Chapter 1: Duality in finite-dimensional Vector Spaces

Note Title 5/9/2013

We restrict attention to the study of Linear Algebra over two fields: the field $\mathbb{R}$ of real numbers and the complex field $\mathbb{C}$.

By construction, $\mathbb{C} = \mathbb{R}^2$ as a set, and one has

- $(a,b)+(c,d) := (a+c, b+d)$
- $(a,b)\cdot(c,d) := (ac-bd, ad+bc)$

$(\cdot)$ distributes over $(+)$!

It is easy to verify the following properties:

1. $(+)$ and $(\cdot)$ are both commutative and associative
2. $(1,0)$ is the multiplicative unit in $\mathbb{C}$
3. for all $z=(x,y)\in\mathbb{C}$, $a\in\mathbb{R}$ one has $(a,0)\cdot z = (ax, ay)$

2 + 3 tell us that $\mathbb{R}\times\{0\} \subseteq \mathbb{C}$ is a subfield that is best identified with $\mathbb{R}$.

4. setting $(x,y) = x\underbrace{(1,0)}_{1} + y\underbrace{(0,1)}_{i} := x+iy$,

one has $i^2 = -1$ and then setting $z = x+iy \overset{\text{def.}}{\Longrightarrow} \bar{z} = x-iy$ one also obtains:

- $\overline{z+w} = \bar{z}+\bar{w}$, $\overline{zw} = \bar{z}\bar{w}$
- $z\bar{z} = |z|^2 := x^2+y^2 \geq 0$ with equality $\Leftrightarrow z=0$
- $z+\bar{z} = 2\,\mathrm{Re}(z)$ $\quad(\mathrm{Re}(x+iy) := x)$

  $z-\bar{z} = 2i\,\mathrm{Im}(z)$ $\quad(\mathrm{Im}(x+iy) := y)$

$z\neq 0$

$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ and in particular $\left|\frac{1}{z}\right| = \frac{1}{|z|}$ in addition to the obvious

$|zw| = |z||w|$, $\left|\frac{z}{w}\right| = |z|/|w|$ etc....

This quick review of the complex numbers is very far from being complete, and we shall keep deepening it as we progress further into discussing the connections between linear algebra and geometry.

(I) Linear Functionals and the Dual space:

Notation: $V_{\mathbb{F}}$ denotes a vector space V over the field $\mathbb{F}$.

- elements of V are "vectors"; elements of $\mathbb{F}$ are "scalars".
- $\mathbb{F}$ is usually $\mathbb{R}$ or $\mathbb{C}$, but not always.

Recall that the set $\mathcal{L}(V,W)$ of all linear transformations from a vector space $V_{\mathbb{F}}$ to the vector space $W_{\mathbb{F}}$ is a vector spc. over $\mathbb{F}$ in its own right:

$$S,T\in \mathcal{L}(V,W),\ \alpha\in\mathbb{F} \Rightarrow \big(\alpha S+T\big)(v) \overset{\text{def}}{=} \alpha S(v)+T(v)$$

A particular case of interest is:

Defn: The dual space V^* of a vector space $V_{\mathbb{F}}$ is defined to be $\mathcal{L}(V,\mathbb{F})$. Elements of V^* are called (linear) functionals on V.

Ex 1: $f\in(\mathbb{F}^n)^* \Rightarrow f(x) = f\left(\sum x_i e_i\right) = \sum x_i f(e_i) = [f(e_1) \cdots f(e_n)]\cdot x$

where e_i are the vectors of the standard basis of $\mathbb{F}^n$.

Thus, $(\mathbb{F}^n)^*$ can be identified with $\mathbb{F}^{1\times n}$. More generally, one then expects $V^*\cong V$ for any finite-dimensional vector space V.

Thm: Let $V_{\mathbb{F}}$ be a finite dimensional vector spc. with basis $(e_1,\dots,e_d)$. Then there exists a unique basis $(e^1,\dots,e^d)$ of $V_{\mathbb{F}}^*$ satisfying $e^i(e_j)=\delta_{ij}$. In particular, $\dim V_{\mathbb{F}}^* = \dim V_{\mathbb{F}}$ and $V_{\mathbb{F}}^*\cong V_{\mathbb{F}}$ whenever $\dim V_{\mathbb{F}}<\infty$.

Defn: $(e^1,\dots,e^d)$ is called the dual basis of $(e_1,\dots,e_d)$.

e_i^* instead of e^i in prof. Gallier's notes.

