Chapter 3

The Hirzebruch-Riemann-Roch Theorem

3.1 Line Bundles, Vector Bundles, Divisors

From now on, $X$ will be a complex, irreducible, algebraic variety (not necessarily smooth). We have

(I) $X$ with the Zariski topology and $O_X$ = germs of algebraic functions. We will write $X$ or $X_{\text{Zar}}$.

(II) $X$ with the complex topology and $O_X$ = germs of algebraic functions. We will write $X_{\mathbb{C}}$ for this.

(III) $X$ with the complex topology and $O_X$ = germs of holomorphic functions. We will write $X_{\text{an}}$ for this.

(IV) $X$ with the complex topology and $O_X$ = germs of $C^\infty$-functions. We will write $X_{C^\infty}$ or $X_{\text{smooth}}$ in this case.

Vector bundles come in four types: Locally trivial in the $\mathbb{Z}$-topology (I); Locally trivial in the $\mathbb{C}$-topology (II, III, IV).

Recall that a rank $r$ vector bundle over $X$ is a space, $E$, together with a surjective map, $p: E \to X$, so that the following properties hold:

1. There is some open covering, $\{U_\alpha \to X\}$, of $X$ and isomorphisms

$$\varphi_\alpha: p^{-1}(U_\alpha) \to U_\alpha \prod C^r \quad (\text{local triviality})$$

We also denote $p^{-1}(U_\alpha)$ by $E \mid U_\alpha$.

2. For every $\alpha$, the following diagram commutes:

$$
\begin{array}{ccc}
\text{p}^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \prod C^r \\
\downarrow p & & \downarrow \text{pr}_1 \\
U_\alpha & & 
\end{array}
$$
(3) Consider the diagram

\[
\begin{array}{c}
p^{-1}(U_{\alpha}) \xrightarrow{\varphi_{\alpha}} U_{\alpha} \prod C^r \\
p^{-1}(U_{\alpha} \cap U_{\beta}) \xrightarrow{\varphi_{\alpha}} (U_{\alpha} \cap U_{\beta}) \prod C^r \\
p^{-1}(U_{\beta} \cap U_{\alpha}) \xrightarrow{\varphi_{\beta}} (U_{\beta} \cap U_{\alpha}) \prod C^r \\
p^{-1}(U_{\beta}) \xrightarrow{\varphi_{\beta}} U_{\beta} \prod C^r
\end{array}
\]

where \( g_{\alpha}^{\beta} = \varphi_{\beta} \circ \varphi_{\alpha}^{-1} \mid p^{-1}(U_{\alpha} \cap U_{\beta}) \). Then,

\[
g_{\alpha}^{\beta} \mid U_{\alpha} \cap U_{\beta} = \text{id} \quad \text{and} \quad g_{\alpha}^{\beta} \mid C^r \in \text{GL}_r(\Gamma(U_{\alpha} \cap U_{\beta}, \mathcal{O}_X))
\]

and the functions \( g_{\alpha}^{\beta} \) in the gluing give type II, III, IV.

On triple overlaps, we have

\[
g_{\alpha}^{\gamma} \circ g_{\beta}^{\alpha} = g_{\gamma}^{\alpha} \quad \text{and} \quad g_{\beta}^{\alpha} = (g_{\alpha}^{\beta})^{-1}.
\]

This means that the \( \{g_{\alpha}^{\beta}\} \) form a 1-cocycle in \( Z^1(\{U_{\alpha} \longrightarrow X\}, \mathcal{G}L_r) \). Here, we denote by \( \mathcal{G}L_r(X) \), or simply \( \mathcal{G}L_r \), the sheaf defined such that, for every open, \( U \subseteq X \),

\[
\Gamma(U, \mathcal{G}L_r(X)) = \text{GL}_r(\Gamma(U, \mathcal{O}_X)),
\]

the group of invertible linear maps of the free module \( \Gamma(U, \mathcal{O}_X)^r \cong \Gamma(U, \mathcal{O}_X^r) \). When \( r = 1 \), we also denote the sheaf \( \mathcal{G}L_1(X) \) by \( \mathbb{G}_m \) or \( \mathcal{O}_X^* \).

Say \( \{\psi_{\alpha}\} \) is another trivialization. We may assume (by refining the covers) that \( \{\varphi_{\alpha}\} \) and \( \{\psi_{\alpha}\} \) use the same cover. Then, we have an isomorphism, \( \sigma_{\alpha} : U_{\alpha} \prod C^r \rightarrow U_{\alpha} \prod C^r \):

\[
\begin{array}{c}
U_{\alpha} \prod C^r \\
p^{-1}(U_{\alpha}) \\
\varphi_{\alpha}
\end{array}
\begin{array}{c}
\sigma_{\alpha} \\
U_{\alpha} \prod C^r
\end{array}
\begin{array}{c}
\psi_{\alpha}
\end{array}
\]

We see that \( \{\sigma_{\alpha}\} \) is a 0-cochain in \( C^0(\{U_{\alpha} \longrightarrow X\}, \mathcal{G}L_r) \). Let \( \{h_{\alpha}^{\beta}\} \) be the gluing data from \( \{\psi_{\alpha}\} \). Then, we have

\[
\begin{aligned}
\varphi_{\beta} &= g_{\alpha}^{\beta} \circ \varphi_{\alpha} \\
\psi_{\beta} &= h_{\alpha}^{\beta} \circ \psi_{\alpha} \\
\psi_{\alpha} &= \sigma_{\alpha} \circ \varphi_{\alpha}.
\end{aligned}
\]

From this, we deduce that \( \sigma_{\beta} \circ \varphi_{\beta} = \psi_{\beta} = h_{\alpha}^{\beta} \circ \sigma_{\alpha} \circ \varphi_{\alpha} \), and then

\[
\varphi_{\beta} = (\sigma_{\beta}^{-1} \circ h_{\alpha}^{\beta} \circ \sigma_{\alpha}) \circ \varphi_{\alpha},
\]
so
\[ \delta_\beta = \sigma_\beta^{-1} \circ h^\beta \circ \sigma_\alpha. \]
This gives an equivalence relation, \( \sim \), on \( Z^1(\{ U_\alpha \to X \}, GL_r) \). Set
\[ H^1(\{ U_\alpha \to X \}, GL_r) = Z^1 / \sim. \]
This is a pointed set. If we pass to the right limit over covers by refinement and call the pointed set from the limit \( \hat{H}^1(X, GL_r) \), we get

**Theorem 3.1** If \( X \) is an algebraic variety of one of the types \( T = I, II, III, IV \), then the set of isomorphism classes of rank \( r \) vector bundles, \( \text{Vect}_{T,r}(X) \), is in one-to-one correspondence with \( \hat{H}^1(X, GL_r) \).

**Remarks:**

1. If \( F \) is some “object” and \( \text{Aut}(F) = \) is the group of automorphisms of \( F \) (in some category), then an \( X \)-torsor for \( F \) is just an “object, \( E \), over \( X \)”, locally (on \( X \)) of the form \( U \coprod F \) and glued by the pairs \((\text{id}, g)\), where \( g \in \text{Maps}(U \cap V, \text{Aut}(F)) \) on \( U \cap V \). The theorem says: \( \hat{H}^1(X, \text{Aut}(F)) \) classifies the \( X \)-torsors for \( F \).

Say \( F = \mathbb{P}_r^c \), we’ll show that in the types I, II, III, \( \text{Aut}(F) = \mathbb{PGL}_r \), where
\[ 0 \to \mathbb{G}_m \to GL_{r+1} \to \mathbb{PGL}_r \to 0 \]
is exact.

2. Say \( 1 \to G' \to G \to G'' \to 1 \) is an exact sequence of sheaves of (not necessarily commutative) groups. Check that
\[
\begin{array}{cccccc}
1 & \to & G'(X) & \to & G(X) & \to & G''(X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(X, G') & \to & H^1(X, G) & \to & H^1(X, G'')
\end{array}
\]
is an exact sequence of pointed sets. To compute \( \delta_0(\sigma) \) where \( \sigma \in G''(X) \), proceed as follows: Cover \( X \) by suitable \( U_\alpha \) and pick \( s_\alpha \in G(U_\alpha) \) mapping to \( \sigma \restriction U_\alpha \) in \( G''(U_\alpha) \). Set
\[ \delta_0(\sigma) = s_\alpha s_\beta^{-1} \text{ on } U_\alpha \cap U_\beta / \sim. \]
We find that \( \delta_0(\sigma) \in \hat{H}^1(X, G') \). When \( G' \subseteq Z(G) \), we get the exact sequence
\[
\begin{array}{cccccc}
1 & \to & G'(X) & \to & G(X) & \to & G''(X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(X, G') & \to & H^1(X, G) & \to & H^1(X, G'') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^2(X, G')
\end{array}
\]

3. Apply the above to the sequence
\[ 0 \to \mathbb{G}_m \to GL_{r+1} \to \mathbb{PGL}_r \to 1. \]
If $X$ is a projective variety, we get
\[ 0 \rightarrow \Gamma(X, \mathcal{O}_X) \rightarrow \mathbb{GL}_{r+1}(\Gamma(X, \mathcal{O}_X)) \rightarrow \mathbb{PGL}_r(\Gamma(X, \mathcal{O}_X)) \rightarrow 0, \]
because $\Gamma(X, \mathcal{O}_X^\times) = \mathbb{C}^*$ and $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$. Consequently, we also have
\[ 0 \rightarrow \check{H}^1(X, \mathcal{O}_X^\times) \rightarrow \check{H}^1(X, \mathbb{GL}_{r+1}) \rightarrow \check{H}^1(X, \mathbb{PGL}_r) \rightarrow \check{H}^2(X, \mathcal{O}_X^\times) = \text{Br}(X), \]
where the last group, $\text{Br}(X)$, is the cohomological Brauer group of $X$ of type $T$. By our theorem, $\check{H}^1(X, \mathcal{O}_X^\times) = \text{Pic}(X)$ classifies type $T$ line bundles, $\check{H}^1(X, \mathbb{GL}_{r+1})$ classifies type $T$ rank $r + 1$ vector bundles and $\check{H}^1(X, \mathbb{PGL}_r)$ classifies type $T$ fibre bundles with fibre $\mathbb{P}^r_C$ (all on $X$).

Let $X$ and $Y$ be two topological spaces and let $\pi: Y \rightarrow X$ be a surjective continuous map. Say we have sheaves of rings $\mathcal{O}_X$ on $X$ and $\mathcal{O}_Y$ on $Y$; we have a homomorphism of sheaves of rings, $\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_Y$. Then, each $\mathcal{O}_Y$-module (or $\mathcal{O}_Y$-algebra), $\mathcal{F}$, gives us the $\mathcal{O}_X$-module (or algebra), $\pi_* \mathcal{F}$ on $X$ (and more generally, $R^q \pi_* \mathcal{F}$) as follows: For any open subset, $U \subseteq X$,
\[ \Gamma(U, \pi_* \mathcal{F}) = \Gamma(\pi^{-1}(U), \mathcal{F}). \]
So, $\Gamma(\pi^{-1}(U), \mathcal{O}_Y)$ acts on $\Gamma(\pi^{-1}(U), \mathcal{F})$ and commutes to restriction to smaller opens. Consequently, $\pi_* \mathcal{F}$ is a $\pi_* \mathcal{O}_Y$-module (or algebra) and then $\mathcal{O}_X$ acts on it via $\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_Y$. Recall also, that $R^q \pi_* \mathcal{F}$ is the sheaf on $X$ generated by the presheaf
\[ \Gamma(U, R^q \pi_* \mathcal{F}) = H^q(\pi^{-1}(U), \mathcal{F}). \]
If $\mathcal{F}$ is an algebra (not commutative), then only $\pi_*$ and $R^1 \pi_*$ are so-far defined.

Let’s look at $\mathcal{F}$ and $\Gamma(Y, \mathcal{F}) = \Gamma(\pi^{-1}(X), \mathcal{F}) = \Gamma(X, \pi_* \mathcal{F})$. Observe that
\[ \Gamma(Y, -) = \Gamma(X, -) \circ \pi_*. \]
So, if $\pi_*$ maps an injective resolution to an exact sequence, then the usual homological algebra gives the spectral sequence of composed functors (Leray spectral sequence)
\[ E_2^{p,q} = H^p(X, R^q \pi_* \mathcal{F}) \Rightarrow H^*(Y, \mathcal{F}). \]
We get the exact sequence of terms of low degree (also called edge sequence)
\[ 1 \rightarrow H^1(X, \pi_* \mathcal{F}) \rightarrow H^1(Y, \mathcal{F}) \rightarrow H^0(X, R^1 \pi_* \mathcal{F}) \rightarrow H^2(X, \pi_* \mathcal{F}) \rightarrow H^2(Y, \mathcal{F}) \rightarrow \cdots \]
In the non-commutative case, we get only
\[ 1 \rightarrow H^1(X, \pi_* \mathcal{F}) \rightarrow H^1(Y, \mathcal{F}) \rightarrow H^0(X, R^1 \pi_* \mathcal{F}). \]

**Application**: Let $X$ be an algebraic variety with the Zariski topology, let $\mathcal{O}_X$ be the sheaf of germs of algebraic functions and let $Y = X_C$ also with $\mathcal{O}_Y = \mathcal{O}_X$ the sheaf of germs of algebraic functions. The map $\pi: Y \rightarrow X$ is just the identity, which is continuous since the Zariski topology is coarser than the $\mathbb{C}$-topology. Take $\mathcal{F} = (\text{possibly noncommutative}) \mathbb{GL}_r$.

**Claim**: $R^1 \text{id}_* \mathbb{GL}_r = (0)$, for all $r \geq 1.$
Proof. It suffices to prove that the stalks are zero. But these are the stalks of the corresponding presheaf
\[
\lim_{U \ni x} H^1_L(U, \mathbb{G}_m)
\]
where \( U \) runs over \( \mathbb{Z} \)-opens and \( H^1 \) is taken in the \( \mathbb{C} \)-topology. Pick \( x \in X \) and some \( \xi \in H^1_L(U, \mathbb{G}_m) \) for some \( \mathbb{Z} \)-open, \( U \ni x \). So, \( \xi \) consists of a vector bundle on \( U \), locally trivial in the \( \mathbb{C} \)-topology. There is some open in the \( \mathbb{C} \)-topology, call it \( U_0 \), with \( x \in U_0 \) and \( U_0 \subseteq U \) where \( \xi \mid U_0 \) is trivial if there exists some sections, \( \sigma_1, \ldots, \sigma_r \), of \( \xi \) over \( U_0 \), and \( \sigma_1, \ldots, \sigma_r \) are linearly independent everywhere on \( U_0 \). The \( \sigma_j \) are algebraic functions on \( U_0 \) to \( \mathbb{C}^r \). Moreover, they are l.i. on \( U_0 \) iff \( \sigma_1 \wedge \cdots \wedge \sigma_r \) is everywhere nonzero on \( U_0 \). But, \( \sigma_1 \wedge \cdots \wedge \sigma_r \) is an algebraic function and its zero set is a \( \mathbb{Z} \)-closed subset in \( X \). So, its complement, \( V \), is \( \mathbb{Z} \)-open and \( x \in U_0 \subseteq V \cap U \). It follows that \( \xi \restriction V \cap U \) is trivial (since the \( \sigma_j \) are l.i. everywhere); so, \( \xi \) indeed becomes trivial on a \( \mathbb{Z} \)-open, as required. \( \square \)

Apply our exact sequence and get

**Theorem 3.2 (Comparison Theorem)** If \( X \) is an algebraic variety, then the canonical map

\[
\text{Vect}^r_{\text{Zar}}(X) \cong \tilde{H}^1(X_{\text{Zar}}, \mathbb{G}_m) \rightarrow H^1(X, \mathbb{G}_m) \cong \text{Vect}^r_{\mathbb{C}}(X)
\]
is an isomorphism for all \( r \geq 1 \) (i.e., a bijection of pointed sets).

