Chapter 2: Inner Product Spaces.

Note Title 5/14/2013

I The most basic way in which the ghost of duality may be summoned is motivated by Euclidean Geometry. We start with the notion of a distance:

Defn: Let X be a non-empty set. A function $d: X \times X \to \mathbb{R}_{\geq 0}$ is a distance on X or a metric if:

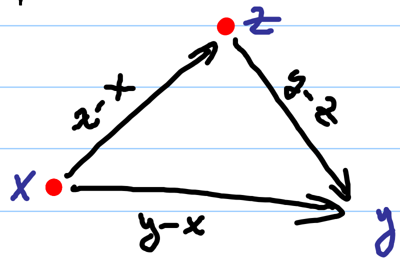
(1) $d(x,y) = d(y,x)$

(2) $d(x,y) = 0 \iff x = y$

(3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

If (1) fails to hold, we say that d is a semi-metric. ▨

In the contex of a real vector space $V_{\mathbb{R}} \cong \mathbb{R}^d$, $d = \dim V_{\mathbb{R}}$ one imagines the picture:



As an illustration for (3) – AKA the triangle inequality – that is also indicative of the need for a notion of length for vectors as a means for generating a notion of distance among points.

Defn: A norm on a real/complex vector spc. V is a function $\|\cdot\|: V \to \mathbb{R}_{\geq 0}$ such that:

(1) $\|v\| = 0 \iff v = 0$;

(2) $\|\lambda v\| = |\lambda| \cdot \|v\|$ for all $v \in V$ and scalars λ;

(3) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$

If (1) fails to hold, we say that $\|\cdot\|$ is a semi-norm. ▨

In this chapter we will explore Euclidean norms - notions of length abiding by the axioms of Euclidean geometry, if you like.

Defn: Let V be a vector space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. An inner product on V is a function

$$V \times V \longrightarrow \mathbb{F}, \quad (x,y) \longmapsto \langle x, y \rangle$$

satisfying: (1) $\langle x + \alpha y, z \rangle \equiv \langle x, z \rangle + \alpha \langle y, z \rangle, \quad x,y,z \in V, \ \alpha \in \mathbb{F}$

$\rightsquigarrow$ linearity in 1st coordinate

(2) $\langle x, y \rangle \equiv \overline{\langle y, x \rangle}$ $\rightsquigarrow$ skew-symmetry

(3) $\langle x, x \rangle \geq 0$ with equality iff $x = 0$ $\rightsquigarrow$ Positivity

Given an inner product, the norm of $x \in V$ is defined as $\|x\| = \sqrt{\langle x, x \rangle}$.

When $\mathbb{F} = \mathbb{R}$ we note that (1)+(2) make $\langle -, - \rangle$ into what is known as a symmetric bilinear form on V: due to (2), property (1) applies to to the second argument of the product as well as to the first and we obtain identities such as

$$\langle x + \alpha y, z + \beta w \rangle \equiv \langle x, z \rangle + \alpha \langle y, z \rangle + \beta \langle x, w \rangle + \alpha\beta \langle y, w \rangle$$

!

When $\mathbb{F} = \mathbb{C}$ we need to take care and observe that

$$\langle x, y + \alpha z \rangle = \langle x, y \rangle + \overline{\alpha} \langle y, z \rangle$$

Specifying an inner product on $\mathbb{R}^n$ or $\mathbb{C}^n$ is not that hard:

Example (Standard Inner Product): Let $V = \mathbb{R}^n$ or $\mathbb{C}^n$. We set

$$x \cdot y := \langle x, y \rangle_{std} := y^* x, \quad \|x\|_{std} = \sqrt{x^* x}$$

where, HENCEFORTH FOR ANY MATRIX $B \in \mathbb{C}^{m \times n}$, $(B^*)_{ij} := \overline{B_{ji}}$, where the bar denotes complex conjugation. Verification of the axioms of an inner product is straightforward

2.1.1. Exercise: Let d be a non-negative integer and let

$$V = \mathbb{C}_d[x] = \{\text{all polynomials in } x \text{ of degree} \leq d\}.$$

Let $z_1, \ldots, z_n \in \mathbb{C}$ be distinct complex numbers and define

$$\langle f, g \rangle = \sum_{k=1}^{n} f(z_i)\overline{g(z_i)}, \quad f, g \in V.$$

Prove that $\langle -, - \rangle$ is an inner product on V if and only if $n \geq d+1$.