Proof: Denote $\mathcal{E} = (e_1, \dots, e_d)$ and recall that the map $V_{\mathbb{F}} \xrightarrow{[\cdot]_{\mathcal{E}}} \mathbb{F}^d$ defined by

$$[v]_{\mathcal{E}} = \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} \iff v = \sum_{i=1}^{d} x_i e_i$$

is a linear isomorphism of $V_{\mathbb{F}}$ with $\mathbb{F}^d$. Let $p_i : \mathbb{F}^d \to \mathbb{F}$ be defined by

$$p_i \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} = x_i$$

Then p_i is linear as well and we define $e^i : V_{\mathbb{F}} \to \mathbb{F}$ by

$$e_i(v) = p_i [v]_{\mathcal{E}} .$$

e^i is linear for all i since it is the composition of linear maps.
Now, we observe that

$$e^i(e_j) = p_i \left(\begin{array}{cc} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{c} \\ \\ \leftarrow j\text{th} \\ \\ \\ \end{array} \end{array} \right) = \delta_{ij}$$

In fact, $e^i\left(\sum_{j=1}^{d} x_j e_j\right) = \sum_{j=1}^{d} x_j e^i(e_j) = \sum_{j=1}^{d} x_j \delta_{ij} = x_i$

Thus we have:

$$v = \sum_{i=1}^{d} e^i(v)\, e_i \quad \text{for all } v \in V$$

We are left to prove that $\mathcal{E}^* = (e^1, \dots, e^d)$ is a basis of V^*.

- for any $f \in V^*$ and $v \in V$ we have:

$$f(v) = f\left(\sum_i e^i(v) e_i \right) = \sum_i e^i(v) \underbrace{f(e_i)}_{\text{scalar}}$$

We conclude: $f = \sum_{i=1}^{d} f(e_i)\, e^i$ for all $f \in V^*$
so $\mathcal{E}^*$ spans V^*.

Computing dual basis for $\mathbb{R}^n$

- To prove independence of ε^*, suppose $\sum \lambda_i e^i = 0$ in V^*.

Then $\displaystyle\sum_{i=1}^{d} \lambda_i e^i(e_j) = 0$ for all $j = 1, \dots, d$

$$\Leftrightarrow \sum_{i=1}^{d} \lambda_i \delta_{ij} = 0 \quad \text{- - - - -}$$

$$\Leftrightarrow \lambda_j = 0 \quad \text{- - - - -},$$

which means ε^* is independent. ▨

Cor: for every $v \in V$, $v \neq 0$ there exists $f \in V^*$ with $f(v) = 1$.

Proof: $v \neq 0 \Rightarrow$ there exists a basis containing v. Then its dual basis contains the functional we seek. ▨

Ex: Compute α^* where $\alpha = \left(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}\right)$ in $\mathbb{R}^2$

Soln: denote $v_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $v_2 = \begin{pmatrix}1\\3\end{pmatrix}$. Letting $\varepsilon = (e_1, e_2)$ be the std. basis of $\mathbb{R}^2$ and $\varepsilon^* = (e^1, e^2)$ be its dual, we are looking for

$$v^i = \sum_{k=1}^{2} x_{ik} e^k \quad \text{such that} \quad v^i(v_j) = \delta_{ij} \text{ for all } i, j.$$

We have $\displaystyle\delta_{ij} = \sum_{k=1}^{2} x_{ik} e^k(v_j)$ for all i, j

$$\Leftrightarrow \underline{I} = \underline{X}\,\underline{V}, \quad \underline{V} = [v_1 | v_2] = \begin{bmatrix}\alpha\\\varepsilon\end{bmatrix},$$

$$\underline{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{bmatrix}$$

The base-change matrix satisfying $[x]_\varepsilon = \begin{bmatrix}\alpha\\\varepsilon\end{bmatrix}[x]_\alpha$

$$\Leftrightarrow \underline{X} = \underline{V}^{-1} = \begin{bmatrix}\alpha\\\varepsilon\end{bmatrix}^{-1} = \begin{bmatrix}1 & 1\\2 & 3\end{bmatrix}^{-1} = \frac{1}{1\cdot 3 - 1\cdot 2}\begin{bmatrix}3 & -1\\-2 & 1\end{bmatrix} = \begin{bmatrix}3 & -1\\-2 & 1\end{bmatrix}$$

Thus, $v^1\begin{pmatrix}x\\y\end{pmatrix} = 3x - y$ and $v^2\begin{pmatrix}x\\y\end{pmatrix} = y - 2x$. ▨

In the interest of not leaving the complex numbers unattended, let us do another example:

Ex 2: Let $\beta = \left(\overset{v_1}{\begin{bmatrix} 1 \\ 1+i \\ 0 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 1 \\ 1-i \\ 0 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} 2+i \\ 3i-4 \\ i+3 \end{bmatrix}} \right)$ in $\mathbb{C}^3$. Find the dual basis β^*.