Thus, to give a rank \( r \) algebraic vector bundle in the \( \mathbb{C} \)-topology is the same as giving a rank \( r \) algebraic vector bundle in the Zariski topology.

If we use \( \mathcal{O}_X = \) holomorphic (analytic) functions, then for many \( X \), we get only an injection

\[
\text{Vect}^r_{\text{Zar}}(X) \hookrightarrow \text{Vect}^r_{\mathbb{C}}(X).
\]

**Connection with the geometry inside \( X \):**

First, assume \( X \) is smooth and irreducible (thus, connected). Let \( V \) be an irreducible subvariety of codimension 1. We know from Chapter 1 that locally on some open, \( U \), there is some \( f \in \Gamma(U, \mathcal{O}_X) = \mathcal{O}_U \) such that \( f = 0 \) cuts out \( V \) in \( U \). Furthermore, \( f \) is analytic if \( V \) is, algebraic if \( V \) is. Form the free abelian group on the \( V \)'s (we can also look at “locally finite” \( \mathbb{Z} \)-combinations in the analytic case); call these objects \( \text{Weil divisors} \) (\( W \)-divisors), and denote the corresponding group, \( \text{WDiv}(X) \).

A divisor \( D \in \text{WDiv}(X) \) is **effective** if \( D = \sum \alpha a_\alpha V_\alpha \), with \( a_\alpha \geq 0 \) for all \( \alpha \). This gives a cone inside \( \text{WDiv}(X) \) and partially orders \( \text{WDiv}(X) \).

Say \( g \) is a holomorphic (or algebraic) function near \( x \). If \( V \) passes through \( x \), in \( \mathcal{O}_{X,x} \)-which is a UFD (by Zariski) we can write

\[
g = f^ag, \quad \text{where } (\tilde{g}, f) = 1.
\]
(The equation \( f = 0 \) defines \( V \) near \( x \) so \( f \) is a prime of \( \mathcal{O}_{X,x} \).) Notice that if \( p = (f) \) in \( \Gamma(U, \mathcal{O}_X) = \mathcal{O}_U \), then \( g = f^ag \) iff \( g \in p^a \) and \( g \notin p^{a+1} \) iff \( g \in p^a(\mathcal{O}_U)_p \) and \( g \notin p^{a+1}(\mathcal{O}_U)_p \). The ring \( (\mathcal{O}_U)_p \) is a local ring of dimension 1 and is regular as \( X \) is a manifold (can be regular even if \( X \) is singular). Therefore, \( a \) is independent of \( x \). The number \( a \) is by definition the **order of vanishing of \( g \) along \( V \)**, denoted \( \text{ord}_V(g) \). If \( g \) is a meromorphic function near \( x \), we write \( g = g_1/g_2 \) locally in \( (\mathcal{O}_U)_p \), with \( (g_1, g_2) = 1 \) and set

\[
\text{ord}_V(g) = \text{ord}_V(g_1) - \text{ord}_V(g_2).
\]

We say that \( g \) has a zero of order \( a \) along \( V \) iff \( \text{ord}_V(g) = a > 0 \) and a pole of order \( a \) iff \( \text{ord}_V(g) = -a < 0 \).

If \( g \in \Gamma(X, \text{Mer}(X)^*) \), set

\[
(g) = \sum_{V \in \text{WDiv}(X)} \text{ord}_V(g) \cdot V.
\]

**Claim.** The above sum is finite, under suitable conditions:
(a) We use algebraic functions.

(b) We use holomorphic functions and restrict $X$ (DX).

Look at $g$, then $1/g$ vanishes on a $Z$-closed, $W_0$. Look at $X - W_0$. Now, $X - W_0$ is $Z$-open so it is a variety and $g \mid X - W_0$ is holomorphic. Look at $V \subseteq X$ and $\text{ord}_V(g) = a \neq 0$, i.e., $V \cap U \neq \emptyset$. Thus, $(g) = p^a$ in $(O_U)_p$, which yields $(g) \subseteq p$ and then $V \cap (X - W_0) = V(p) \subseteq V((g))$. But, $V(g)$ is a union of irreducible components (algebraic case) and $V$ is codimension 1, so $V$ is equal to one of these components. Therefore, there are only finitely many $V$’s arising from $X - W_0$.

The function $1/g$ vanishes on $W_0$, so write $W_0$ as a union of irreducible components. Again, there are only finitely many $V$ arising from $W_0$. So, altogether, there are only finitely many $V$’s associated with $g$ where $g$ has a zero or a pole. We call $(g) \in \text{WDiv}(X)$ a principal divisor. Given any two divisors $D, E \in \text{WDiv}(X)$, we define linear (or rational) equivalence by

$$D \sim E \iff (\exists g \in \text{Mer}(X))(D - E = (g)).$$

The equivalence classes of divisors modulo $\sim$ is the Weil class group, $\text{WCl}(X)$.

**Remark:** All goes through for any $X$ (of our sort) for which, for all primes, $p$, of height 1, the ring $(O_U)_p$ is a regular local ring (of dimension 1, i.e., a P.I.D.) This is, in general, hard to check (but, OK if $X$ is normal).

Cartier had the idea to use a general $X$ but consider only the $V$’s given locally as $f = 0$. For every open, $U \subseteq X$, consider $A_U = \Gamma(U, O_X)$. Let $S_U$ be the set of all non-zero divisors of $A_U$, a multiplicative set. We get a presheaf of rings, $U \mapsto S_U^{-1} A_U$, and the corresponding sheaf, $\text{Mer}(X)$, is the total fraction sheaf of $O_X$. We have an embedding $O_X \hookrightarrow \text{Mer}(X)$ and we let $\text{Mer}(X)^*$ be the sheaf of invertible elements of $\text{Mer}(X)$. Then, we have the exact sequence

$$0 \rightarrow O_X^* \rightarrow \text{Mer}(X)^* \rightarrow \mathcal{D}_X \rightarrow 0,$$

where $\mathcal{D}_X$ is the sheaf cokernel.

We claim that if we define $\mathcal{D}_X = \text{Coker}(O_X^* \rightarrow \text{Mer}(X)^*)$ in the $\mathbb{C}$-topology, then it is also the kernel in the $Z$-topology.

Take $\sigma \in \Gamma(U, \mathcal{D}_X)$ and replace $X$ by $U$, so that we may assume that $U = X$. Then, as $\sigma$ is liftable locally in the $\mathbb{C}$-topology, there exist a $\mathbb{C}$-open cover, $U_{\alpha}$ and some $\sigma_{\alpha} \in \Gamma(U, \text{Mer}(X)^*)$ so that $\sigma_{\alpha} \mapsto \sigma \mid U_{\alpha}$. Make the $U_{\alpha}$ small enough so that $\sigma_{\alpha} = f_{\alpha}/g_{\alpha}$, where $f_{\alpha}, g_{\alpha}$ are holomorphic. It follows that $\sigma_{\alpha}$ is defined on a $Z$-open, $\tilde{U}_{\alpha} \supseteq U_{\alpha}$. Look at $\tilde{U}_{\alpha} \cap \tilde{U}_{\beta} \supseteq U_{\alpha} \cap U_{\beta}$. We know $\sigma_{\alpha}/\sigma_{\beta}$ is invertible holomorphic on $U_{\alpha} \cap U_{\beta}$ and so,

$$\frac{\sigma_{\alpha}}{\sigma_{\beta}} : \frac{\sigma_{\beta}}{\sigma_{\alpha}} \equiv 1 \quad \text{on} \quad U_{\alpha} \cap U_{\beta}.$$

It follows that $\sigma_{\alpha}/\sigma_{\beta}$ is invertible on $\tilde{U}_{\alpha} \cap \tilde{U}_{\beta}$ and then, restricting slightly further we get a $Z$-open cover and $\sigma_{\alpha}$’s on it lifting $\sigma$. $\square$

**Definition 3.1** A Cartier divisor (for short, C-divisor) on $X$ is a global section of $\mathcal{D}_X$. Two Cartier divisors, $\sigma, \tau$ are rationally equivalent, denoted $\sigma \sim \tau$, if $\sigma/\tau \in \Gamma(X, \text{Mer}(X)^*)$. Of course, this means there is a $\mathbb{C}$ or $Z$-open cover, $U_{\alpha}$, of $X$ and some $\sigma_{\alpha}, \tau_{\alpha} \in \Gamma(U_{\alpha}, \text{Mer}(X)^*)$ with $\sigma_{\alpha}/\tau_{\alpha}$ invertible holomorphic on $U_{\alpha} \cap U_{\beta}$. The group of Cartier divisors is denoted by $\text{CDiv}(X)$ and the corresponding group of equivalence classes modulo rational equivalence by $\text{Cl}(X)$ (the class group).

The idea is that if $\{(U_{\alpha}, \sigma_{\alpha})\}_{\alpha}$ defines a $C$-divisor, then we look on $U_{\alpha}$ at

$$\sigma_{\alpha}^0 - \sigma_{\alpha}^\infty = (\text{locus } \sigma_{\alpha} = 0) - (\text{locus } \frac{1}{\sigma_{\alpha}} = 0).$$
When we have the situation where \( \text{WDiv}(X) \) exists, then the map

\[
\{(U_\alpha, \sigma_\alpha)\}_\alpha \mapsto \{\sigma_\alpha^0 - \sigma_\alpha^\infty\}
\]

takes \( \mathbb{C} \)-divisors to Weil divisors. Say \( \sigma_\alpha \) and \( \sigma'_\alpha \) are both liftings of the same \( \sigma \), then on \( U_\alpha \) we have

\[
\sigma'_\alpha = \sigma_\alpha g_\alpha \quad \text{where} \quad g_\alpha \in \Gamma(X, \mathcal{O}_X^*)
\]

Therefore,

\[
\sigma'_\alpha^0 - \sigma'_\alpha^\infty = \sigma_\alpha^0 - \sigma_\alpha^\infty
\]

and the Weil divisors are the same (provided they make sense). If \( \sigma, \tau \in \text{CDiv}(X) \) and \( \sigma \sim \tau \), then there is a global meromorphic function, \( f \), with \( \sigma = f\tau \). Consequently

\[
\sigma_\alpha^0 - \tau_\alpha^0 = (f)^0 - (f)^\infty + \tau_\alpha^0 - \tau_\alpha^\infty,
\]

which shows that the corresponding Weil divisors are linearly equivalent. We get

**Proposition 3.3** If \( X \) is an algebraic variety, the sheaf \( D_X \) is the same in either the Zariski or \( \mathbb{C} \)-topology and if \( X \) allows Weil divisors (non-singular in codimension 1), then the map \( \text{CDiv}(X) \to \text{WDiv}(X) \) given by \( \sigma \mapsto \sigma_\alpha^0 - \sigma_\alpha^\infty \) is well-defined and we get a commutative diagram with injective rows

\[
\begin{array}{ccc}
\text{CDiv}(X) & \to & \text{WDiv}(X) \\
\downarrow & & \downarrow \\
\text{Cl}(X) & \to & \text{WCl}(X)
\end{array}
\]

If \( X \) is a manifold then our rows are isomorphisms.

**Proof.** We only need to prove the last statement. Pick \( D = \sum \alpha n_\alpha V_\alpha \), a Weil divisor, where each \( V_\alpha \) is irreducible of codimension 1. As \( X \) is manifold, each \( V_\alpha \) is given by \( f_\alpha = 0 \) on a small enough open, \( U \); take for \( \sigma | U \), the product \( \prod_\alpha f_\alpha^{n_\alpha} \) and this gives our \( \mathbb{C} \)-divisor.

We can use the following in some computations.

**Proposition 3.4** Assume \( X \) is an algebraic variety and \( Y \hookrightarrow X \) is a subvariety. Write \( U = X - Y \), then the maps

\[
\sigma \in \text{CDiv}(X) \mapsto \sigma | U \in \text{CDiv}(U),
\]

resp.

\[
\sum \alpha n_\alpha V_\alpha \in \text{WDiv}(X) \mapsto \sum \alpha n_\alpha (V_\alpha \cap U) \in \text{WDiv}(U)
\]

are surjections from \( \text{CDiv}(X) \) or \( \text{WDiv}(X) \) to the corresponding object in \( U \). If \( \text{codim}_X(Y) \geq 2 \), then our maps are isomorphisms. If \( \text{codim}_X(Y) = 1 \) and \( Y \) is irreducible and locally principal, then the sequences

\[
\mathbb{Z} \to \text{CDiv}(X) \to \text{CDiv}(U) \to 0 \quad \text{and} \quad \mathbb{Z} \to \text{WDiv}(X) \to \text{WDiv}(U) \to 0
\]

are exact (where the left hand map is \( n \mapsto nY \)).

**Proof.** The maps clearly exist. Given an object in \( U \), take its closure in \( X \), then restriction to \( U \) gives back the object. For \( Y \) of codimension at least 2, all procedures are insensitive to such \( Y \), so we don’t change anything by removing \( Y \). A divisor \( \xi \in \text{CDiv}(X) \) (or \( \text{WDiv}(X) \)) goes to zero iff its “support” is contained in \( Y \). But, \( Y \) is irreducible and so are the components of \( \xi \). Therefore, \( \xi = nY \), for some \( n \). □
Recall that line bundles on \(X\) are in one-to-one correspondence with invertible sheaves, that is, rank 1, locally free \(\mathcal{O}_X\)-modules. If \(L\) is a line bundle, we associate to it, \(\mathcal{O}_X(L)\), the sheaf of sections (algebraic, holomorphic, \(C^\infty\)) of \(L\).