Specifying a form satisfying requirements (1) & (2) of an inner product on $\mathbb{C}^n$ is not that tricky:

Pick any matrix $A \in \mathbb{C}^{n \times n}$ satisfying $A = A^*$ and a basis $\beta = (v_1, \ldots, v_n)$ of V and set

$$\langle x, y \rangle = [y]_\beta^* A [x]_\beta.$$

It is the positivity requirement that poises non-trivial restrictions. For example:

Ex: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Then A defines an inner product on $\mathbb{R}^2$ in the above manner iff
$$b = c \text{ and } x^T A x > 0$$
whenever $x \in \mathbb{R}^2$ is a non-zero vector.
In particular, if $x \in \mathbb{R}^2$ is an eigenvector of A with $Ax = \lambda x$ we must have
$$0 < x^T A x = x^T \lambda x = \lambda x^T x = \lambda \|x\|^2_{std}$$
and we conclude that A can only have positive eigenvalues, which is equivalent to $a+d > 0$ <u>and</u> $ad - bc > 0$.

II Elementary properties of inner products:

We start out with the Cauchy-Schwartz (-Gauss-Bunyakovski-......) inequality. It will be convenient to introduce a new notion:

Defn: Let V be an inner product space. We say that $S \subseteq V$ is an <u>orthogonal</u> set if for any $u, v \in S$ one has $\langle u, v \rangle = 0$ unless $u = v$.
S is <u>orthonormal</u> if, in addition, $\|u\| = 1$ for all $u \in S$.

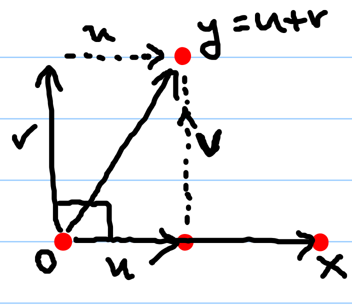
denoted $u \perp v$

Observation: $x \perp y$, $x \perp z$, $\alpha \in \mathbb{C} \Rightarrow x \perp (y + \alpha z)$.

The following picture motivates everything we do next:

Given $x, y \in V$, we seek a decomposition of y as the sum of two vectors u and v with:

$$- \; u \in \mathrm{Span}(x)$$
$$- \; v \perp x$$



We try $u = \frac{\langle y,x\rangle}{\|x\|} \cdot \frac{x}{\|x\|}$, $v = y-u$, and it works:

Obviously, $u \in \mathrm{Span}(x)$ by construction, and

$$\langle y-u, x\rangle = \langle y,x\rangle - \langle u,x\rangle$$
$$= \langle y,x\rangle - \left\langle \frac{\langle y,x\rangle}{\|x\|^2}x, x\right\rangle$$
$$= \langle y,x\rangle - \frac{\langle y,x\rangle}{\|x\|^2}\langle x,x\rangle = 0.$$

Since $v \perp x$ and $u \in \mathrm{Span}(x)$ we also have $u \perp v$.

Now we observe that $\|v\|^2 = \langle v,v\rangle \geq 0$:

$$0 \leq \langle v,v\rangle = \langle y-u, v\rangle = \langle y,v\rangle - \langle u,v\rangle$$
$$= \langle y,v\rangle = \langle y, y-u\rangle = \langle y,y\rangle - \langle y,u\rangle$$
$$= \|y\|^2 - \left\langle y, \frac{\langle y,x\rangle}{\|x\|^2}x\right\rangle = \|y\|^2 - \frac{\overline{\langle y,x\rangle}}{\|x\|^2}\langle y,x\rangle$$
$$= \|y\|^2 - \frac{|\langle y,x\rangle|^2}{\|x\|^2}$$

We conclude:

$$|\langle y,x\rangle|^2 \leq \|x\|^2\|y\|^2$$

Cauchy-Schwartz Inequality

In particular:

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$

So we obtain (MEMORIZE):

$$\begin{aligned}\|x+y\|^2 &= \|x\|^2+\|y\|^2+2\,\mathrm{Re}\langle x,y\rangle \\ &\le \|x\|^2+\|y\|^2+2|\langle x,y\rangle| \\ &\le \|x\|^2+\|y\|^2+2\|x\|\,\|y\| \\ &= (\|x\|+\|y\|)^2\end{aligned}$$

... and we have deduced the triangle inequality for Euclidean Norms:

$$\|x+y\| \le \|x\|+\|y\|.$$

The C-S inequality motivates the following definition of angle:

$$\cos\measuredangle(x,y) := \frac{\mathrm{Re}\langle x,y\rangle}{\|x\|\cdot\|y\|}, \quad \measuredangle(x,y)\in[0,\pi]$$

Which turns the identities $\|x\pm y\|^2=\|x\|^2+\|y\|^2\pm 2\,\mathrm{Re}\langle x,y\rangle$ into nothing more than the Euclidean law of cosines.