Soln: As before, setting $B = \begin{bmatrix} 1 & 1 & 2+i \\ 1+i & 1-i & 3i-4 \\ 0 & 0 & i+3 \end{bmatrix}$ our goal is to compute

$X = B^{-1}$, whose rows contain the coefficients of the elements of β^* with respect to the dual $\mathcal{E}^*$ of the std. basis $\mathcal{E} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$ of $\mathbb{C}^3$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2+i & 1 & 0 & 0 \\ 1+i & 1-i & 3i-4 & 0 & 1 & 0 \\ 0 & 0 & i+3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3/(i+3)]{R_2-(1-i)R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 2+i & 1 & 0 & 0 \\ 2 & 0 & 0 & 1-i & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{3-i}{10} \end{array}\right]$$

$$\Big\downarrow\ R_2 - 2R_1 \text{ then } R_2/2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1+i}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 2+i & \frac{1-i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{3-i}{10} \end{array}\right] \xleftarrow{R_1-R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 2+i & 1 & 0 & 0 \\ 0 & 1 & 2+i & \frac{1-i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{3-i}{10} \end{array}\right]$$

$$\Big\downarrow\ R_2 - (2+i)R_3$$

$$\left[\begin{array}{c|ccc} & \frac{1+i}{2} & -\frac{1}{2} & 0 \\ I & \frac{1-i}{2} & \frac{1}{2} & -\frac{i+4}{10} \\ & 0 & 0 & \frac{3-i}{10} \end{array}\right] \Rightarrow X = \frac{1}{10}\begin{bmatrix} 5+5i & -5 & 0 \\ 5-5i & 5 & i+4 \\ 0 & 0 & 3-i \end{bmatrix}$$

We conclude:

$$v^1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2}(x-y) + \frac{i}{2}x$$

$$v^2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{10}(5x+5y+4z) + \frac{i}{10}(z-5x)$$

$$v^3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{3-i}{10}z$$

1.II.1 Exercise: Let V be the set of solutions of the ODE
$x^{(4)} - 4x^{(3)} + 3x^{(2)} + 4x^{(1)} - 4x^{(0)} = 0.$
Find a basis for V and prove it is a basis; then compute the dual basis.

(II) The double dual: Define $V^{**} \overset{\text{def.}}{=} (V^*)^*$. Since working with elements of V^* often requires evaluating them at vectors of V, it is convenient to use the notation

$$\langle f | v \rangle \overset{\text{def.}}{=} f(v) \quad \text{for } v \in V,\ f \in V^*$$

Now look at $\varphi \in V^{**}$... what is a good way of thinking of such an animal?

Defn: The evaluation map $\mathrm{ev} : V \to V^{**}$ is defined by

$$\langle \mathrm{ev}(v) \,|\, f \rangle \overset{\text{def.}}{=} \langle f \,|\, v \rangle$$

We check that $\mathrm{ev}(v) \in V^{**}$: for $f, g \in V^*$, $\lambda \in \mathbb{F}$

$$\langle \mathrm{ev}(v) \,|\, f + \alpha g \rangle = \langle f + \alpha g \,|\, v \rangle = \langle f | v \rangle + \alpha \langle g | v \rangle$$

$$= \langle \mathrm{ev}(v) | f \rangle + \alpha \langle \mathrm{ev}(v) | g \rangle$$

Also note that ev is linear: for $v, w \in V$ and $\lambda \in \mathbb{F}$

$$\langle \mathrm{ev}(v + \lambda w) \,|\, f \rangle = \langle f \,|\, v + \lambda w \rangle \overset{f \text{ linear}}{=} \langle f | v \rangle + \lambda \langle f | w \rangle$$

$$= \langle \mathrm{ev}(v) | f \rangle + \lambda \langle \mathrm{ev}(w) | f \rangle$$

$$\Longrightarrow \mathrm{ev}(v + \lambda w) \equiv \mathrm{ev}(v) + \lambda\, \mathrm{ev}(w)$$

In addition, $\mathrm{ev} : V \to V^{**}$ is injective:

By the corollary to the existence of a dual basis.

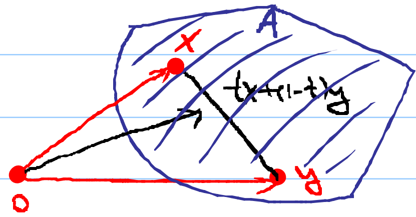
$$v \in \mathrm{Ker}(\mathrm{ev}) \iff \forall_{f \in V^*} \langle \mathrm{ev}(v), f \rangle = 0 \iff \forall_{f \in V^*} f(v) = 0 \Rightarrow v = 0$$

Conclusion: if $V_{\mathbb{F}}$ is finite-dimensional, then $\mathrm{ev} : V \to V^{**}$ is an isomorphism.