In the other direction, if \(\mathcal{L}\) is a rank 1 locally free \(\mathcal{O}_X\)-module, first make \(\mathcal{L}^D\) and the \(\mathcal{O}_X\)-algebra, \(\text{Sym}_{\mathcal{O}_X}(\mathcal{L}^D)\), where

\[
\text{Sym}_{\mathcal{O}_X}(\mathcal{L}^D) = \prod_{n \geq 0} (\mathcal{L}^D)^{\otimes n} / (a \otimes b - b \otimes a).
\]

On a small enough open, \(U\),

\[
\text{Sym}_{\mathcal{O}_X}(\mathcal{L}^D) \mid U = \mathcal{O}_U[T],
\]

so we form \(\text{Spec}(\text{Sym}_{\mathcal{O}_X}(\mathcal{L}^D) \mid U) \cong U \prod \mathbb{C}^1\), and glue using the data for \(\mathcal{L}^D\). We get the line bundle, \(\text{Spec}(\text{Sym}_{\mathcal{O}_X}(\mathcal{L}^D))\).

Given a Cartier divisor, \(D = \{(U_\alpha, f_\alpha)\}\), we make the submodule, \(\mathcal{O}_X(D)\), of \(\mathcal{M}\text{er}(X)\) given on \(U_\alpha\) by

\[
\mathcal{O}_X(D) \mid U_\alpha = \frac{1}{f_\alpha} \mathcal{O}_X \mid U_\alpha \subseteq \mathcal{M}\text{er}(X) \mid U_\alpha.
\]

If \(\{(U_\alpha, g_\alpha)\}\) also defines \(D\) (we may assume the covers are the same by refining the covers if necessary), then there exist \(h_\alpha \in \Gamma(U_\alpha, \mathcal{M}\text{er}(X)^*)\), with

\[
f_\alpha h_\alpha = g_\alpha.
\]

Then, the map \(\xi \mapsto \frac{1}{f_\alpha} \xi\) takes \(\frac{1}{f_\alpha}\) to \(\frac{1}{g_\alpha}\); so, \(\frac{1}{f_\alpha}\) and \(\frac{1}{g_\alpha}\) generate the same submodule of \(\mathcal{M}\text{er}(X) \mid U_\alpha\). On \(U_\alpha \cap U_\beta\), we have

\[
\frac{f_\alpha}{f_\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X^*),
\]

and as

\[
\frac{f_\alpha}{f_\beta} \cdot \frac{1}{f_\alpha} = \frac{1}{f_\beta},
\]

we get

\[
\frac{1}{f_\alpha} \mathcal{O}_{U_\alpha} \mid U_\alpha \cap U_\beta = \frac{1}{f_\beta} \mathcal{O}_{U_\beta} \mid U_\alpha \cap U_\beta.
\]

Consequently, our modules agree on the overlaps and so, \(\mathcal{O}_X(D)\) is a rank 1, locally free subsheaf of \(\mathcal{M}\text{er}(X)\).

Say \(D\) and \(E\) are Cartier divisors and \(D \sim E\). So, there is a global meromorphic function, \(f \in \Gamma(X, \mathcal{M}\text{er}(X)^*)\) and on \(U_\alpha\),

\[
f_\alpha f = g_\alpha.
\]

Then, the map \(\xi \mapsto \frac{1}{f} \xi\) is an \(\mathcal{O}_X\)-isomorphism

\[
\mathcal{O}_X(D) \cong \mathcal{O}_X(E).
\]

Therefore, we get a map from \(\text{Cl}(X)\) to the invertible submodules of \(\mathcal{M}\text{er}(X)\).

Given an invertible submodule, \(\mathcal{L}\), of \(\mathcal{M}\text{er}(X)\), locally, on \(U\), we have \(\mathcal{L} \mid U = \frac{1}{f_U} \mathcal{O}_U \subseteq \mathcal{M}\text{er}(X) \mid U\). Thus, \(\{(U, f_U)\}\) gives a \(C\)-divisor describing \(\mathcal{L}\). Suppose \(\mathcal{L}\) and \(\mathcal{M}\) are two invertible submodules of \(\mathcal{M}\text{er}(X)\) and \(\mathcal{L} \cong \mathcal{M}\); say \(\varphi: \mathcal{L} \to \mathcal{M}\) is an \(\mathcal{O}_X\)-isomorphism. Locally (possibly after refining covers), on \(U_\alpha\), we have

\[
\mathcal{L} \mid U_\alpha \cong \frac{1}{f_\alpha} \mathcal{O}_{U_\alpha} \quad \text{and} \quad \mathcal{M} \mid U_\alpha \cong \frac{1}{g_\alpha} \mathcal{O}_{U_\alpha}.
\]

So, \(\varphi: \mathcal{L} \mid U_\alpha \to \mathcal{M} \mid U_\alpha\) is given by some \(\tau_\alpha\) such that

\[
\varphi\left(\frac{1}{f_\alpha}\right) = \tau_\alpha \frac{1}{g_\alpha}.
\]
Consequently, $\varphi_\alpha \restriction U_\alpha$ is multiplication by $\tau_\alpha$ and $\varphi_\beta \restriction U_\beta$ is multiplication by $\tau_\beta$. Yet $\varphi_\alpha \restriction U_\alpha$ and $\varphi_\beta \restriction U_\beta$ agree on $U_\alpha \cap U_\beta$, so $\tau_\alpha = \tau_\beta$ on $U_\alpha \cap U_\beta$. This shows that the $\tau_\alpha$ patch and define a global $\tau$ such that

$$\tau \restriction U_\alpha = g_\alpha \varphi \left( \frac{1}{f_\alpha} \right)$$

and $\tau \restriction U_\beta = g_\beta \varphi \left( \frac{1}{f_\beta} \right)$

on overlaps. Therefore, we can define a global $\Phi$ via

$$\Phi = g_\alpha \varphi \left( \frac{1}{f_\alpha} \right) \in \mathcal{M}er(X),$$

and we find $\xi \mapsto \frac{1}{\Phi} \xi$ gives the desired isomorphism.

**Theorem 3.5** If $X$ is an algebraic variety (or holomorphic or $C^\infty$ variety) then there is a canonical map, $\text{CDiv}(X) \to \text{rank 1, locally free submodules of } \mathcal{M}er(X)$. It is surjective. Two Cartier divisors $D$ and $E$ are rationally equivalent iff the corresponding invertible sheaves $\mathcal{O}_X(D)$ and $\mathcal{O}_X(E)$ are (abstractly) isomorphic. Hence, there is an injection of the class group, $\text{Cl}(X)$ into the group of rank 1, locally free $\mathcal{O}_X$-submodules of $\mathcal{M}er(X)$ modulo isomorphism. If $X$ is an algebraic variety and we use algebraic functions and if $X$ is irreducible, then every rank 1, locally free $\mathcal{O}_X$-module is an $\mathcal{O}_X(D)$. The map $D \mapsto \mathcal{O}_X(D)$ is just the connecting homomorphism in the cohomology sequence,

$$H^0(X, \mathcal{D}_X) \xrightarrow{\delta} H^1(X, \mathcal{O}_X^*).$$

**Proof.** Only the last statement needs proof. We have the exact sequence

$$0 \to \mathcal{O}_X^* \to \mathcal{M}er(X)^* \to \mathcal{D}_X \to 0.$$ 

Apply cohomology (we may use the $\mathbb{Z}$-topology, by the comparison theorem): We get

$$\Gamma(X, \mathcal{M}er(X)^*) \to \text{CDiv}(X) \to \text{Pic}(X) \to H^1(X, \mathcal{M}er(X)^*) = (0).$$

Note that this shows that there is a surjection $\text{CDiv}(X) \to \text{Pic}(X)$.

How is $\delta$ defined? Given $D \in H^0(X, \mathcal{D}_X) = \text{CDiv}(X)$, if $\{(U_\alpha, f_\alpha)\}$ is a local lifting of $D$, the map $\delta$ associates the cohomology class $[f_\beta/f_\alpha]$, where $f_\beta/f_\alpha$ is viewed as a 1-cocycle on $\mathcal{O}_X^*$. On the other hand, when we go through the construction of $\mathcal{O}_X(D)$, we have the isomorphisms

$$\mathcal{O}_X(D) \mid U_\alpha = \frac{1}{f_\alpha} \mathcal{O}_U \cong \mathcal{O}_{U_\alpha} \supseteq \mathcal{O}_{U_\alpha} \cap \mathcal{O}_{U_\beta} \quad \text{(mult. by } f_\alpha)$$

and

$$\mathcal{O}_X(D) \mid U_\beta = \frac{1}{f_\beta} \mathcal{O}_U \cong \mathcal{O}_{U_\alpha} \supseteq \mathcal{O}_{U_\alpha} \cap \mathcal{O}_{U_\beta} \quad \text{(mult. by } f_\beta)$$

and we see that the transition function, $g_\alpha^\delta$, on $\mathcal{O}_{U_\alpha} \cap \mathcal{O}_{U_\beta}$ is nonother that multiplication by $f_\beta/f_\alpha$. But then, both $\mathcal{O}_X(D)$ and $\delta(D)$ are line bundles defined by the same transition functions (multiplication by $f_\beta/f_\alpha$) and $\delta(D) = \mathcal{O}_X(D)$. \[\square\]

Say $D = \{(U_\alpha, f_\alpha)\}$ is a Cartier divisor on $X$. Then, the intuition is that the geometric object associated to $D$ is

$$\text{(zeros of } f_\alpha - \text{poles of } f_\alpha \text{) on } U_\alpha.$$ 

This leads to saying that the Cartier divisor $D$ is an effective divisor iff each $f_\alpha$ is holomorphic on $U_\alpha$. In this case, $f_\alpha = 0$ gives on $U_\alpha$ a locally principal, codimension 1 subvariety and conversely. Now each subvariety, $V$, has a corresponding sheaf of ideals, $\mathcal{I}_V$. If $V$ is locally principal, given by the $f_\alpha$’s, then $\mathcal{I}_V \mid U_\alpha = f_\alpha \mathcal{O}_X \mid U_\alpha$. But, $f_\alpha \mathcal{O}_X \mid U_\alpha$ is exactly $\mathcal{O}_X(-D)$ on $U_\alpha$ if $D = \{(U_\alpha, f_\alpha)\}$. Hence, $\mathcal{I}_X = \mathcal{O}_X(-D)$. We get
Proposition 3.6 If $X$ is an algebraic variety, then the effective Cartier divisors on $X$ are in one-to-one correspondence with the locally principal codimension 1 subvarieties of $X$. If $V$ is one of the latter and if $D$ corresponds to $V$, then the ideal cutting out $V$ is exactly $\mathcal{O}_X(-D)$. Hence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_V \to 0$$

is exact.

What are the global sections of $\mathcal{O}_X(D)$?

Such sections are holomorphic maps $\sigma : X \to \mathcal{O}_X(D)$ such that $\pi \circ \sigma = \text{id}$ (where $\pi : \mathcal{O}_X(D) \to X$ is the canonical projection associated with the bundle $\mathcal{O}_X(D)$). If $D$ is given by $\{(u_\alpha, f_\alpha)\}$, the diagram

\[
\begin{array}{ccc}
\mathcal{O}_X(D) \upharpoonright U_\alpha & \xrightarrow{f_\alpha} & \mathcal{O}_X \upharpoonright U_\alpha \\
\downarrow & & \downarrow \\
\mathcal{O}_X(D) \upharpoonright U_\alpha \cap U_\beta & \xrightarrow{f_\alpha \cap f_\beta} & \mathcal{O}_X \upharpoonright U_\alpha \cap U_\beta \\
\downarrow & & \downarrow \\
\mathcal{O}_X(D) \upharpoonright U_\beta \cap U_\alpha & \xrightarrow{f_\beta} & \mathcal{O}_X \upharpoonright U_\beta \cap U_\alpha \\
\downarrow & & \downarrow \\
\mathcal{O}_X(D) \upharpoonright U_\beta & \xrightarrow{f_\beta} & \mathcal{O}_X \upharpoonright U_\beta \\
\end{array}
\]

implies that

$$\sigma_\alpha = f_\alpha \sigma : U_\alpha \to \mathcal{O}_X \upharpoonright U_\alpha$$

and $$\sigma_\beta = f_\beta \sigma : U_\beta \to \mathcal{O}_X \upharpoonright U_\beta.$$  

However, we need

$$\sigma_\beta = g_\alpha^\beta \sigma_\alpha,$$

which means that a global section, $\sigma$, is a family of local holomorphic functions, $\sigma_\alpha$, so that $\sigma_\beta = g_\alpha^\beta \sigma_\alpha$. But, as $g_\alpha^\beta = f_\beta / f_\alpha$, we get

$$\frac{\sigma_\alpha}{f_\alpha} = \frac{\sigma_\beta}{f_\beta} \text{ on } U_\alpha \cap U_\beta.$$

Therefore, the meromorphic functions, $\sigma_\alpha / f_\alpha$, patch and give a global meromorphic function, $F_\sigma$. We have

$$f_\alpha (F_\sigma \upharpoonright U_\alpha) = \sigma_\alpha$$

a holomorphic function. Therefore, $(f_\alpha \upharpoonright U_\alpha) + (F_\sigma \upharpoonright U_\alpha) \geq 0$, for all $\alpha$ and as the pieces patch, we get

$$D + (F_\sigma) \geq 0.$$  

Conversely, say $F \in \Gamma(X, \text{Mer}(X))$ and $D + (F) \geq 0$. Locally on $U_\alpha$, we have $D = \{(u_\alpha, f_\alpha)\}$ and $(f_\alpha F) \geq 0$. If we set $\sigma_\alpha = f_\alpha F$, we get a holomorphic function on $U_\alpha$. But,

$$g_\alpha^\beta \sigma_\alpha = \frac{f_\beta}{f_\alpha} f_\alpha F = f_\beta F = \sigma_\beta,$$

so the $\sigma_\alpha$’s give a global section of $\mathcal{O}_X(D)$.

Proposition 3.7 If $X$ is an algebraic variety, then

$$H^0(X, \mathcal{O}_X(D)) = \{0\} \cup \{F \in \Gamma(X, \text{Mer}(X)) \mid (F) + D \geq 0\}.$$

in particular,

$$|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D))) = \{E \mid E \geq 0 \text{ and } E \sim D\},$$

the complete linear system of $D$, is naturally a projective space and $H^0(X, \mathcal{O}_X(D)) \neq (0)$ iff there is some Cartier divisor, $E \geq 0$, and $E \sim D$.  