It is interesting to remark that the notion of angle is independent of the base field, so that a complex inner product "conveys" more information about V than "just" the geometry of lengths and angles.

2.II.1. Exercise: Suppose $V=V_{\mathbb{C}}$ and $\langle -,-\rangle$ is a complex-valued inner product on V. Show that then

$$(x,y) = \mathrm{Re}\langle x,y\rangle$$

is a real-valued inner product on V, where the scalars are restricted to the real field.

Thus, the notions of distance and angle -

$$\operatorname{dist}(x,y) = \|x-y\| \ , \quad \measuredangle(x,y) = \operatorname{Arccos} \frac{\langle x,y\rangle}{\|x\|\|y\|}$$

- are only manifestations of the real part of $\langle -,- \rangle$. Are there others? FROM NOW ON, A REAL INNER PRODUCT SPACE WILL BE CALLED EUCLIDEN; A COMPLEX INNER PRODUCT SPACE WILL BE CALLED UNITARY.

Example: (triviality criterion) Let V be an inner product space and let $v \in V$. Then:

$$v = 0 \iff \forall_{w\in V} \operatorname{Re}\langle v, w\rangle = 0$$

Indeed, if $v=0$ and $w\in V$ is arbitrary then setting $x = \langle 0, w\rangle$ we obtain

$$x = \langle 0,w\rangle = \langle 0+0,w\rangle = \langle 0,w\rangle + \langle 0,w\rangle = x+x,$$

and for x being an element of a field this means $x=0$. Conversely,

$$\begin{aligned}\forall_{w\in V} \operatorname{Re}\langle v,w\rangle = 0 \quad &\Rightarrow \operatorname{Re}\langle v,v\rangle = 0\\ &\iff \|v\| = 0\\ &\iff v = 0. \quad \blacksquare\end{aligned}$$

But now what does this tell us about $[x,y] \overset{\text{def}}{=} \operatorname{Im}\langle x,y\rangle$ when V is a unitary space? We observe:

- $[v,v] = \operatorname{Im}\langle v,v\rangle = \operatorname{Im}\|v\|^2 = 0$ for all $v\in V$;

- $\forall_{w\in V} [v,w] = 0 \iff \forall_{w\in V} \operatorname{Re}\langle iv,w\rangle = 0 \iff iv = 0 \iff v = 0$;

Also, $[-,-]$ is linear in each of its coordinates (separately), over the field $\mathbb{R}$. This is a totally new kind of form!

2.II.2. **Exercise:** Let V be a normed vector space with norm $\|\cdot\|$. Prove that the norm $\|\cdot\|$ comes from an inner product if and only if

(⊛) $\|x+y\|^2+\|x-y\|^2 = 2\left[\|x\|^2+\|y\|^2\right]$ for all $x,y\in V$

What is the geometric meaning of this identity?

(Hint: you could start by reconstructing $\mathrm{Re}\langle x,y\rangle$ from the expressions in (⊛) using the cosine law; then proceed to tweak those for the complex case)

2.II.3. **Exercise:** Let $V=\mathcal{C}(a,b)$, the space of continuous functions from $[a,b]$ to $\mathbb{C}$.

(1) Prove that $\langle f,g\rangle = \int_a^b f(t)\overline{g(t)}\,dt$ is an inner product.

(2) Verify that, for every positive integer n, the set

$$\mathcal{F}_n := \{1, \cos x, \cos 2x, \dots, \cos nx, \sin x, \sin 2x, \dots, \sin nx\}$$

is an orthogonal set in $\mathcal{C}(0,2\pi)$.

Orthogonality is a fundamental concept in algebra, because:

Lemma: Let V be an inner product space and suppose $S\subseteq V$ is an orthogonal set of non-zero vectors. Then S is lin. indep.