III Some reasons to study functionals:

- **Convexity**: a subset $A \subseteq V_{\mathbb{R}}$ is convex, if $x, y \in A$, $t \in [0,1] \Rightarrow tx + (1-t)y \in A$.

A basic problem has been to characterize convex subsets "from above", i.e.: as intersections of standardized convex sets.



Defn: A halfspace in $V_{\mathbb{R}}$ is a set of the form

$$[f \leq 0] \overset{\text{def.}}{=} \{ v \in V \mid f(v) \leq 0 \}$$

for some $f \in V^*$, $f \neq 0$. Also, a hyperplane is the kernel of a non-zero functional; we also sometimes refer to open halfspaces which are sets of the form

$$[f < 0] \overset{\text{def}}{=} \{ v \in V \mid f(v) < 0 \}$$

One expects that a convex set in V is the intersection of halfspaces (open and/or closed). Even in $\mathbb{R}^n$ this is not an entirely trivial theorem, while for ∞-dimensional vector spaces such theorems require deep results in functional analysis such as the Hahn-Banach theorem.

Remark: in the case of $V_{\mathbb{C}}$, there is a distinction between a (linear) hyperplane which is the kernel of a functional $V \to \mathbb{C}$, and a REAL hyperplane. After all $V_{\mathbb{C}}$ can be considered as a vector space (of double dimension) over $\mathbb{R}$, so that $(V_{\mathbb{R}})^*$ and $(V_{\mathbb{C}})^*$ are quite different.

1.III.1 Exercise: Prove that the map $R: (V_{\mathbb{C}})^* \to (V_{\mathbb{R}})^*$ given by $\langle R(f), v \rangle = \mathrm{Re}\langle f, v \rangle$ is $\mathbb{R}$-linear. Is it bijective? If so, what is its inverse?

Limits and differentiation: Many fundamental operations on functions are linear -

$f \mapsto \lim_{x \to x_0} f(x)$ is a linear functional (partially) defined on the vector space of functions, say, $f: U \to \mathbb{R}$, U a nbd. of $x_0 \in \mathbb{R}^n$.

$f \mapsto \left.\frac{\partial f}{\partial x_i}\right|_{x_0}$ is another such functional, defined on a smaller subspace.

For example, real vs. complex differentiation: Given a function of a complex variable $f(z)$, defined and continuous in an open set $U \subseteq \mathbb{C}$ containing a point p, we could take one of two approaches to defining differentiability

Real-valued differentiation: Say that f is diff'ble at p iff there exists an $\mathbb{R}$-linear transformation $A: \mathbb{R}^2 \to \mathbb{R}^2$ (recall that $\mathbb{C} = \mathbb{R}^2$ in our defn.) such that

$$f(p+z) - f(p) = Az + o(|z|)$$

Complex-valued differentiation: Say that f is diff'ble at p iff there exists a $\mathbb{C}$-linear transformation $B: \mathbb{C} \to \mathbb{C}$ such that

$$f(p+z) - f(p) = Bz + o(|z|)$$

When are these the same thing? Well, since $\dim_{\mathbb{C}}(\mathbb{C}) = 1$, we have that the two definitions coinciding for a function f means that, as a linear transformation of $\mathbb{R}^2$, the map $z \mapsto Az$ must coincide with the result of multiplying $z \in \mathbb{C}$ by a complex scalar.

What does THAT imply? Well, let us look at $\mathbb{C}$ once again:

the matrix A is known: writing $f(z)=u(z)+iv(z)$ with u,v real and $z=x+iy$ we know that A is the matrix of partial derivatives:

$$A=\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}$$

However now we also have:

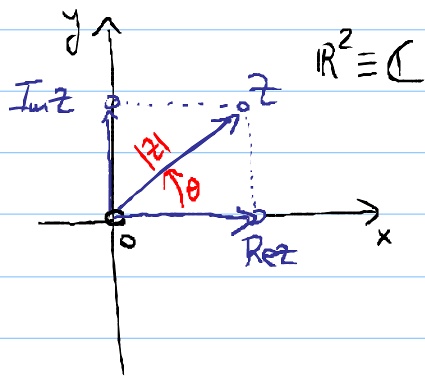
$$A\begin{bmatrix}1\\0\end{bmatrix}=B\cdot 1 \ , \ A\begin{bmatrix}0\\1\end{bmatrix}=B\cdot i$$

with $B=a+ib$ and complex numbers written in column form:

$$\begin{bmatrix}u_x\\u_y\end{bmatrix}=A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix} \ , \ \begin{bmatrix}v_x\\v_y\end{bmatrix}=A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-b\\a\end{bmatrix}$$

$$\Longrightarrow \boxed{u_x=v_y \text{ and } u_y=-v_x}$$ Cauchy-Riemann equations of $\mathbb{C}$-differentiability.

