Problem list from Chapter 1:

Note Title 5/11/2013

1.I.1. <u>Exercise:</u> Let V be the set of solutions to the ODE

$$x^{(4)} - 4x^{(3)} + 3x^{(2)} + 4x^{(1)} - 4x = 0$$

Find a basis for V and prove it is a basis; then compute its dual.

1.II.1 <u>Exercise:</u> Prove that the map $R: (V_{\mathbb{C}})^* \to (V_{\mathbb{R}})^*$ given by $\langle R(f), v\rangle = \mathrm{Re}\langle f, v\rangle$ is $\mathbb{R}$-linear. Is it bijective? If so, what is its inverse?

1.III.2. <u>Exercise:</u> verify the formula for $A, B \geq 0$ and $\alpha, \beta \in [0, 2\pi)$

$$\left.\begin{aligned} z &= A(\cos\alpha + i\sin\beta) \\ w &= B(\cos\alpha + i\sin\beta) \end{aligned}\right\} \Longrightarrow z\cdot w = AB\left(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right)$$

1.II.3. <u>Exercise:</u> consider the set $\Omega_n = \{z \in \mathbb{C} \mid z^n = 1\}$ for n a pos. integer.

(a) Prove $\Omega_n = \left\{ \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n} \;\middle|\; k = 0, \dots, n-1 \right\}$

(b) Provide a formula for all solutions of the equation $az^n - b = 0$ where $a, b \in \mathbb{C}$ are given complex numbers, $a \neq 0$.

(c) Define $\omega_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$
The number ω_n is called the fundamental n-th root of unity.
Compute the sum of all roots of any equation of the form $az^n - b = 0$, $a \neq 0$. What is the geometric meaning of your answer?

1.IV.1. __Exercise:__ Verify that S° and T° are subspaces of V^* and V, respectively. Also, verify that $S^{\circ\circ} \supseteq \operatorname{Span} S$, $T^{\circ\circ} \supseteq \operatorname{Span}(T)$.

1.IV.2. __Exercise:__ Prove that $\operatorname{codim}(W)$ is well-defined by proving $\operatorname{codim}(W) = \dim(W^\circ)$. Then verify that $W \leq V$ has codimension $\leq k$ iff W can be written as an intersection of the form $W = \bigcap_{i=1}^{k} \operatorname{Ker}(f_i)$, $f_i \in V^*$

1.IV.3. __Exercise:__ Let $V = \mathbb{F}^{n\times n}$, the space of $n\times n$ matrices, AKA $M_n(\mathbb{F})$ when considered as a ring with respect to matrix multiplication.

(1) Let $B \in V$ be fixed. Show that $f(X) = \operatorname{trace}(B^T X)$ is a linear functional on V.

(2) Show that $f(XY) = f(YX)$ for any $X, Y \in V$ and f as in (1).

(3) Verify that the set W of all functionals of the form defined in (1) is a vector subspace of V^* and give an upper bound on its dimension.

1.IV.4. __Exercise:__ Let $V = C^\infty(\mathbb{R})$ be the space of all C^∞-smooth real-valued functions on the real line, that is: $f \in V$ if and only if it has continuous derivatives of all orders at every point $p \in \mathbb{R}$.

Let $D: V \to V$ denote the differentiation operator $D(f) = \frac{df(t)}{dt}$, and let $\eta: V \to \mathbb{R}$ denote the linear functional $\eta(f) = f(0)$.

We define $\eta_n(f) = (\eta \circ D^n)(f)$, for all non-negative integers n, and set

$$Z = \operatorname{Span}(\eta_n)_{n\geq 0} \subseteq V^*$$

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1.IV.4. (cntd.) (1) Prove that $(\eta_n)_{n\geq 0}$ is a basis for Z; where have you seen it used?

(2) Prove that Z is a proper subspace of V^* by exhibiting a non-zero element in Z°.

(3) How large do you think Z° really is? Can it possibly be ∞-dimensional?

(4) How much larger is V^{**} in comparison to V? Try producing a lower bound in the form of an independent subset $S \subseteq V^{**} - ev(V)$, with cardinality as high as possible.

(VI) Additional problems:

1.VI.1. Lagrange Interpolation: Consider the sp. $V = \mathbb{F}_d[t]$ of polynomials of degree $\leq d$ with coefficients in the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

① Let $a_0, \ldots, a_d \in \mathbb{F}$ be distinct scalars. Given any $b_0, \ldots, b_d \in \mathbb{F}$, find a polynomial $p(x) \in V$ satisfying $p(a_i) = b_i$ for all $i = 0, \ldots, d$. Be sure to produce an explicit formula.

② With the same $a_0, \ldots, a_d$, let us define $\beta = (\vartheta^0, \ldots, \vartheta^d)$ in V^* by setting

$$\vartheta^i(p) := p(a_i)$$

Find a basis α of V such that $\beta = \alpha^*$ (concluding that β is, in fact, a basis of V^*).

1.VI.2. Let U, W be subspaces of $V_{\mathbb{F}}$. Which of the following hold true, and under what conditions?

(1) $(U+W)^{\circ} = U^{\circ} \cap W^{\circ}$

(2) $(U \cap W)^{\circ} = U^{\circ} + W^{\circ}$

1.VI.3. Let $P \subseteq \mathbb{R}^2$ be a finite subset and let V be the vector space of all vector fields $F : \mathbb{R}^2 - P \to \mathbb{R}^2$ that are C^1-smooth and irrotational. Let $W \subseteq V$ be the subspace of all fields $F \in V$ that are conservative.

(1) Prove that $\dim V$ is infinite by presenting an infinite independent subset of W^*

(2) Prove that $\operatorname{codim} W = |P|$.

(PLEASE SEE REVIEW MATERIAL IN THE NOTES)