Recall that an $\mathcal{O}_X$-module, $\mathcal{F}$, is a $Z$-QC (resp. $\mathbb{C}$-QC, here QC = quasi-coherent) iff everywhere locally, i.e., for small $(Z, \text{resp. } \mathbb{C})$ open, $U$, there exist sets $I(U)$ and $J(U)$ and some exact sequence

$$(\mathcal{O}_X \upharpoonright U)^{I(U)} \xrightarrow{\mathcal{F}|_U} (\mathcal{O}_X \upharpoonright U)^{J(U)} \longrightarrow \mathcal{F} \upharpoonright U \longrightarrow 0.$$  

Since $\mathcal{O}_X$ is coherent (usual fact that the rings $\Gamma(U_\alpha, \mathcal{O}_X) = A_\alpha$, for $U_\alpha$ open affine, are noetherian) or Oka’s theorem in the analytic case, a sheaf, $\mathcal{F}$, is coherent iff it is QC and finitely generated iff it is finitely presented, i.e., everywhere locally,

$$(\mathcal{O}_X \upharpoonright U)^q \xrightarrow{\mathcal{F}|_U} (\mathcal{O}_X \upharpoonright U)^q \longrightarrow \mathcal{F} \upharpoonright U \longrightarrow 0 \quad \text{is exact.} \quad (*)$$  

(Here, $p, q$ are functions of $U$ and finite).

In the case of the Zariski topology, $\mathcal{F}$ is QC iff for every affine open, $U$, the sheaf $\mathcal{F} \upharpoonright U$ has the form $\widetilde{M}$, for some $\Gamma(U, \mathcal{O}_X)$-module, $M$. The sheaf $M$ is defined so that, for every open $W \subseteq U$,

$$\Gamma(W, \widetilde{M}) = \left\{ \sigma : W \longrightarrow \bigcup_{\xi \in W} M_{\xi} \left| \begin{array}{l}
(1) \sigma(\xi) \in M_{\xi} \\
(2) (\forall \xi \in W) (\exists W \text{(open)} \subseteq W, \exists f \in M, \exists g \in \Gamma(V, \mathcal{O}_X))(g \neq 0 \text{ on } V) \\
(3) (\forall y \in V) \left( \sigma(y) = \text{image} \left( \begin{bmatrix} 1 \\ y \end{bmatrix} \right) \text{ in } M_y \right) \end{array} \right\}.$$  

**Proposition 3.8** Say $X$ is an algebraic variety and $\mathcal{F}$ is an $\mathcal{O}_X$-module. Then, $\mathcal{F}$ is $Z$-coherent iff $\mathcal{F}$ is $\mathbb{C}$-coherent.

**Proof.** Say $\mathcal{F}$ is $Z$-coherent, then locally $Z$, the sheaf $\mathcal{F}$ satisfies $(\dagger)$. But, every $Z$-open is also $\mathbb{C}$-open, so $\mathcal{F}$ is $\mathbb{C}$-coherent.

Now, assume $\mathcal{F}$ is $\mathbb{C}$-coherent, then locally $\mathbb{C}$, we have $(\dagger)$, where $U$ is $\mathbb{C}$-open. The map $\varphi_U$ is given by a $p \times q$ matrix of holomorphic functions on $U$. Each is algebraically defined on a $Z$-open containing $U$. The intersection of these finitely many $Z$-opens is a $Z$-open, $U$ and $\bar{U} \supseteq U$. So, we get a sheaf

$$\widetilde{\mathcal{F}} \upharpoonright \bar{U} = \text{Coker} ((\mathcal{O}_X \upharpoonright \bar{U})^g \longrightarrow (\mathcal{O}_X \upharpoonright \bar{U})^g).$$

The sheaves $\widetilde{\mathcal{F}} \upharpoonright \bar{U}$ patch (easy–DX) and we get a sheaf, $\widetilde{\mathcal{F}}$. On $U$, the sheaf $\widetilde{\mathcal{F}}$ is equal to $\mathcal{F}$, so $\widetilde{\mathcal{F}} = \mathcal{F}$. □

We have the continuous map $X_C \xrightarrow{\text{id}} X_{\text{Zar}}$ and we get (see Homework)

**Theorem 3.9** (Comparison Theorem for cohomology of coherent sheaves) If $X$ is an algebraic variety and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then the canonical map

$$H^q(X_{\text{Zar}}, \mathcal{F}) \longrightarrow H^q(X_C, \mathcal{F})$$

is an isomorphism for all $q \geq 0$.

Say $V$ is a closed subvariety of $X = \mathbb{P}_C^n$. Then, $V$ is given by a coherent sheaf of ideals of $\mathcal{O}_X$, say $\mathcal{I}_V$ and we have the exact sequence

$$0 \longrightarrow \mathcal{I}_V \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_V \longrightarrow 0,$$

where $\mathcal{O}_V$ is the sheaf of germs of holomorphic functions on $V$ and has support on $V$. If $V$ is a hypersurface, then $V$ is given by $f = 0$, where $f$ is a form of degree $d$. If $D$ is a Cartier divisor of $f$, then $\mathcal{I}_V = \mathcal{O}_X(-D)$. Similarly another hypersurface, $W$, is given by $g = 0$ and if $\deg(f) = \deg(g)$, then $f/g$ is a global meromorphic function on $\mathbb{P}_C^n$. Therefore, $(f/g) = V - W$, which implies $V \sim W$. In particular, $g = (\text{linear form})^d$ and so, $V \sim dH$, where $H$ is a hyperplane. Therefore the set of effective Cartier disisors of $\mathbb{P}_C^n$ is in one-to-one correspondence with forms of varying degrees $d \geq 0$ and

$$\text{Cl}(\mathbb{P}_C^n) \cong \mathbb{Z},$$
namely, $V \mapsto \deg(V) = \delta(V)$ (our old notation) = $(\deg(f)) \cdot H \in H^2(\mathbb{P}^n, \mathbb{Z})$. We deduce,

$$\text{Pic}^0(\mathbb{P}^n) = (0) \quad \text{and} \quad \text{Pic}(\mathbb{P}^n) = \text{Cl}(\mathbb{P}^n) = \mathbb{Z}.$$

Say $V$ is a closed subvariety of $\mathbb{P}^n$, then we have the exact sequence

$$0 \rightarrow \mathcal{I}_V \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_V \rightarrow 0.$$

Twist with $\mathcal{O}_{\mathbb{P}^n}(d)$, i.e., tensor with $\mathcal{O}_{\mathbb{P}^n}(d)$ (Recall that by definition, $\mathcal{O}_{\mathbb{P}^n}(d) = \mathcal{O}_{\mathbb{P}^n}(dH)$, where $H$ is a hyperplane). We get the exact sequence

$$0 \rightarrow \mathcal{I}_V(d) \rightarrow \mathcal{O}_{\mathbb{P}^n}(d) \rightarrow \mathcal{O}_V(d) \rightarrow 0$$

(with $\mathcal{I}_V(d) = \mathcal{I}_V \otimes \mathcal{O}_{\mathbb{P}^n}(d)$ and $\mathcal{O}_V(d) = \mathcal{O}_V \otimes \mathcal{O}_{\mathbb{P}^n}(d)$) and we can apply cohomology, to get

$$0 \rightarrow H^0(\mathbb{P}^n, \mathcal{I}_V(d)) \rightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \rightarrow H^0(V, \mathcal{O}_V(d)) \quad \text{is exact},$$

as $\mathcal{O}_V(d)$ has support $V$. Now,

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \{0\} \cup \{E \geq 0, E \sim dH\}.$$

If $E = \sum_Q a_Q Q$, where $\dim(Q) = n - 1$ and $a_Q \geq 0$, we set $\deg(E) = \sum_Q a_Q \deg(Q)$. If $E \geq 0$, then $\deg(E) \geq 0$, from which we deduce

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \begin{cases} (0) & \text{if } d < 0 \\ \mathbb{C}^{n+1} & \text{i.e., all forms of degree } d \text{ in } X_0, \ldots, X_n, \text{ if } d \geq 0. \end{cases}$$

We deduce,

$$H^0(\mathbb{P}^n, \mathcal{I}_V(d)) = \{\text{all forms of degree } d \text{ vanishing on } V\} \cup \{0\},$$

that is, all hypersurfaces, $Z \subseteq \mathbb{P}^n$, with $V \subseteq Z$ (and 0).

Consequently, to give $\xi \in H^0(\mathbb{P}^n, \mathcal{I}_V(d))$ is to give a hypersurface of $\mathbb{P}^n$ containing $V$. Therefore,

$$H^0(\mathbb{P}^n, \mathcal{I}_V(d)) = (0) \quad \text{iff} \quad \text{no hypersurface of degree } d \text{ contains } V.$$

(In particular, $V$ is nondegenerate iff $H^0(\mathbb{P}^n, \mathcal{I}_V(d)) = (0)$.)

We now compute the groups $H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, for all $n, q, d$. First, consider $d \geq 0$ and use induction on $n$. For $\mathbb{P}^0$, we have

$$H^q(\mathbb{P}^0, \mathcal{O}_{\mathbb{P}^0}(d)) = \begin{cases} (0) & \text{if } q > 0 \\ \mathbb{C} & \text{if } q = 0. \end{cases}$$

Next, $\mathbb{P}^1$. The sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\mathbb{P}^0} \rightarrow 0$$

is exact.

By tensoring with $\mathcal{O}_{\mathbb{P}^1}(d)$, we get

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(d-1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(d) \rightarrow \mathcal{O}_{\mathbb{P}^0}(d) \rightarrow 0$$

is exact.

by taking cohomology, we get

$$0 \rightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1)) \overset{\alpha}{\rightarrow} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \overset{\beta}{\rightarrow} H^0(\mathbb{P}^0, \mathcal{O}_{\mathbb{P}^0}(d)) \rightarrow 0$$

$$\rightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1)) \rightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \rightarrow 0$$
since $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = (0)$, by hypothesis. Now, if we pick coordinates, the embedding $\mathbb{P}^0 \hookrightarrow \mathbb{P}^1$ corresponds to $x_0 = 0$. Consequently, the map $\alpha$ is multiplication by $x_0$ and the map $\beta$ is $x_0 \mapsto 0$. Therefore,

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d - 1)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)), \quad \text{for all } d \geq 0,$$

and we deduce

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C} = (0),$$

and $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = (0)$, too. We know that

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = \mathbb{C}^{d+1}; \quad d \geq 0;$$

and we just proved that

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = (0); \quad d \geq -1.$$

In order to understand the induction pattern, let us do the case of $\mathbb{P}^2$. We have the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(d - 1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^2}(d) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^1}(d) \rightarrow 0$$

and by taking cohomology, we get

$$
\begin{array}{c}
0 \\
\downarrow \\
H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d - 1)) \xrightarrow{\alpha} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \xrightarrow{\beta} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \\
\downarrow \\
H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d - 1)) \rightarrow H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \rightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \\
\downarrow \\
H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d - 1)) \rightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \rightarrow 0
\end{array}
$$

By the induction hypothesis, $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = (0)$ if $d \geq -1$, so

$$H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d - 1)) \cong H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)), \quad \text{for all } d \geq -1.$$ 

Therefore,

$$H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \cong H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}), \quad \text{for all } d \geq -2.$$ 

But, the dimension of the right hand side is $h^{0,1} = 0$ (the irregularity, $h^{0,1}$, of $\mathbb{P}^2$ is zero). We conclude that

$$H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) = (0) \quad \text{for all } d \geq -2.$$ 

A similar reasoning applied to $H^2$ shows

$$H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \cong H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}), \quad \text{for all } d \geq -2.$$ 

The dimension of the right hand side group is $H^{0,2} = p_2(\mathbb{P}^2) = 0$, so we deduce

$$H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) = (0) \quad \text{for all } d \geq -2.$$ 

By induction, we get

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \begin{cases} \mathbb{C}^{n+d} & \text{if } d \geq 0 \\ (0) & \text{if } d < 0 \end{cases}$$

and

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = (0) \quad \text{if } d \geq -n, \text{ for all } q > 0.$$
For the rest of the cases, we use Serre duality and the Euler sequence. Serre duality says
\[ H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))^D \cong H^{n-q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d) \otimes \Omega^0_{\mathbb{P}^n}). \]

From the Euler sequence
\[ 0 \to \mathcal{O}_{\mathbb{P}^n} \to \prod_{n+1 \text{ times}} \mathcal{O}_{\mathbb{P}^n}(1) \to T^{1,0}_{\mathbb{P}^n} \to 0, \]
by taking the highest wedge, we get
\[ \bigwedge^{n+1} \left( \prod_{n+1} \mathcal{O}_{\mathbb{P}^n}(1) \right) \cong \bigwedge^n T^{1,0}_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}, \]
from which we conclude
\[ (\Omega^0_{\mathbb{P}^n})^D \cong \bigwedge^{n+1} \left( \bigwedge^n \mathcal{O}_{\mathbb{P}^n}(1) \right) \cong \mathcal{O}_{\mathbb{P}^n}(n+1). \]

Therefore
\[ \omega_{\mathbb{P}^n} = \Omega^0_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-(n+1)) = \mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n}), \]
where \( K_{\mathbb{P}^n} \) is the canonical divisor on \( \mathbb{P}^n \), by definition. Therefore, we have
\[ H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))^D \cong H^{n-q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d-n-1))^D. \]

If \( 1 \leq q \leq n-1 \) and \( d \geq n \), then we know that the left hand side is zero. As \( 1 \leq n-q \leq n-1 \), it follows that
\[ H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d-n-1)) = (0) \quad \text{when } d \geq n. \]

Therefore,
\[ H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = (0) \quad \text{for all } d \text{ and all } q \text{ with } 1 \leq q \leq n-1. \]

We also have
\[ H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))^D \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d-n-1)), \]
and the right hand side is (0) if \( -d-(n+1) < 0 \), i.e., \( d \geq n \). Thus, if \( d \leq -(n+1) \), then we have \( \delta = -d-(n+1) \geq 0 \), so
\[ H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\delta))^D = \mathbb{C}^{(n+\delta)}, \quad \text{where } \delta = -(d+n+1). \]

The pairing is given by
\[ \frac{1}{f} \otimes \frac{f}{x_0 x_1 \cdots x_n} \to \int_{\mathbb{P}^n} \frac{dx_0 \wedge \cdots \wedge dx_n}{x_0 \cdots x_n}, \]
where \( \deg(f) = -d \), with \( d \leq -n-1 \). Summarizing all this, we get

**Theorem 3.10** The cohomology of line bundles on \( \mathbb{P}^n \) satisfies
\[ H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = (0) \quad \text{for all } n, d \text{ and all } q \text{ with } 1 \leq q \leq n-1. \]

Furthermore,
\[ H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \mathbb{C}^{(n+d)}, \quad \text{if } d \geq 0, \text{ else } (0), \]
and
\[ H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \mathbb{C}^{(n+\delta)}, \quad \text{where } \delta = -(d+n+1) \text{ and } d \leq -n-1, \text{ else } (0). \]

We also proved that
\[ \omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-(n+1)) = \mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n}). \]
3.2 Chern Classes and Segre Classes

The most important spaces (for us) are the Kähler manifolds and unless we explicitly mention otherwise, $X$ will be Kähler. But, we can make Chern classes if $X$ is worse.