What do these equations mean? To understand this, we will need to revisit the complex numbers.



Writing $z=a+ib$, $\bar{z}=a-ib$
and $|z|^2=a^2+b^2$

We observe: $z=|z|\underbrace{\left[\frac{a}{\sqrt{a^2+b^2}}\cdot 1+\frac{b}{\sqrt{a^2+b^2}}\cdot i\right]}_{\frac{z}{|z|} \text{ is a unit vector in } \mathbb{R}^2}$

$\leadsto$ exists a unique $\theta\in[0,2\pi)$, also known as $\theta=\operatorname{Arg}(z)$,

such that $z=|z|(\cos\theta+i\sin\theta)$

1.III.2. Exercise: verify the formula for $A,B\geq 0$ and $\alpha,\beta\in[0,2\pi)$

$$\left.\begin{array}{l} z=A(\cos\alpha+i\sin\alpha) \\ w=B(\cos\beta+i\sin\beta) \end{array}\right\} \Longrightarrow z\cdot w=AB(\cos(\alpha+\beta)+i\sin(\alpha+\beta))$$

Returning to our original question regarding the meaning of the Cauchy-Riemann equations for $\mathbb{C}$-differentiability, we conclude:

f is $\mathbb{C}$-diff'ble at p $\iff$ $Df|_p$ acts on $\mathbb{R}^2$ by rotation & rescaling

What we want to take away from this short raid on complex function theory is the understanding that a complex-linear functional is NOT THE SAME as a pair of real-linear ones. This is the proper time to revisit exercise 1.III.1.

Finally, here are some $\mathbb{C}$-muscle-growing exercises:

1.II.3. Exercise: consider the set $\Omega_n = \{z \in \mathbb{C} \mid z^n = 1\}$ for n a pos. integer.

(a) Prove $\Omega_n = \left\{ \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n} \;\middle|\; k = 0, \dots, n-1 \right\}$

(b) Provide a formula for all solutions of the equation $az^n - b = 0$ where $a, b \in \mathbb{C}$ are given complex numbers, $a \neq 0$.

(c) Define $\omega_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$
The number ω_n is called the fundamental n-th root of unity.
Compute the sum of all roots of any equation of the form $az^n - b = 0$, $a \neq 0$. What is the geometric meaning of your answer?

(IV) Linear Independence, Annihilators:

Defn: Let $V_{\mathbb{F}}$ be a vec. spc. and let $S \subseteq V$, $T \subseteq V^*$ be subsets. We define the annihilators of S and T to be:

$$S^\circ = \{ f \in V^* \mid f(s) = 0 \text{ for all } s \in S \}$$
$$T^\circ = \{ v \in V \mid t(v) = 0 \text{ for all } t \in T \}$$

1.IV.1. Exercise: Verify that S° and T° are subspaces of V^* and V, respectively. Also, verify that $S^{\circ\circ} \supseteq \mathrm{Span}\, S$, $T^{\circ\circ} \supseteq \mathrm{Span}(T)$.

Observing that a linear functional $f \in V^*$ defines a homogeneous linear equation $f(v) = 0$ on V, we expect the following results to hold:

Thm (Properties of annihilators): Let V be a finite-dimensional space and let $W \subseteq V$ be a subspace. Then

$$W^{\circ\circ} = W \quad \text{and} \quad \dim W + \dim W^\circ = \dim V$$

Proof: Let $\varepsilon = (v_1, \dots, v_d, v_{d+1}, \dots, v_n)$ be a basis of V such that $(v_1, \dots, v_d)$ is a basis of W, and consider the dual basis ε^*. We observe that

$$v^k(v_i) = 0 \text{ whenever } i \le d < k \Rightarrow v^{d+1}, \dots, v^n \in W^\circ,$$

hence $W^\circ \supseteq Sp(v^{d+1}, \dots, v^n)$. Now, taking any $f \in W^\circ$ we write

$$f = \sum_{i=1}^{n} f(v_i)\, v^i = \sum_{i=d+1}^{n} f(v_i)\, v^i \in Sp(v^{d+1}, \dots, v^n)$$

And we obtain $W^\circ = Sp(v^{d+1}, \dots, v^n)$ and in particular $\dim W^\circ = n - d$.