Proof: If $\sum_{i=1}^n \lambda_i s_i = 0$ for distinct elements $s_1,\dots,s_n\in S$, then for every j: $0=\left\langle \sum_i \lambda_i s_i, s_j\right\rangle = \sum \lambda_i \langle s_i,s_j\rangle = \lambda_j$. ▤

The process which facilitated our proof of the C-S inequality deserves special attention: we've obtained the inequality by first decomposing a vector $y \in V$ as the sum of an orthogonal pair of vectors. An iterated version of this process is not hard to prove:

<u>Thm (Gram-Schmidt Process).</u> Let $(u_1, \ldots, u_m)$ be an independent sequence in an inner product space V. Then the sequence $(w_1, \ldots, w_m)$ constructed below is a sequence of pairwise-orthogonal vectors spanning $\mathrm{Span}_{\mathbb{F}}(u_1, \ldots, u_m)$:

$$w_1 = u_1, \quad w_2 = u_2 - \frac{\langle u_2, w_1 \rangle}{\|w_1\|^2} w_1$$

$$\vdots$$

$$w_{k+1} = u_{k+1} - \sum_{i=1}^{k} \frac{\langle u_{k+1}, w_i \rangle}{\|w_i\|^2} w_i$$

$$\vdots$$

In particular, every finite-dimensional inner product space has an orthonormal basis. ▨

Suppose V has an orthonormal basis $\mathcal{E} = (v_1, \ldots, v_d)$. Let $\mathcal{E}^* = (v^1, \ldots, v^d)$ denote the dual basis. Then for every $w \in V$ and $i \in \{1, \ldots, d\}$:

- $w = \sum_{j=1}^{d} v^j(w) v_j$
- $\langle w, v_i \rangle = \left\langle \sum_{j=1}^{d} v^j(w) v_j, v_i \right\rangle = \sum_{j=1}^{d} v^j(w) \langle v_j, v_i \rangle$
  $= v^i(w)$

We conclude:

Lemma: In a finite-dimensional inner product space V, the dual basis to an orthonormal basis $\mathcal{E} = (v_1, \dots, v_d)$ is given by the formula

$$v^i(w) = \langle w, v_i \rangle \qquad \blacksquare$$

The obvious corollary is Parseval's Identity:

$$\forall_{v \in V} \quad v = \sum_{i=1}^{d} \langle v, v_i \rangle v_i$$

In the case when $\mathcal{E} = (v_1, \dots, v_d)$ is an orthogonal basis:

$$\forall_{v \in V} \quad v = \sum_{i=1}^{d} \frac{\langle v, v_i \rangle}{\|v_i\|^2} v_i$$

Fourier coefficients of v wrt. $\mathcal{E}$

Moreover:

$$\|v\|^2 = \langle v, v \rangle = \left\langle \sum_{i=1}^{d} \frac{\langle v, v_i \rangle}{\|v_i\|^2} v_i, \sum_{j=1}^{d} \frac{\langle v, v_j \rangle}{\|v_j\|^2} v_j \right\rangle$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\langle v, v_i \rangle}{\|v_i\|^2} \frac{\overline{\langle v, v_j \rangle}}{\|v_j\|^2} \langle v_i, v_j \rangle$$

$$= \sum_{i=1}^{d} \frac{|\langle v, v_i \rangle|^2}{\|v_i\|^2}$$

We have:

$$\|v\|^2 = \sum_{i=1}^{d} \left| \frac{\langle v, v_i \rangle}{\|v_i\|} \right|^2$$

whenever $(v_1, \dots, v_d)$ is an orthogonal basis of V.

Obviously, if $(v_1, \dots, v_d)$ is an orthogonal set and $v \in V$ then

$$W = \mathrm{Span}(v_1, \dots, v_d, v)$$

is an inner product space and is finite-dimensional. Applying the Gram-Schmidt process to the sequence $(v_1, \dots, v_d, v)$ we obtain an orthogonal basis of W of the form $(v_1, \dots, v_d, v_{d+1})$ (*) and so

There is a small omission here; Find it.

$$\|v\|^2 = \sum_{i=1}^{d+1} \left| \frac{\langle v, v_i \rangle}{\|v_i\|} \right|^2 \overset{\circledast}{\geq} \sum_{i=1}^{d} \left| \frac{\langle v, v_i \rangle}{\|v_i\|} \right|^2$$

we note that $\circledast$ is an equality if and only if $v \in \mathrm{Span}(v_1, \dots, v_d)$.