Remark: The material in this Section is also covered in Hirzebruch [8] and under other forms in Chern [4], Milnor and Stasheff [11], Bott and Tu [3], Madsen and Tornehave [9] and Griffith and Harris [6].

Let $X$ be admissible iff

1. $X$ is $\sigma$-compact, i.e.,
   (a) $X$ is locally compact and
   (b) $X$ is a countable union of compacts.
2. The combinatorial dimension of $X$ is finite.

Note that (1) implies that $X$ is paracompact. Consequently, everything we did on sheaves goes through.

Say $X$ is an algebraic variety and $F$ is a QC $\mathcal{O}_X$-module. Then, $H^0(X, F)$ encodes the most important geometric information contained in $F$. For example, $F$ = a line bundle or a vector bundle, then $H^0(X, F)$ = space of global sections of given type.

If $F = \mathcal{I}_V(d)$, where $V \subseteq \mathbb{P}^n$, then

$$H^0(X, F) = \text{space of global sections containing } V.$$ 

This leads to the Riemann-Roch (RR) problem.

Given $X$ and a QC $\mathcal{O}_X$-module, $F$,

(a) Determine when $H^0(X, F)$ has finite dimension and

(b) If so, compute the dimension, $\dim_{\mathbb{C}} H^0(X, F)$.

Some answers:

(a) Finiteness Theorem: If $X$ is a compact, complex, analytic manifold and $F$ is a coherent $\mathcal{O}_X$-module, then $H^q(X, F)$ has finite dimension for every $q \geq 0$.

(b) It was noticed in the fifties (Kodaira and Spencer) that if $\{X_t\}_{t \in S}$ is a reasonable family of compact algebraic varieties ($\mathbb{C}$-analytic manifolds), ($S$ is just a $\mathbb{R}$-differentiable smooth manifold and the $X_t$ are a proper flat family), then

$$\chi(X_t, \mathcal{O}_{X_t}) = \sum_{i=0}^{\dim X_t} (-1)^i \dim(H^i(X_t, \mathcal{O}_{X_t}))$$

was independent of $t$.

The Riemann-Roch problem goes back to Riemann and the finiteness theorem goes back to Oka, Cartan-Serre, Serre, Grauert, Grothendieck, ... .

Examples. (1) Riemann (1850’s): If $X$ is a compact Riemann surface, then

$$\chi(X, \mathcal{O}_X) = 1 - g$$
where \( g \) is the number of holes of \( X \) (as a real surface).

(2) Max Noether (1880’s): If \( X \) is a compact, complex surface, then
\[
\chi(X, O_X) = \frac{1}{12}(K_X^2 + \text{top Euler char.}(X)).
\]
(Here, \( K_X^2 = O_X(K_X) \cup O_X(K_X) \) in the cohomology ring, an element of \( H^4(X, \mathbb{Z}) \).)

(3) Severi, Eger-Todd (1920, 1937) conjectured:
\[
\chi(X, O_X) = \text{some polynomial in the Euler-Todd class of } X,
\]
for \( X \) a general compact algebraic, complex manifold.

(4) In the fourties and fifties (3) was reformulated as a statement about Chern classes–no proof before Hirzebruch.

(5) September 29, 1952: Serre (letter to Kodaira and Spencer) conjectured: If \( F \) is a rank \( r \) vector bundle over the compact, complex algebraic manifold, \( X \), then
\[
\chi(X, F) = \text{polynomial in the Chern classes of } X \text{ and those of } F.
\]
Serre’s conjecture (5) was proved by Hirzebruch a few months later.

To see this makes sense, we’ll prove

**Theorem 3.11** (Riemann-Roch for a compact Riemann Surface and for a line bundle) If \( X \) is a compact Riemann surface and if \( L \) is a complex analytic line bundle on \( X \), then there is an integer, \( \deg(L) \), such that
\[
\chi(D) = \deg(L) + 1 - g
\]
where \( g = \dim H^0(X, \omega_X) = \dim H^1(X, O_X) \) is the genus of \( X \).

**Proof.** First, we know \( X \) is an algebraic variety (a curve), by Riemann’s theorem (see Homework). From another Homework (from Fall 2003), \( X \) is embeddable in \( \mathbb{P}^N_C \), for some \( N \), and by GAGA (yet to come!), \( L \) is an algebraic line bundle. It follows that \( L = O_X(D) \), for some Cartier divisor, \( D \). Now, if \( f \in \text{Mer}(X) \), we showed (again, see Homework) that \( f: X \to \mathbb{P}^1_C = S^2 \) is a branched covering map and this implies that
\[
\#(f^{-1}(\infty)) = \#(f^{-1}(0)) = \text{degree of the map},
\]
so \( \deg(f) = \#(f^{-1}(0)) - \#(f^{-1}(\infty)) = 0 \). As a consequence, if \( E \sim D \), then \( \deg(E) = \deg(D) \) and the first statement is proved. Serre duality says
\[
H^0(X, \omega_X \otimes L^D) \cong H^1(X, L).\]
Thus, the left hand side of the Riemann-Roch formula is just \( \chi(X, O_X(D)) \), where \( L = O_X(D) \). Observe that \( \chi(X, O_X(D)) \) is an Euler function in the bundle sense (this is always true of Euler-Poincaré characteristics). Look at any point \( P \), on \( X \), we have the exact sequence
\[
0 \to O_X(-P) \to O_X \to \kappa_P \to 0,
\]
where \( \kappa_P \) is the skyscraper sheaf at \( P \), i.e.,
\[
(\kappa_P)_x = \begin{cases} 
0 & \text{if } x \neq P \\
\mathbb{C} & \text{if } x = P.
\end{cases}
\]
If we tensor with $\mathcal{O}_X(D)$, we get the exact sequence

$$0 \rightarrow \mathcal{O}_X(D - P) \rightarrow \mathcal{O}_X(D) \rightarrow \kappa_P \otimes \mathcal{O}_X(D) \rightarrow 0.$$ 

When we apply cohomology, we get

$$\chi(X, \kappa_P \otimes \mathcal{O}_X(D)) + \chi(X, \mathcal{O}_X(D - P)) = \chi(X, \mathcal{O}_X(D)).$$

There are three cases.

(a) $D = 0$. The Riemann-Roch formula is a tautology, by definition of $g$ and the fact that $H^0(X, \mathcal{O}_X) = \mathbb{C}$.

(b) $D > 0$. Pick any $P$ appearing in $D$. Then, $\deg(D - P) = \deg(D) - 1$ and we can use induction. The base case holds, by (a). Using the induction hypothesis, we get

$$1 + \deg(D - P) + 1 - g = \chi(X, \mathcal{O}_X(D)),$$

which says

$$\chi(X, \mathcal{O}_X(D)) = \deg(D) + 1 - g,$$

proving the induction step when $D > 0$.

(c) $D$ is arbitrary. In this case, write $D = D^+ - D^-$, with $D^+, D^- \geq 0$; then

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(D^-) \rightarrow \kappa_{D^-} \rightarrow 0$$

is exact and

$$\deg(\kappa_{D^-}) = \deg(D^-) = \chi(X, \mathcal{O}_X(D^-)).$$

If we tensor the above exact sequence with $\mathcal{O}_X(D)$, we get

$$0 \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_X(D + D^-) \rightarrow \kappa_{D^-} \rightarrow 0$$

is exact.

When we apply cohomology, we get

$$\chi(X, \mathcal{O}_X(D)) + \deg(D^-) = \chi(X, \mathcal{O}_X(D + D^-)) = \chi(X, \mathcal{O}_X(D^+)).$$

However, by (b), we have $\chi(X, \mathcal{O}_X(D^+)) = \deg(D^+) + 1 - g$, so we deduce

$$\chi(X, \mathcal{O}_X(D)) = \deg(D^+) - \deg(D^-) + 1 - g = \deg(D) + 1 - g,$$

which finishes the proof.

We will show:

(a) $\mathcal{L}$ possesses a class, $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$.

(b) If $X$ is a Riemann surface and $[X] \in H_2(X, \mathbb{Z}) = \mathbb{Z}$ is its fundamental class, then $\deg(\mathcal{L}) = c(\mathcal{L})[X] \in \mathbb{Z}$.

Then, the Riemann-Roch formula becomes

$$\chi(X, \mathcal{L}) = c_1(\mathcal{L})[X] + 1 - g = \left[c_1(\mathcal{L}) + \frac{1}{2}(2 - 2g)\right][X] = \left[c_1(\mathcal{L}) + \frac{1}{2}c_1(T_X^{1,0})\right][X].$$

This is Hirzebruch’s form of the Riemann-Roch theorem for Riemann surfaces and line bundles.

What about vector bundles?
Theorem 3.12 (Atiyah-Serre on vector bundles) Let \( X \) be either a compact, complex \( C^\infty \)-manifold or an algebraic variety. If \( E \) is a rank \( r \) vector bundle on \( X \), of class \( C^\infty \) in case \( X \) is just \( C^\infty \), algebraic if \( X \) is algebraic, in the latter case assume \( E \) is generated by its global sections (that is, the map, \( \Gamma_{\text{alg}}(X, \mathcal{O}_X(E)) \to E_x \), given by \( \sigma \mapsto \sigma(x) \), is surjective for all \( x \)), then, there is a trivial bundle of rank \( r-d \) (where \( d = \dim_{\mathbb{C}} X \)) denoted \( \mathbb{I}^{r-d} \), and a bundle exact sequence
\[
0 \to \mathbb{I}^{r-d} \to E \to E'' \to 0
\]
and the rank of the bundle \( E'' \) is at most \( d \).

Proof. Observe that if \( r < d \), there is nothing to prove and \( \text{rk}(E'') = \text{rk}(E) \) and also if \( r = d \) take \( (0) \) for the left hand side. So, we may assume \( r > d \). In the \( C^\infty \)-case, we always have \( E \) generated by its global \( C^\infty \)-sections (partition of unity argument).

Pick \( x \), note \( \dim E_x = r \), so there is a finite dimensional subspace of \( \Gamma(X, \mathcal{O}_X(E)) \) surjecting onto \( E_x \). By continuity (or algebraicity), this holds \( \mathbb{C} \)-near (resp. \( \mathbb{Z} \)-near) \( x \). Cover by these opens and so

(a) In the \( C^\infty \)-case, finitely many of these opens cover \( X \) (recall, \( X \) is compact).

(b) In the algebraic case, again, finitely many of these opens cover \( X \), as \( X \) is quasi-compact in the \( \mathbb{Z} \)-topology.

Therefore, there exists a finite dimensional space, \( W \subseteq \Gamma(X, \mathcal{O}_X(E)) \), and the map \( W \to E_x \) given by \( \sigma \mapsto \sigma(x) \) is surjective for all \( x \in X \). Let
\[
\ker(x) = \ker(W \to E_x).
\]
Consider the projective space \( \mathbb{P}(\ker(x)) \to \mathbb{P} = \mathbb{P}(W) \). Observe that \( \dim \ker(x) = \dim W - r \) is independent of \( x \). Now, look at \( \bigcup_{x \in X} \mathbb{P}(\ker(x)) \) and let \( Z \) be its \( \mathbb{Z} \)-closure. We have
\[
\dim Z = \dim X + \dim W - r - 1 = \dim W + d - r - 1,
\]
so, \( \text{codim}(Z \hookrightarrow \mathbb{P}) = r - d \). Thus, there is some projective subspace, \( T \), of \( \mathbb{P} \) with \( \dim T = r - d - 1 \), so that
\[
T \cap Z = \emptyset.
\]
Then, \( T = \mathbb{P}(S) \), for some subspace, \( S \), of \( W \) (\( \dim S = r - d \)). Look at
\[
X \prod S = X \prod \mathbb{C}^{r-d} = \mathbb{I}^{r-d}.
\]
Send \( \mathbb{I}^{r-d} \) to \( E \) via \( (x, s) \mapsto s(x) \in E \). As \( T \cap Z = \emptyset \), the value \( s(x) \) is never zero. Therefore, for any \( x \in X \), \( \text{Im}(\mathbb{I}^{r-d} \to E) \) has full rank; set \( E'' = E/\text{Im}(\mathbb{I}^{r-d} \to E) \) a vector bundle of rank \( d \), then
\[
0 \to \mathbb{I}^{r-d} \to E \to E'' \to 0
\]
as a bundle sequence. \( \square \)

Remarks:

(a) If \( 0 \to E' \to E \to E'' \to 0 \) is bundle exact, then
\[
c_1(E) = c_1(E') + c_1(E'').
\]

(b) If \( E \) is the trivial bundle, \( \mathbb{I}^r \), then \( c_j(E) = 0 \), for \( j = 1, \ldots, r \).

(c) If \( \text{rk}(E) = r \), then \( c_1(E) = c_1(\bigwedge^r E) \).
In view of (a)–(c), Atiyah-Serre can be reformulated as
\[ c_1(E) = c_1(\bigwedge^\mathit{rk} E) = c_1(E'') = c_1(\bigwedge^\mathit{rk} E''). \]

We now use the Atiyah-Serre theorem to prove a version of Riemann-Roch first shown by Weil.

**Theorem 3.13** (Riemann-Roch on a Riemann surface for a vector bundle) If \( X \) is a compact Riemann surface and \( E \) is a complex analytic rank \( r \) vector bundle on \( X \), then
\[
\dim \mathcal{H}^0(X, \mathcal{O}_X(E)) - \dim \mathcal{H}^1(X, \omega_X \otimes \mathcal{O}_X(E)) = \chi(X, \mathcal{O}_X) = c_1(E) + \mathit{rk}(E)(1 - g).
\]

**Proof.** The first equality is just Serre Duality. As before, by Riemann’s theorem \( X \) is projective algebraic and by GAGA, \( E \) is an algebraic vector bundle. Now, as \( X \hookrightarrow \mathbb{P}^N \), it turns out (Serre) that for \( \delta > 0 \), the “twisted bundle”, \( E \otimes \mathcal{O}_X(\delta) (= E \otimes \mathcal{O}_{\mathbb{P}^N}^\delta) \) is generated by its global holomorphic sections. We can apply Atiyah-Serre to \( E \otimes \mathcal{O}_X(\delta) \). We get
\[
0 \longrightarrow I^r_{r-1} \longrightarrow E \otimes \mathcal{O}_X(\delta) \longrightarrow E'' \longrightarrow 0 \text{ is exact,}
\]
where \( \mathit{rk}(E'') = 1 \). If we twist with \( \mathcal{O}_X(-\delta) \), we get the exact sequence
\[
0 \longrightarrow \bigoplus_{r-1} \mathcal{O}_X(-\delta) \longrightarrow E \longrightarrow E''(-\delta) \longrightarrow 0.
\]
(Here, \( E''(-\delta) = E'' \otimes \mathcal{O}_X(-\delta) \).) Now, use induction on \( r \). The case \( r = 1 \) is ordinary Riemann-Roch for line bundles. Assume the induction hypothesis for \( r - 1 \). As \( \chi \) is an Euler function, we have
\[
\chi(X, \mathcal{O}_X(E)) = \chi(X, E''(-\delta)) + \chi\left(\bigoplus_{r-1} \mathcal{O}_X(-\delta)\right).
\]
The first term on the right hand side is
\[
c_1(E''(-\delta)) + 1 - g.
\]
by ordinary Riemann-Roch and the second term on the right hand side is
\[
c_1\left(\bigoplus_{r-1} \mathcal{O}_X(-\delta)\right) + (r - 1)(1 - g).
\]
by the induction hypothesis. We deduce that
\[
\chi(X, \mathcal{O}_X(E)) = c_1(E''(-\delta)) + c_1\left(\bigoplus_{r-1} \mathcal{O}_X(-\delta)\right) + r(1 - g).
\]
But, we know that
\[
c_1(E) = c_1(E''(-\delta)) + c_1\left(\bigoplus_{r-1} \mathcal{O}_X(-\delta)\right),
\]
so we conclude that
\[
\chi(X, \mathcal{O}_X(E)) = c_1(E) + r(1 - g),
\]
establishing the induction hypothesis and the theorem. \( \square \)

**Remark:** We can write the above as
\[
\chi(X, \mathcal{O}_X(E)) = c_1(E) + \frac{\mathit{rk}(E)}{2}c_1(T_X^{-1,0}),
\]
which is Hirzebruch’s form of Riemann-Roch.