In order to verify $W = W^{\circ\circ}$, we apply the double dual:

Let $\overline{W}$ denote the annihilator of W° in V^{**}.
Then
$$\dim \overline{W} + \dim W^{\circ} = \dim V^{*} = \dim V$$
$$\Rightarrow \dim \overline{W} = \dim W.$$

Now, looking at the definition of $W^{\circ\circ}$ we see:

$$v \in W^{\circ\circ} \iff f(v) = 0 \text{ for all } f \in W^{\circ}$$
$$\iff \langle f, v \rangle = 0 \text{ for all } f \in W^{\circ}$$
$$\iff ev(v) \in \overline{W}$$

Thus $W^{\circ\circ} = ev^{-1}(\overline{W})$ and, since ev is an isomorphism, we have
$$\dim W^{\circ\circ} = \dim \overline{W} = \dim W.$$

At the same time, $W \subseteq W^{\circ\circ}$ and we must conclude $W = W^{\circ\circ}$.

$\boxtimes$

Remark: This is not necessarily the most direct proof of the identity $W = W^{\circ\circ}$ for finite-dimensional vector spaces, but it does "put a finger" on the reason why this identity fails for ∞-dimensional ones.

Defn: The co-dimension of a subspace $W < V$ is defined as
$$\operatorname{codim}(W) := \dim W_1, \text{ for any } W_1 < V \text{ satisfying } V = W \oplus W_1.$$

1.IV.2. Exercise: Prove that $\operatorname{codim}(W)$ is well-defined by proving $\operatorname{codim}(W) = \dim(W^{\circ})$. Then verify that $W < V$ has codimension $\leq k$ iff W can be written as an intersection of the form
$$W = \bigcap_{i=1}^{k} \operatorname{Ker}(f_i), \quad f_i \in V^{*}$$

1.IV.3. <u>Exercise</u>: Let $V=\mathbb{F}^{n\times n}$, the space of $n\times n$ matrices, AKA $M_n(\mathbb{F})$ when considered as a ring with respect to matrix multiplication.

(1) Let $B\in V$ be fixed. Show that $f(X)=\mathrm{trace}(B^TX)$ is a linear functional on V.

(2) Show that $f(XY)=f(YX)$ for any $X,Y\in V$ and f as in (1).

(3) Verify that the set W of all functionals of the form defined in (1) is a vector subspace of V^* and give an upper bound on its dimension.

1.IV.4. <u>Exercise</u>: Let $V=\mathcal{C}^\infty(\mathbb{R})$ be the space of all $\mathcal{C}^\infty$-smooth real-valued functions on the real line, that is: $f\in V$ if and only if it has continuous derivatives of all orders at every point $p\in\mathbb{R}$.

Let $D: V\to V$ denote the differentiation operator $D(f)=\frac{df(t)}{dt}$, and let $\eta: V\to\mathbb{R}$ denote the linear functional $\eta(f)=f(0)$.

We define $\eta_n(f)=(\eta\circ D^n)(f)$, for all non-negative integers n, and set

$$Z=\mathrm{Span}(\eta_n)_{n\geq 0}\subseteq V^*$$

(1) Prove that $(\eta_n)_{n\geq 0}$ is a basis for Z; where have you seen it used?

(2) Prove that Z is a proper subspace of V^* by exhibiting a non-zero element in Z°.

(3) How large do you think Z° really is? Can it possibly be ∞-dimensional?

(4) How much larger is V^{**} in comparison to V? Try producing a lower bound in the form of an independent subset $S\subseteq V^{**}-\mathrm{ev}(V)$, with cardinality as high as possible.

(V) Duality for Operators:

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[dr, dotted, "A^*\varphi"'] & W \arrow[d, "\varphi"] \\ & \mathbb{F} \end{tikzcd}$$

One of the main utilities of the dual space is in its producing a solid framework within which the significance of the transposition operation on matrices becomes apparent.

Defn: Let $A: V_{\mathbb{F}} \to W_{\mathbb{F}}$ be a linear transformation. Then $A^*: W^* \to V^*$ is defined by

$$\langle A^*\varphi \mid v \rangle = \langle \varphi \mid Av \rangle \quad \text{(see diagram above)}$$

Equivalently,

$$A^*\varphi = \varphi \circ A. \quad \blacksquare$$

A^* is denoted A^T in Gallier's notes.