2.II.4. Verify the last statement, that is: prove that for every $v \in V$ and every orthogonal set $(v_1, \dots, v_d)$ one has

$$\|v\|^2 \geq \sum_{i=1}^{d} \left| \frac{\langle v, v_i \rangle}{\|v_i\|} \right|^2$$

with equality iff $v \in \mathrm{Span}(v_1, \dots, v_d)$.

Bessel's Inequality

Our most recent analysis can be blamed on the following:

2.II.5. Riesz's Representation theorem, baby version: Let V be a finite-dimensional inner product space. Prove that every $f \in V^*$ has some $v_f \in V$ such that

$$f(u) = \langle u, v_f \rangle$$

holds for all $u \in V$. Also answer the following questions:

(1) Is v_f uniquely determined by f?

(2) Is the mapping $f \mapsto v_f$ linear? $\mathbb{R}$-linear?

(III) Orthogonal complements, sums and projections:

Let V be an inner product space and $S \subseteq V$ be a subset. We define:

$$S^{\perp} = \{ v \in V \mid v \perp s \text{ for all } s \in S \}$$

Defn: $S^{\perp}$ is called the orthogonal complement of S in V.

2.III.1. Exercise: Prove the following properties of orthogonal complements

(1) $S^{\perp}$ is a subspace of V;

(2) $S \subseteq S^{\perp\perp}$;

(3) $S \cap S^{\perp} \subseteq \{0\}$.

Lemma: Let W be a finite-dimensional subspace of an inner product space V. Then $V = W \oplus W^{\perp}$.

Proof: Since $W \cap W^{\perp} = \{0\}$, it remains to prove that every $v \in V$ can be written down as a sum $v = w + w_0$ where $w \in W$ and $w_0 \in W^{\perp}$.

Let $(v_1, \ldots, v_d)$ be an orthonormal basis of W. Set

$$w = \sum_{i=1}^{d} \langle v, v_i \rangle v_i , \qquad w_0 = v - w$$

Clearly, $w \in W$ and we need to check that $w_0 \in W^{\perp}$. It will suffice to show that $w_0 \perp v_j$ for every $j = 1, \ldots, d$.

$$\langle w_0, v_j \rangle = \langle v - \sum_{i=1}^{d} \langle v, v_i \rangle v_i , v_j \rangle =$$

$$= \langle v, v_j \rangle - \sum_{i=1}^{d} \langle v, v_i \rangle \langle v_i, v_j \rangle = \langle v, v_j \rangle - \langle v, v_j \rangle = 0$$

and we are done. $\square$

2.III.2. Exercise: Prove the following, for W, U subspaces of a FDIPS V:

$$W^{\perp\perp} = W, \quad (U+W)^{\perp} = U^{\perp} \cap W^{\perp}, \quad (U \cap W)^{\perp} = U^{\perp} + W^{\perp}.$$

Restricting to the finite dimensional case we summarize:

Theorem and Definition: Let V be a finite-dimensional inner product space and let W be a subspace of V. Then there exists a linear operator $P_W : V \to V$ satisfying the following:

(1) $P_W^2 = P_W$

(2) $\mathrm{Im}(P_W) = W$

(3) $\mathrm{Ker}(P_W) = W^{\perp}$

The orthogonal projection to W

Moreover: (4) $I - P_W = P_{W^{\perp}}$,

(5) for any orthonormal basis $(v_1, \dots, v_d)$ of W we have the formula:

$$P_W(v) = \sum_{i=1}^{d} \langle v, v_i \rangle v_i, \quad \text{for all } v \in V$$

(6) for all $v \in V$, $\|P_W(v)\| \leq \|v\|$. ▨

2.III.3. Exercise: Let V be a finite-dimensional inner prod. space and U, W are subspaces. Then (1) $(U+W)^{\perp} = U^{\perp} \cap W^{\perp}$, (2) $(U \cap W)^{\perp} = U^{\perp} + W^{\perp}$.

2.III.4. Exercise: Let W be a subspace of a fin. dim'l inner prod. spc. V and let $v \in V$.

Show that $\|v - P_W(v)\| \leq \|v - w\|$ for all $w \in W$

[$P_W(v)$ is the best approximation of v in W]

2.III.5. Exercise: Let $U, W < V$ as before. Find a necessary and sufficient condition for P_U and P_W to commute ($P_U P_W = P_W P_U$), and prove it.