We will need later some properties of $\chi(X, \mathcal{O}_X)$ and $p_g(X)$. Recall that $p_g(X) = \dim \mathcal{H}^n(X, \mathcal{O}_X) = \dim \mathcal{H}^0(X, \Omega^2_X)$, where $\Omega^2_X = \wedge^2 T^* X$. (The vector spaces $\mathcal{H}^0(X, \Omega^2_X)$ were what the Italian geometers (in fact, all geometers) of the nineteenth century understood.)

**Proposition 3.14** The functions $\chi(X, \mathcal{O}_X)$ and $p_g(X)$ are multiplicative on compact, Kähler manifolds, i.e.,

$$\chi(X \prod Y, \mathcal{O}_{X \prod Y}) = \chi(X, \mathcal{O}_X) \chi(Y, \mathcal{O}_Y)$$

$$p_g(X \prod Y) = p_g(X) p_g(Y).$$

**Proof.** Remember that $\dim \mathcal{H}^l(X, \mathcal{O}_X) = \dim \mathcal{H}^0(X, \Omega^l_X) = h^0(X, \Omega^l_X) = h^l_0$. Then,

$$\chi(X, \mathcal{O}_X) = \sum_{j=0}^n (-1)^j \dim \mathcal{H}^0(X, \Omega^j_X) = \sum_{j=0}^n (-1)^j h^j_0.$$

Also recall the Künneth formula

$$\bigoplus_{p+q' = a} H^p(X, \Omega^p_X) \otimes H^q'(X, \Omega^{p'}_X) \cong H^b(X \prod Y, \Omega^a_{X \prod Y}).$$

Set $b = 0$, then $q = q' = 0$ and we get

$$\sum_{p+p' = a} h^{p,0}(X) h^{p',0}(Y) = h^{a,0}(X \prod Y).$$

Then,

$$\chi(X, \mathcal{O}_X) \chi(Y, \mathcal{O}_Y) = \left( \sum_{r=0}^m (-1)^r h^{r,0}(X) \right) \left( \sum_{s=0}^n (-1)^s h^{s,0}(Y) \right)$$

$$= \sum_{r,s=0}^{m+n} (-1)^{r+s} h^{r,0}(X) h^{s,0}(Y)$$

$$= \sum_{k=0}^{m+n} (-1)^k \sum_{r+s=k} h^{r,0}(X) h^{s,0}(Y)$$

$$= \sum_{k=0}^{m+n} (-1)^k \chi(X \prod Y, \mathcal{O}_{X \prod Y}).$$

The second statement is obvious from Künneth. □

Next, we introduce Hirzebruch’s axiomatic approach.

Let $E$ be a complex vector bundle on $X$, where $X$ is one of our spaces (admissible). It will turn out that $E$ is a unitary bundle (a $U(q)$-bundle, where $q = \text{rk}(E)$).

Chern classes are cohomology classes, $c_l(E)$, satisfying the following axioms:
Axiom (I). (Existence and Chern polynomial). If $E$ is a rank $q$ unitary bundle over $X$ and $X$ is admissible, then there exist cohomology classes, $c_l(E) \in H^{2l}(X, \mathbb{Z})$, the Chern classes of $E$ and we set

$$c(E)(t) = \sum_{l=0}^{\infty} c_l(E) t^l \in H^*(X, \mathbb{Z})[[t]],$$

with $c_0(E) = 1$.

As $\dim_X X = d < \infty$, we get $c_l(E) = 0$ for $l > d$, so $C(E)(t)$ is in fact a polynomial in $H^*(X, \mathbb{Z})[t]$ called the Chern polynomial of $E$ where $\deg(t) = 2$.

Say $\pi: Y \to X$ and $E$ is a $U(q)$-bundle over $X$, then we have two maps

$$H^*(X, \mathbb{Z}) \xrightarrow{\pi^*} H^*(Y, \mathbb{Z}) \text{ and } H^1(X, U(q)) \xrightarrow{\pi^*} H^1(Y, U(q)).$$

Axiom (II). (Naturality). For every $E$, a $U(q)$-bundle on $X$ and map, $\pi: Y \to X$, (with $X,Y$ admissible), we have

$$c(\pi^* E)(t) = \pi^*(c(E))(t),$$

as elements of $H^*(Y, \mathbb{Z})[[t]]$.

Axiom (III). (Whitney coproduct axiom). If $E$, a $U(q)$-bundle is a coproduct (in the $C$ or $C\infty$-sense),

$$E = \coprod_{j=1}^{\text{rk}(E)} E_j$$

of $U(1)$-bundles, then

$$c(E)(t) = \prod_{j=1}^{\text{rk}(E)} c(E_j)(t).$$

Axiom (IV). (Normalization). If $X = \mathbb{P}^n_C$ and $O_X(1)$ is the $U(1)$-bundle corresponding to the hyperplane divisor, $H$, on $\mathbb{P}^n_C$, then

$$c(O_X(1))(t) = 1 + Ht,$$

where $H$ is considered in $H^2(X, \mathbb{Z})$.

Remark: If $i: \mathbb{P}^{n-1}_C \hookrightarrow \mathbb{P}^n_C$, then

$$i^*O_{\mathbb{P}^{n}}(1) = O_{\mathbb{P}^{n-1}}(1)$$

and $i^*(H)$ in $H^2(\mathbb{P}^{n-1}_C, \mathbb{Z})$ is $H_{\mathbb{P}^{n-1}}$. By Axiom (II) and Axiom (IV)

$$i^*(1 + H_{\mathbb{P}^n}(t)) = i^*(c(O_{\mathbb{P}^n})(t)) = c(i^*(O_{\mathbb{P}^n}))(t)) = 1 + H_{\mathbb{P}^{n-1}}.$$ 

Therefore, we can use any $n$ to normalize.

Some Remarks on bundles. First, on $\mathbb{P}^n = \mathbb{P}^n_C$: Geometric models of $O_{\mathbb{P}^n}(\pm 1)$.

Consider the map

$$\mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n.$$ 

If we blow up $0$ in $\mathbb{C}^{n+1}$, we get $B_0(\mathbb{C}^{n+1})$ as follows: In $\mathbb{C}^{n+1} \coprod \mathbb{P}^n$, look at the subvariety given by

$$\{(z);(\xi) \mid z_i \xi_j = z_j \xi_i, 0 \leq i,j \leq n\}.$$
By definition, this is $B_0(\mathbb{C}^{n+1})$, an algebraic variety over $\mathbb{C}$. We have the two projections

$$\begin{array}{ccc}
B_0(\mathbb{C}^{n+1}) & \xrightarrow{pr_1} & \mathbb{C}^{n+1} \\
& \xrightarrow{pr_2} & \mathbb{P}^n.
\end{array}$$

Look at the fibre, $pr_1^{-1}(\langle z \rangle)$ over $z \in \mathbb{C}^{n+1}$. There are two cases:

(a) $\langle z \rangle = 0$, in which case, $pr_1^{-1}(\langle z \rangle) = \mathbb{P}^n$.

(b) $\langle z \rangle \neq 0$, so, there is some $j$ with $z_j \neq 0$. We get $\xi_i = \frac{z_i}{z_j} \xi_j$, for all $i$, which implies:

(α) $\xi_j \neq 0$.

(β) All $\xi_i$ are determined by $\xi_j$.

(γ) $\frac{\xi_i}{\xi_j} = \frac{z_i}{z_j}$.

This implies

$$\langle \xi \rangle = \left( \frac{\xi_0}{\xi_j}, \frac{\xi_1}{\xi_j}, \ldots, \frac{\xi_n}{\xi_j} \right) = \left( \frac{z_0}{z_j}, \frac{z_1}{z_j}, \ldots, \frac{z_n}{z_j} \right).$$

Therefore, $pr_1^{-1}(\langle z \rangle) = \langle \langle z \rangle; (z) \rangle$, a single point.

Let us now look at $pr_2^{-1}(\langle \xi \rangle)$, for $\langle \xi \rangle \in \mathbb{P}^n$. Since $\langle \xi \rangle \in \mathbb{P}^n$, there is some $j$ such that $\xi_j \neq 0$. A point $\langle \langle z \rangle; (\xi) \rangle$ above $\langle \xi \rangle$ is given by all $\langle z_0; z_1; \ldots; z_n \rangle$ so that

$$z_i = \frac{\xi_i}{\xi_j} z_j.$$

Let $z_j = t$, then the fibre above $\xi$ is the complex line

$$z_0 = \frac{\xi_0}{\xi_j} t, \quad z_1 = \frac{\xi_1}{\xi_j} t, \quad \ldots, \quad z_j = t, \quad \ldots, \quad z_n = \frac{\xi_n}{\xi_j} t.$$

We get a line family over $\mathbb{P}^n$. Thus, $pr_2 : B_0(\mathbb{C}^{n+1}) \rightarrow \mathbb{P}^n$ is a line family.

(A) What kinds of maps, $\sigma : \mathbb{P}^n \rightarrow B_0(\mathbb{C}^{n+1})$, exist with $\sigma$ holomorphic and $pr_2 \circ \sigma = \text{id}$?

If $\sigma$ exists, then $pr_1 \circ \sigma : \mathbb{P}^n \rightarrow \mathbb{C}^{n+1}$ is holomorphic; this implies that $pr_1 \circ \sigma$ is a constant map. But, $\sigma(\langle \xi \rangle)$ belongs to a line through $\langle \xi \rangle = (\xi_0; \ldots; \xi_n)$, for all $\langle \xi \rangle$, yet $pr_1 \circ \sigma = \text{const}$, so this point must lie on all line. This can only happen if $\sigma(\langle \xi \rangle) = 0$ in the line through $\xi$.

(B) I claim $B_0(\mathbb{C}^{n+1})$ is locally trivial, i.e., a line bundle. If so, (A) says $B_0(\mathbb{C}^{n+1})$ has no global holomorphic sections and we will know that $B_0(\mathbb{C}^{n+1}) = \mathcal{O}_{\mathbb{P}^n}(-q)$, for some $q > 0$.

To show that $B_0(\mathbb{C}^{n+1})$ is locally trivial over $\mathbb{P}^n$, consider the usual cover, $U_0, \ldots, U_n$, of $\mathbb{P}^n$ (recall, $U_j = \{ \langle \xi \rangle \in \mathbb{P}^n | \xi_j \neq 0 \}$). If $v \in B_0(\mathbb{C}^{n+1}) \mid U_j$, then $v = \langle \langle z \rangle; (x) \rangle$, with $\xi_j \neq 0$. Define $\varphi_j$ as the map

$$v \mapsto \langle \langle \xi \rangle; z_j \rangle \in U_j \prod \mathbb{C}$$

and the backwards map

$$\langle \langle \xi \rangle; t \rangle \in U_j \prod \mathbb{C} \mapsto \langle \langle z \rangle; (\xi) \rangle, \quad \text{where} \quad z_i = \frac{\xi_i}{\xi_j} t, \quad i = 0, \ldots, n.$$
The reader should check that the point of \( \mathbb{C}^{n+1} \prod \mathbb{P}^n \) so constructed is in \( B_0(\mathbb{C}^{n+1}) \) and that the maps are inverses of one another.

We can make a section, \( \sigma_j \), of \( B_0(\mathbb{C}^{n+1}) \upharpoonright U_j \), via

\[
\sigma((\xi)) = \left< \xi_0, \ldots, \xi_{j-1}, 1, \ldots, \xi_j \right> \left< \xi \right>,
\]

and we see that \( \varphi(\sigma((\xi))) = ((\xi); 1) \in U_j \prod \mathbb{C} \), which shows that \( \sigma \) is a holomorphic section which is never zero. The transition function, \( g_j^j \), renders the diagram

\[
\begin{array}{ccc}
B_0 \upharpoonright U_i & \xrightarrow{\varphi_i} & U_i \prod \mathbb{C} \\
\uparrow & & \uparrow \\
B_0 \upharpoonright U_i \cap U_j & \xrightarrow{g_j^j} & U_j \prod \mathbb{C}
\end{array}
\]

commutative. It follows that

\[
\varphi_j(v) = g_j^j(\varphi_i(v)) = g_j^j(((\xi); z_i)) = ((\xi); z_j)
\]

and we conclude that \( g_j^j(z_i) = z_j \), which means that \( g_j^j \) is multiplication by \( z_j/z_i = \xi_j/\xi_i \).