The following results then becomes fundamental for all computations with functionals:

Thm: (Properties of the transpose) Let $U_{\mathbb{F}} \xrightarrow{A} V_{\mathbb{F}} \xrightarrow{B} W_{\mathbb{F}}$ be linear transformations over the field $\mathbb{F}$. Then:

(1) $(BA)^* = A^* B^*$;

(2) if $U_{\mathbb{F}} \xrightarrow{A_1} V_{\mathbb{F}}$ is linear, too, then $(A + \lambda A_1)^* = A^* + \lambda A_1^*$ for all $\lambda \in \mathbb{F}$;

(3) if α, β are bases for U and V, respectively, then

$$[A^*]_{\alpha^*}^{\beta^*} = \left([A]_{\beta}^{\alpha}\right)^T$$

Thus, the transpose of a matrix is nothing more than the matrix representing its dual transformation with respect to the std. coordinate basis.

We should ask ourselves how dual operators interact with the double dual identification. That is, given $T: V \to W$ we have $T^*: W^* \to V^*$ and hence also

$$\begin{array}{ccccc} T^{**}: & V^{**} & \longrightarrow & W^{**} \\ & \uparrow ev & & \uparrow ev \\ T: & V & \longrightarrow & W \end{array}$$

Taking $v \in V$ we compare:

$$(T^{**} \circ ev)(v) \stackrel{?}{=} (ev \circ T)(v) \qquad ⛤$$

Pick any $\varphi \in W^*$ and calculate:

$$\langle T^{**}(ev(v)) \mid \varphi \rangle = \langle ev(v) \mid T^*\varphi \rangle = \langle T^*\varphi, v \rangle$$
$$= \langle \varphi \mid Tv \rangle = \langle ev(Tv) \mid \varphi \rangle = \langle (ev \circ T)(v) \mid \varphi \rangle$$

Thus, equality does hold in ⛤

<u>Thm:</u> (Naturality of the double dual) For any linear transformation $T: V_{\mathbb{F}} \to W_{\mathbb{F}}$ there is a commutative diagram

$$\begin{array}{ccccc} T^{**}: & V^{**} & \longrightarrow & W^{**} \\ & \uparrow ev & \circlearrowleft & \uparrow ev \\ T: & V & \longrightarrow & W \end{array}$$

$\blacksquare$

Theorem: Let $T: V \to W$ be a linear transformation, then:

(1) $\operatorname{Ker} T^* = (\operatorname{Im} T)^\circ$

(2) If $\dim V < \infty$ then $\operatorname{Im} T^* = (\operatorname{Ker} T)^\circ$

Proof: Let $x \in W^*$; then

$$x \in \operatorname{Ker} T^* \Leftrightarrow T^* x = 0 \Leftrightarrow \langle T^* x | v \rangle = 0 \text{ for all } v \in V$$

$$\Leftrightarrow \langle x | Tv \rangle = 0 \text{ for all } v \in V$$

$$\Leftrightarrow \langle x | w \rangle = 0 \text{ for all } w \in \operatorname{Im} T$$

$$\Leftrightarrow x \in (\operatorname{Im} T)^\circ,$$

and this proves (1). For (2) we observe

$$y \in \operatorname{Im} T^* \Leftrightarrow \text{exists } x \in W^* \text{ such that } y = T^* x$$

$\Rightarrow$ for any $v \in \operatorname{Ker} T$ one has

$$\langle y | v \rangle = \langle T^* x | v \rangle = \langle x | Tv \rangle = \langle x | 0 \rangle = 0$$

$$\Rightarrow y \in (\operatorname{Ker} T)^\circ.$$

Thus we have $\operatorname{Im} T^* \subseteq (\operatorname{Ker} T)^\circ$. We also have

This is where $\dim V < \infty$ is being used!

$$\dim (\operatorname{Ker} T)^\circ = \dim V - \dim \operatorname{Ker} T = \dim \operatorname{Im} T = \dim \operatorname{Im} T^*,$$

which completes the proof, up to verifying the last equality. ∎

Lemma: Suppose $T: V \to W$ and $\dim V < \infty$. Then $\dim \operatorname{Im} T = \dim \operatorname{Im} T^*$

Proof: We already know that $\operatorname{Im} T^* \subseteq (\operatorname{Ker} T)^\circ$ for any T.

