces, we construct

$$\mathbb{C}\otimes T : \mathbb{C}\otimes V \longrightarrow \mathbb{C}\otimes W$$

by setting

$$\mathbb{C}\otimes T(v+iw) = T(v)+i\,T(w).$$

(3) Letting $\alpha=(v_1,\ldots,v_n)$, $\beta=(w_1,\ldots,w_m)$ be bases of V & W, respectively, prove that:

$$\left[\mathbb{C}\otimes T\right]^{\mathbb{C}\otimes\alpha}_{\mathbb{C}\otimes\beta} = \left[T\right]^{\alpha}_{\beta} \quad \text{where:} \left\{\begin{array}{c} \alpha=(v_1,\ldots,v_n) \\ \Downarrow \text{def.} \\ \mathbb{C}\otimes\alpha=(\mathbb{C}\otimes v_1,\ldots,\mathbb{C}\otimes v_n)\end{array}\right.$$

(4) Show that $(\mathbb{C}\otimes T)^* = \mathbb{C}\otimes(T^*)$.

In view of the above, we $\boxed{\text{MAY ALWAYS ASSUME } T: V\to W \text{ IS AN OPERATOR BETWEEN UNITARY SPACES.}}$

2.IV.4. <u>Exercise</u>: Let $V=\mathcal{C}(0,1)$, the space of continuous real-valued functions in $[0,1]$.
Let $W<V$ be the subspace of polynomial functions of degree ≤ 3.
Let $u(t)=\cos(2\pi t)$ and $v(t)=\sin(2\pi t)$. Find the best approximations of u & of v in W with respect to the norm

$$\|f\| \overset{\text{def.}}{=} \int_0^1 |f(t)|^2 dt.$$