We now make another bundle on \( \mathbb{P}^n \), which will turn out to be \( \mathcal{O}_{\mathbb{P}^n}(1) \). Embed \( \mathbb{P}^n \) in \( \mathbb{P}^{n+1} \) by viewing \( \mathbb{P}^n \) as the hyperplane defined by \( z_{n+1} = 0 \) and let \( P = (0: \ldots : 1) \in \mathbb{P}^{n+1} \). Clearly, \( P \notin \mathbb{P}^n \). We have the projection, \( \pi: (\mathbb{P}^{n+1} - \{P\}) \to \mathbb{P}^n \), from \( P \) onto \( \mathbb{P}^r \), where

\[
\pi(z_0: \ldots : z_n: z_{n+1}) = (z_0: \ldots : z_n).
\]

We get a line family over \( \mathbb{P}^n \), where the fibre over \( Q \in \mathbb{P}^n \) is just the line \( l_{PQ} \) (since \( P \notin \mathbb{P}^n \), this line is always well defined). The parametric equations of this line are

\[
(u: t) \mapsto (uz_0: \ldots : uz_n : t),
\]

where \( (u: t) \in \mathbb{P}^1 \) and \( Q = (z_0: \ldots : z_n) \). When \( t = 0 \), we get \( Q \) and hence \( u = 0 \), we get \( P \). Next, we prove that \( \mathbb{P}^{n+1} - \{P\} \) is locally trivial. Make a section, \( \sigma_j \), of \( \pi \) over \( U_j \subseteq \mathbb{P}^n \) by setting

\[
\sigma_j((\xi)) = (\xi; \xi_j).
\]

This point corresponds to the point \((1: \xi_j) \) on \( l_{PQ} \) and \( \xi_j \neq 0 \), so it is well-defined. As \( Q \) is the point of \( l_{PQ} \) for which \( t = 0 \), we have \( \sigma_j((\xi)) \neq Q \). We make an isomorphism, \( \psi_j: (\mathbb{P}^{n+1} - \{P\}) \upharpoonright U_j \to U_j \prod \mathbb{C} \), via

\[
(z_0: \ldots : z_{j-1}: z_j: z_{j+1}: \ldots : z_{n+1}) \mapsto \begin{pmatrix} z_0: \ldots : z_n: \frac{z_{n+1}}{z_j} \end{pmatrix}.
\]

Observe that

\[
s_j((\xi)) = \psi_j \circ \sigma_j((\xi)) = \psi_j((\xi); \xi_j) = ((\xi); 1) \in U_j \prod \mathbb{C}.
\]

For any \((z_0: \ldots : z_{n+1}) \in (\mathbb{P}^{n+1} - \{P\}) \upharpoonright U_i \cap U_j\), we have \( z_i \neq 0 \) and \( z_j \neq 0 \); moreover

\[
\psi_i(z_0: \ldots : z_{n+1}) = \begin{pmatrix} z_0: \ldots : z_n: \frac{z_{n+1}}{z_i} \end{pmatrix} \quad \text{and} \quad \psi_j(z_0: \ldots : z_{n+1}) = \begin{pmatrix} z_0: \ldots : z_n: \frac{z_{n+1}}{z_j} \end{pmatrix}.
\]
This means that the transition function, \( h_i^j \), on \( U_i \cap U_j \), is multiplication by \( z_i/z_j \). These are the inverses of the transition functions of our previous bundle, \( B_0(\mathbb{C}^{n+1}) \), which means that the bundle \( \mathbb{P}^{n+1} - \{ P \} \) is the dual bundle of \( B_0(\mathbb{C}^{n+1}) \). We will use geometry to show that the bundle \( \mathbb{P}^{n+1} - \{ P \} \) is in fact \( \mathcal{O}_{\mathbb{P}^n}(1) \).

Look at the hyperplanes, \( H \), of \( \mathbb{P}^{n+1} \). They are given by linear forms,

\[
H : \sum_{j=0}^{n+1} a_j Z_j = 0.
\]

The hyperplanes through \( P \) form a \( \mathbb{P}^n \), since \( P \in H \) iff \( a_{n+1} = 0 \). The rest of the hyperplanes are in the affine space, \( \mathbb{C}^{n+1} = \mathbb{P}^{n+1} - \mathbb{P}^n \). Indeed such hyperplanes, \( H_\alpha \), are given by

\[
H_\alpha : \sum_{j=0}^n \alpha_j Z_j + Z_{n+1} = 0, \quad (\alpha_0, \ldots, \alpha_n) \in \mathbb{C}^{n+1}.
\]

Given any hyperplane, \( H_\alpha \) (with \( \alpha \in \mathbb{C}^{n+1} \)), find the intersection, \( \sigma_\alpha(Q) \), of the line \( l_{PQ} \) with \( H_\alpha \). Note that \( \sigma_\alpha \) is a global section of \( \mathbb{P}^{n+1} - \{ P \} \). The affine line obtained from \( l_{PQ} \) by deleting \( P \) is given by

\[
\tau \mapsto (z_0 : \cdots : z_n : \tau),
\]

where \( Q = (z_0 : \cdots : z_n) \). This lines cuts \( H_\alpha \) iff

\[
\sum_{j=0}^n \alpha_j z_j + \tau = 0,
\]

so we deduce \( \tau = -\sum_{j=0}^n \alpha_j z_j \) and

\[
\sigma_\alpha(z_0 : \cdots : z_n) = \left( z_0 : \cdots : z_n : -\sum_{j=0}^n \alpha_j z_j \right),
\]

which means that \( \sigma_\alpha \) is a holomorphic section. Now, consider a holomorphic section, \( \sigma : \mathbb{P}^n \to (\mathbb{P}^{n+1} - \{ P \}) \to \mathbb{P}^{n+1} \), of \( \pi : (\mathbb{P}^{n+1} - \{ P \}) \to \mathbb{P}^n \). As \( \sigma \) is an algebraic map and \( \mathbb{P}^r \) is proper, \( \sigma(\mathbb{P}^n) \) is \( \mathbb{C} \)-closed, irreducible and has dimension \( n \) in \( \mathbb{P}^{n+1} \). Therefore, \( \sigma(\mathbb{P}^n) \) is a hypersurface. But, our map factors through \( \mathbb{P}^{n+1} - \{ P \} \), so \( \sigma(\mathbb{P}^n) \subseteq \mathbb{P}^{n+1} - \{ P \} \). This hypersurface has some degree, \( d \), but all the lines \( l_{PQ} \) cut \( \sigma(\mathbb{P}^n) \) in a single point, which implies that \( d = 1 \), i.e., \( \sigma(\mathbb{P}^n) \) is a hyperplane not through \( P \). Putting all these facts together, we have shown that space of global sections \( \Gamma(\mathbb{P}^n, \mathbb{P}^{n+1} - \{ P \}) \) is in one-to-one correspondence with the hyperplanes \( H_\alpha \), i.e., the linear forms \( \sum_{j=0}^n \alpha_j z_j \) (a \( \mathbb{C}^{n+1} \)). Therefore, we conclude that \( \mathbb{P}^{n+1} - \{ P \} \) is \( \mathcal{O}_{\mathbb{P}^n}(1) \). Since \( B_0(\mathbb{C}^{n+1}) \) is the dual of \( \mathbb{P}^{n+1} - \{ P \} \), we also conclude that \( B_0(\mathbb{C}^{n+1}) = \mathcal{O}_{\mathbb{P}^n}(-1) \).

In order to prove that Chern classes exist, we need to know more about bundles. The reader may wish to consult Atiyah [2], Milnor and Stasheff [11], Hirsh [7], May [10] or Morita [12] for a more detailed treatment of bundles.

Recall that if \( G \) is a group, then \( H^1(X, G) \) classifies the \( G \)-torsors over \( X \), e.g., (in our case) the fibre bundles, fibre \( F \), over \( X \) (your favorite topology) with \( \text{Aut}(F) = G \). When \( F = G \) and \( G \) acts by left translation to make it \( \text{Aut}(F) \), the fibre bundle is called a principal bundle. Look at \( \varphi : G' \to G \), a homomorphism of groups. Now, we know that we get a map

\[
H^1(X, G') \to H^1(X, G).
\]

We would like to see this geometrically and we may take as representations principal bundles. Say \( E' \in H^1(X, G') \) a principal bundle with fibre \( G' \) and group \( G' \). Consider \( G \amalg E' \) and make an equivalence relation \( \sim \) via: For all \( \sigma \in G' \), all \( g \in G \), all \( e' \in E' \)

\[
(\varphi g(\sigma), e') \sim (g e' \sigma^{-1}).
\]
3.2. CHERN CLASSES AND SEGRE CLASSES

Set $E'_{G,G} = \varphi_*(E') = G\prod E'/\sim$.

Let us check that the fibre over $x \in X$ is $G$. Since $E'$ is locally trivial, we have $E' \mid U \cong U \prod G'$, for some small enough open, $U$. The action of $G'$ is such that: For $\sigma \in G'$ and $(u, \tau) \in U \prod G'$,

$$\sigma(u, \tau) = (u, \sigma \tau).$$

Over $U$, we have $(G \prod E') \mid U = G \prod U \prod G'$, so our $\varphi_*(E')$ is still locally trivial and the action is on the left on $G'$, its fibre. It follows that

$$E' \mapsto \varphi_*(E')$$

is our map $H^1(X, G') \rightarrow H^1(X, G)$.

Next, say $\theta : Y \rightarrow X$ is a map (of spaces), then we get a map

$$H^1(X, G) \xrightarrow{\theta^*} H^1(Y, G).$$

Given $E \in H^1(X, G)$, we have the commutative diagram

$$\begin{array}{ccc}
E \prod_X Y & \xrightarrow{E} & E \\
\downarrow & & \downarrow \pi_E \\
Y & \xrightarrow{\theta} & X,
\end{array}$$

so we get a space, $\theta^*(E) = E \prod_X Y$, over $Y$. Over a “small” open, $U$, of $X$, we have $E \mid U \cong G \prod U$ and

$$\theta^*(E) \mid \theta^{-1}(U) \cong G \prod \theta^{-1}(U),$$

and this gives

$$H^1(X, G) \xrightarrow{\theta^*} H^1(Y, G).$$

Say $G$ is a (Lie) group and we have a linear representation, $\varphi : G \rightarrow \text{GL}(r, \mathbb{C})$. By the above, we get a map

$$E \mapsto E_G := \varphi_*(E)$$

from principal $G$-bundles over $X$ to principal $\text{GL}(r, \mathbb{C})$-bundles over $X$. But if $V$ is a fixed vector space of dimension $r$, the construction above gives a rank $r$ vector bundle $\text{GL}(r, \mathbb{C}) \prod V/\sim$. If $\mathcal{V}$ is a rank $r$ vector bundle over $\mathbb{C}$, then look at the sheaf, $\mathcal{I}som(\mathcal{V}, \mathcal{V})$, whose fibre at $x$ is the space $\text{Isom}(\mathcal{V}_x, \mathcal{V}_x)$. This sheaf defines a $\text{GL}(r, \mathbb{C})$-bundle.

Say $G' \subseteq G$ is a closed subgroup of the topological group, $G$.

If $G$ is a real Lie group and $G'$ is a closed subgroup, then $G'$ is also a real Lie group (E. Cartan). But, if $G$ is a complex Lie group and $G'$ is a closed subgroup, then $G'$ need not be a complex Lie group. For example, look at $G = \mathbb{C}^* = \text{GL}(1, \mathbb{C})$ and $G' = \mathbb{U}(1) = \{z \in \mathbb{C} \mid |z| = 1 \}$.

Convention: If $G$ is a complex Lie group, when we say $G'$ is a closed subgroup we mean a complex Lie group, closed in $G$.

Say $G$ is a topological group and $G'$ is a closed subgroup of $G$. Look at the space $G/G'$ and at the continuous map, $\pi : G \rightarrow G/G'$. We say $\pi$ has a local section iff there is some some $V \subseteq G/G'$ with $1_G \cdot G' \in V$ and a continuous map

$$s : V \rightarrow G,$$

such that $\pi \circ s = \text{id}_V$.

When we untwist this definition we find that it means $s(v) \in v$, where $v$ is viewed as a coset. Generally, one must assume the existence of a local section—this is not true in general.
Theorem 3.15 If $G$ and $G'$ are topological groups and $G'$ is a closed subgroup of $G$, assume a local section exists. Then

1. The map $G \rightarrow G/G'$ makes $G$ a continuous principal bundle with fibre and group $G'$ and base $G/G'$.

2. If $G$ is a real Lie group and $G'$ is a closed subgroup, then a local smooth section always exists and $G$ is a smooth principal bundle over $G/G'$, with fibre (and group) $G'$.

3. If $G$ is a complex Lie group and $G'$ is a closed complex Lie subgroup, then a complex analytic local section always exists and makes $G$ a complex holomorphic principal bundle over $G/G'$, with fibre (and group) $G'$.

Proof. The proof of (1) is deferred to the next theorem.

(2) & (3). Use local coordinates, choosing coordinates transverse to $G'$ after choosing coordinates in $G'$ near $1_{G'}$. The rest is (DX)– because we get a local section and we repeat the proof for (1) to prove the bundle assertion.

Now, say $E$ is a fibre bundle, with group $G$ over $X$ (and fibre $F$) and say $G'$ is a closed subgroup of $G$. Then, we have a new bundle, $E/G'$. The bundle $E/G'$ is obtained from $E$ by identifying in each fibre the elements $x$ and $x\sigma$, where $\sigma \in G'$. Then, the group of $E/G'$ is still $G$ and the fibre is $F/G'$. In particular, if $E$ is principal, then the group of $E/G'$ is $G$ and its fibre is $G/G'$. We have a map $E \rightarrow E/G'$ and a diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\pi_E} & E/G' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\pi} & E/G' \\
\end{array}
\]

Theorem 3.16 If $G \rightarrow G/G'$ possesses a local section, then for a principal $G$-bundle $E$ over $X$

1. $E/G'$ is a fibre bundle over $X$, with fibre $G/G'$.

2. $E \rightarrow E/G'$ is in a natural way a principal bundle (over $E/G'$) with group and fibre $G'$. If $\xi \in H^1(X, G)$ represents $E$, write $\xi_{G'}$ for the element of $H^1(E/G', G')$ whose bundle is just $E \rightarrow E/G'$.

3. From the diagram of bundles

\[
\begin{array}{ccc}
E & \xrightarrow{\pi_E} & E/G' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\pi_{E/G'}} & E/G' \\
\end{array}
\]

we get the commutative diagram

\[
\begin{array}{ccc}
H^1(X, G') & \xrightarrow{\pi^*_{E/G'}} & H^1(X, G) \\
\downarrow \quad \pi^*_{E/G'} & & \downarrow \quad \pi^*_{E/G'} \\
H^1(E/G', G') \xrightarrow{i_*} H^1(E/G', G')
\end{array}
\]

(Here $i: G' \hookrightarrow G$ is the inclusion map) and $i_*(\xi_{G'}) = \pi^*_{E/G'}(\xi)$, that is, when $E$ is pulled back to the new base $E/G'$, it arises from a bundle whose structure group is $G'$.}
3.2. CHERN CLASSES AND SEGRE CLASSES

Figure 3.1: The fibre bundle $E$ over $E/G'$

Proof. (1) is already proved (there is no need for our hypothesis on local sections).