Now, consider $T^*: W^* \to V^*$: By applying (1) of the preceding argument to T^* we obtain:

$$(\operatorname{Im}(T^*))^\circ = \operatorname{Ker} T^{**}$$

For any $\varphi \in V^{**}$ we have: $\varphi \in \operatorname{Ker} T^{**} \Leftrightarrow T^{**}\varphi = 0$

But now $\dim V < \infty$ implies every such φ can be written as $\varphi = ev(v)$

$$\Leftrightarrow T^{**}(ev(v)) = 0 \Leftrightarrow ev(T(v)) = 0$$

$$\Leftrightarrow T(v) = 0 \Leftrightarrow v \in \operatorname{Ker} T$$

$$\Leftrightarrow \varphi \in ev(\operatorname{Ker} T)$$

and so $\operatorname{Ker} T^{**} = ev(\operatorname{Ker} T)$ and since ev is an isomorphism:

$$\dim(\operatorname{Im} T^*)^\circ = \dim \operatorname{Ker} T^{**} = \dim \operatorname{Ker} T = \dim V - \dim \operatorname{Im} T$$

$\Rightarrow \dim \operatorname{Im} T^* = \dim \operatorname{Im} T$, as desired. $\blacksquare$

Remark: This is an extension of the theorem stating that the row rank of a matrix equals its column rank.

(VI) Additional Problems:

1.VI.1. Lagrange Interpolation: Consider the sp. $V = \mathbb{F}_d[t]$ of polynomials of degree $\leq d$ with coefficients in the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

(1) Let $a_0, \ldots, a_d \in \mathbb{F}$ be distinct scalars. Given any $b_0, \ldots, b_d \in \mathbb{F}$, find a polynomial $p(x) \in V$ satisfying $p(a_i) = b_i$ for all $i = 0, \ldots, d$. Be sure to produce an explicit formula.

(2) With the same $a_0, \ldots, a_d$, let us define $\beta = (v^0, \ldots, v^d)$ in V^* by setting

$$v^i(p) := p(a_i)$$

Find a basis α of V such that $\beta = \alpha^*$ (concluding that β is, in fact, a basis of V^*).

1.VI.2. Let U, W be subspaces of $V_{\mathbb{F}}$. Which of the following hold true, and under what conditions?

(1) $(U+W)^0 = U^0 \cap W^0$

(2) $(U \cap W)^0 = U^0 + W^0$

1.VI.3. Let $P \subseteq \mathbb{R}^2$ be a finite subset and let V be the vector space of all vector fields $F: \mathbb{R}^2 - P \to \mathbb{R}^2$ that are $\mathcal{C}^1$ smooth and irrotational. Let $W \subseteq V$ be the subspace of all fields $F \in V$ that are conservative.

(1) Prove that $\dim V$ is infinite by presenting an infinite independent subset of W^*

(2) Prove that $\operatorname{codim} W = |P|$.

(next page)

Here are some reminders:

- A vector field $F(x,y) = P(x,y)e_1 + Q(x,y)e_2$ (here $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$) is said to be <u>irrotational</u> in a region $\mathcal{U} \subseteq \mathbb{R}^2$ if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ throughout $\mathcal{U}$.
- $F : \mathcal{U} \to \mathbb{R}^2$ is said to be <u>conservative</u>, if all path integrals of the form

$$\int_\gamma P dx + Q dy \overset{\text{def}}{=} \int_a^b \langle F(t), \gamma'(t) \rangle dt \quad , \quad \gamma : [a,b] \to \mathbb{R}^2$$

only depend on the endpoints $\gamma(a)$ and $\gamma(b)$ of γ. The conservativity of $\mathcal{C}^1$-continuous vector field F in $\mathcal{U} \subseteq \mathbb{R}^2$ is equivalent to F coinciding on $\mathcal{U}$ with the gradient field of a $\mathcal{C}^2$ function $\varphi : \mathcal{U} \to \mathbb{R}$. Such functions are called <u>potentials</u>.
- Perhaps the most significant property of $\mathcal{C}^1$ vector fields $F : \mathcal{U} \to \mathbb{R}^2$, $\mathcal{U} \subseteq \mathbb{R}^2$ open is <u>Green's theorem</u>:

<u>Thm</u>: Let $\mathcal{U} \subseteq \mathbb{R}^2$ be open and $F : \mathcal{U} \to \mathbb{R}^2$ be a $\mathcal{C}^1$ vector field on $\mathcal{U}$. Then for any compact domain D with piecewise smooth boundary ∂D such that $\bar{D} = D \cup \partial D$ is contained in $\mathcal{U}$, one has:

$$\int_{\partial D} P dx + Q dy = \iint_{\bar{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \,,$$

where $F \equiv P e_1 + Q e_2$ and ∂D is taken with positive orientation.

- From the characterization via potentials it is easy to see (by the mixed partials theorem) that a conservative field is irrotational. The converse is, in general, false, but the extent to which it is false is governed by the specific shape of $\mathcal{U}$, via Green's theorem. This being a very condensed review of the basic relevant facts, you are advised to seek a more complete review in your calculus text.