(2) Pick a cover $\{U_\alpha\}$, of $C$ where $E \mid U_\alpha$ is trivial so that

$$E \mid U_\alpha \cong U_\alpha \prod G.$$ 

Now, consider $G \rightarrow G'/G'$ and the local section $s: V(\subseteq G/G') \rightarrow G$ (with $1_{G/G'} \in V$). We know $s(v) \in v$ (as a coset) and look at $\pi^{-1}(V)$. If $x \in \pi^{-1}(V)$, set

$$\theta(x) = (x^{-1}s(\pi(x)), \pi(x)) \in G' \prod V.$$

This gives an isomorphism (in the appropriate category), $\pi^{-1}(V) \cong G' \prod V$. If we translate $V$ around $G/G'$, we get $G$ as a fibre bundle over $G'/G'$ and group $G'$ giving (1) of the previous theorem. But, $U_\alpha \prod V$ and the $U_\alpha \prod (\text{translate of } V)$ give a cover of $E/G'$ and we have

$$E \mid U_\alpha \cong U_\alpha \prod \pi^{-1}(V) \cong U_\alpha \prod V \prod G',$$

giving $E$ as fibre bundle over $E/G'$ with group and fibre $G'$. Here, the diagrams are obvious and the picture of Figure 3.1 finishes the proof. Both sides of the last formula are “push into the board” (by definition for $i_*$ and by elementary computation in $\pi_E^{G}(\xi)$). □

Definition 3.2 If $E$ is a bundle over $X$ with group $G$ and if $G'$ is a closed subgroup of $G$ so that the cohomology representative of $G$, say $\xi$ actually arises as $i_*(\eta)$ for some $\eta \in H^1(X, G')$, then $E$ can have its structure group reduced to $G'$.

If we restate (3) of the previous theorem in this language, we get

Corollary 3.17 Every bundle $E$ over $X$ with group $G$ when pulled back to $E/G'$ has its structure group reduced to $G'$.

Theorem 3.18 Let $E$ be a bundle over $X$, with group $G$ and let $G'$ be a closed subgroup of $G$. Then, $E$ as a bundle over $X$ can have its structure group reduced to $G'$ iff the bundle $E/G'$ admits a global section over $X$. In this case if $s: X \rightarrow E/G'$ is the global section of $E/G'$, then $s^*(E)$ where $E$ is considered as bundle over $E/G'$ with group $G'$ is the element $\eta \in H^1(X, G')$ which gives the structure group reduction. In terms of cocycles, $E$ admits a reduction to group $G'$ iff there exists an open cover $\{U_\alpha\}$ of $X$ so that the transition functions

$$g^\alpha_{\beta}: U_\alpha \cap U_\beta \rightarrow G$$

map $U_\alpha \cap U_\beta$ into the subgroup $G'$. The section of $E/G'$ is given in the cover by maps $s_\alpha: U_\alpha \rightarrow U_\alpha \prod G/G'$, where $s_\alpha(u) = (u, 1_{G/G'})$. The cocycle $g^\alpha_{\beta}$ represents $s^*(E)$ when its values are considered to be in $G'$ and represents $E$ when its values are considered to be in $G$.

Proof. Consider the picture of Figure 3.1 above. Suppose $E$ can have structure group reduced to $G'$, then there is a principal bundle, $F$, for $G'$ and its transition functions give $E$ too. This $F$ can be embedded in $E$, the fibres are $G'$. Apply $\pi_{E \rightarrow E/G'}$ to $F$, we get get a space over $X$ whose points lie in the bundle $E/G'$, one point for each point of $X$. Thus, the map $s: X \rightarrow \text{point of } \pi_{E \rightarrow E/G'}(F)$ over $x$, is our section of $E/G'$ over $X$.

Conversely, given a section, $s: X \rightarrow E/G'$, we have $E$ as principal bundle over $E/G'$, with fibre and group $G'$. So, $s^*(E)$ gives a bundle, $F$, principal for $G'$, lying over $X$. Note, $F$ is the bundle given by $s^*(\xi_{G'})$,
where $\xi$ represents $E$. This shows the $F$ constructed reduces to the group $G'$. The rest (with cocycles) is standard. \qed 

Look at $\mathbb{C}^q$ and $\text{GL}(q, \mathbb{C})$. Write $\mathbb{C}^q_r$ for the span of $e_1, \ldots, e_r$ (the first $r$ canonical basis vectors) = $\text{Ker} \pi_r$, where $\pi_r$ is projection on the last $q-r$ basis vectors, $e_{r+1}, \ldots, e_q$. Let $\text{Grass}(r, q; \mathbb{C})$ denote the complex Grassmannian of $r$-dimensional linear subspaces in $\mathbb{C}^q$. There is a natural action of $\text{GL}(q, \mathbb{C})$ on $\text{Grass}(r, q; \mathbb{C})$ and it is clearly transitive. Let us look for the stabilizer of $\mathbb{C}^q_r$. It is the subgroup, $\text{GL}(r, q-r; \mathbb{C})$, of $\text{GL}(q, \mathbb{C})$, consisting of all matrices of the form

$$
\begin{pmatrix}
A & B \\
0 & C
\end{pmatrix}
$$

where $A$ is $r \times r$. It follows that, as a homogeneous space,

$$
\text{GL}(q, \mathbb{C})/\text{GL}(r, q-r; \mathbb{C}) \cong \text{Grass}(r, q; \mathbb{C}).
$$

If we restrict the action to $\text{U}(q)$, the above matrices must be of the form

$$
\begin{pmatrix}
A & 0 \\
0 & C
\end{pmatrix}
$$

where $A \in \text{U}(r)$ and $C \in \text{U}(q-r)$, so

$$
\text{U}(q)/\text{U}(r) \prod \text{U}(q-r) \cong \text{Grass}(r, q; \mathbb{C}).
$$

**Remark:** Note, in the real case we obtain

$$
\text{GL}(q, \mathbb{R})/\text{GL}(r, q-r; \mathbb{R}) \cong \text{O}(q)/\text{O}(r) \prod \text{O}(q-r) \cong \text{Grass}(r, q; \mathbb{R}).
$$

If one looks at oriented planes, then this becomes

$$
\text{GL}^+(q, \mathbb{R})/\text{GL}^+(r, q-r; \mathbb{R}) \cong \text{SO}(q)/\text{SO}(r) \prod \text{SO}(q-r) \cong \text{Grass}^+(r, q; \mathbb{R}).
$$

**Theorem 3.19** *(Theorem A)* If $X$ is paracompact, $f$ and $g$ are two maps $X \rightarrow Y$ and $E$ is a bundle over $Y$, then when $f$ is homotopic to $g$ and not for holomorphic bundles, we have $f^*E \cong g^*E$.

**Theorem 3.20** *(Theorem B)* Suppose $X$ is paracompact and $E$ is a bundle over $X$ whose fibre is a cell. If $Z$ is any closed subset of $X$ (even empty) then any section (continuous, smooth, but not holomorphic) of $E$ over $Z$ admits an extension to a global section (continuous or smooth) of $E$. That is, the sheaf $\mathcal{O}_X(E)$ is a soft sheaf. (Note this holds when $E$ is a vector bundle and it is Tietze’s Extension Theorem).

**Theorem 3.21** *(Theorem C)* Say $G'$ is a closed subgroup of $G$ and $X$ is paracompact. If $G/G'$ is a cell, then the natural map

$$
H^1_{\text{top}}(X, G') \rightarrow H^1_{\text{top}}(X, G) \quad \text{or} \quad H^1_{\text{diff}}(X, G') \rightarrow H^1_{\text{diff}}(X, G)
$$

is a bijection. That is, every principal $G$-bundle can have its structure group reduced to $G'$ and comes from a unique principal $G'$-bundle.

**Proof.** Suppose $E$ is a principal $G$-bundle and look at $E/G'$ over $X$. The fibre of $E/G'$ over $X$ is $G/G'$, a cell. Over a small closed set, say $Z$, the bundle $E/G'$ has a section; so, by Theorem B our section extends to a global section ($G/G'$ is a cell). Then, by Theorem 3.18, the bundle $E$ comes from $H^1(X, G')$ and surjectivity is proved.
Now, say $E$ and $F$ are principal $G'$-bundles and that they become isomorphic as $G$-bundles. Take a common covering $\{ U_\alpha \}$, where $E$ and $F$ are trivialized. Then $g_\alpha^0(E), g_\alpha^0(F)$, their transition functions become cohomologous in the $G$-bundle theory. This means that there exist maps, $h_\alpha : U_\alpha \rightarrow G$ so that

$$g_\alpha^0(F) = h_\alpha^{-1} g_\alpha^0(E) h_\alpha^{-1}.$$  

Consider $X \coprod I$ where $I = [0,1]$ and cover $X \coprod I$ by the opens

$$U^0_\alpha = U_\alpha \coprod [0,1) \quad \text{and} \quad U^1_\alpha = U_\alpha \coprod (0,1].$$

Make a principal bundle on $X \coprod I$ using the following transition functions:

$$g^\beta_\alpha : U^0_\alpha \cap U^0_\beta \longrightarrow G$$

via $g^\beta_\alpha (x,t) = g^\beta_\alpha (E)(x)$;

$$g^\beta_\alpha : U^1_\alpha \cap U^1_\beta \longrightarrow G$$

via $g^\beta_\alpha (x,t) = g^\beta_\alpha (F)(x)$;

$$g^\beta_\alpha : U^0_\alpha \cap U^1_\beta \longrightarrow G$$

via $g^\beta_\alpha (x,t) = h_\beta (x) g^\beta_\alpha (E)(x) = g^\beta_\alpha (E)(x) h_\alpha (x)$. Call this new bundle $(E,F)$ and let

$$Z = X \coprod \{0\} \cup X \coprod \{1\} \hookrightarrow X \coprod I$$

a closed subset. Note that $(E,F)$ over $Z$ is a $G'$-bundle. Thus, Theorem 3.18 says $(E,F)/G'$ has a global section over $Z$. But, its fibre is $G/G'$, a cell. Therefore, by Theorem B, the bundle $(E,F)/G'$ has a global section over all of $X$. By Theorem 3.18, again, the bundle $(E,F)$ comes from a $G'$-bundle, $(\tilde{E},\tilde{F})$. Write $f_0 : X \rightarrow X \coprod I$ for the function given by

$$f_0(x) = (x,0)$$

and $f_1 : X \rightarrow X \coprod I$ for the function given by

$$f_1(x) = (x,1).$$

If $(\tilde{E},\tilde{F}) | X \coprod \{0\} = (\tilde{E},\tilde{F})_0$, then $f_0^*(\tilde{E},\tilde{F})_0 = E$, i.e., $f_0^*(\tilde{E},\tilde{F}) = E$ and similarly, $f_1^*(\tilde{E},\tilde{F}) = F$; and $f_0$ is homotopic to $f_1$. By Theorem A, we get $E \cong F$ as $G'$-bundles. $\square$

There is a theorem of Steenrod stating: If $X$ is a differentiable manifold and $E$ is a fibre bundle over $X$, then every continuous section of $E$ may be approximated (with arbitrary $\epsilon$) on compact subsets of $X$ by a smooth section. When $E$ is a vector bundle, this is easy to prove by an argument involving a partition of unity and approximation techniques using convolution. This proves

**Theorem 3.22 (Theorem D)** If $X$ is a differentiable manifold and $G$ is a Lie group, then the map

$$H^1_{cont}(X,G) \longrightarrow H^1_{cont}(X,G)$$

is a bijection.

We get the

**Corollary 3.23** If $X$ is a differentiable manifold, then in the diagram below, for the following pairs $(G',G)$

$(\alpha)$ $G' = U(q), G = GL(q,C)$.

$(\beta)$ $G' = U(r) \coprod U(q-r), G = GL(r,q-r;\mathbb{C})$ or $G = GL(r,\mathbb{C}) \coprod GL(q-r,\mathbb{C})$. 

ξ gives rise to

Here,

\[
\Delta(q, \mathbb{C}) = \bigcap_{r=1}^{q} \text{GL}(r, q-r; \mathbb{C})
\]

the upper triangular matrices.

Proof. Observe that \(G/G'\) is a cell in all cases and that \(\Delta(q, \mathbb{C}) \cap U(q) = T^q\). □

Suppose \(\xi\) corresponds to a \(\text{GL}(q)\)-bundle which has group reduced to \(\text{GL}(r, q-r; \mathbb{C})\). Then, the maps

\[
M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mapsto A \quad \text{and} \quad M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mapsto C
\]

give surjections \(\text{GL}(r, q-r; \mathbb{C}) \rightarrow \text{GL}(r, q-r; \mathbb{C})\) and \(\text{GL}(r, q-r; \mathbb{C}) \rightarrow \text{GL}(q-r, \mathbb{C})\), so \(\xi\) comes from \(\tilde{\xi}\) and \(\xi\) gives rise to \(\xi'\) and \(\xi''\) which are \(\text{GL}(r, \mathbb{C})\) and \(\text{GL}(q-r, \mathbb{C})\)-bundles, respectively. In this case one says: the \(\text{GL}(q, \mathbb{C})\)-bundle \(\xi\) admits a reduction to a (rank \(r\)) subbundle \(\xi'\) and a (rank \(q-r\)) quotient bundle \(\xi''\).

When we use \(\Delta(q, \mathbb{C})\) and \(\text{GL}(q, \mathbb{C})\) then we get \(q\) maps, \(\varphi_j : \Delta(q, \mathbb{C}) \rightarrow \mathbb{C}^*,\) namely

\[
\varphi_j : \begin{pmatrix} a_1 & * & \cdots & * & * \\ 0 & a_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{q-1} & * \\ 0 & 0 & \cdots & 0 & a_q \end{pmatrix} \mapsto a_j.
\]

So, if \(\tilde{\xi}\) is our \(\Delta(q, \mathbb{C})\)-bundle, we get \(q\) line bundles \(\xi_1, \ldots, \xi_q\) from \(\tilde{\xi}\) and one says \(\xi\) has \(\xi_1, \ldots, \xi_q\) as diagonal line bundles.

Set

\[
F_q = \text{GL}(q; \mathbb{C})/\Delta(q; \mathbb{C}) = \text{GL}(q; \mathbb{C})/\bigcap_{r=1}^{q} \text{GL}(r, q-r; \mathbb{C}),
\]

the flag manifold, i.e., the set of all flags

\[
\{0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_q = V \mid \text{dim}(V_j) = j\}.
\]

Since \(F_q = \text{GL}(q; \mathbb{C})/\bigcap_{r=1}^{q} \text{GL}(r, q-r; \mathbb{C})\), we see that \(F_q\) is embedded in \(\prod_{r=1}^{q} \text{Grass}(r, q; \mathbb{C})\). Thus, as the above is a closed immersion, \(F_q\) is an algebraic variety, even a projective variety (by Segre). If \(V\) is a rank \(q\) vector bundle over \(X\), say \(E(V) = \text{Isom}(\mathbb{C}^q, V)\) is the associated principal bundle, then write

\[
[r]V = E(V)/\text{GL}(r, q-r; \mathbb{C}),
\]

a bundle over \(X\) whose fibres are \(\text{Grass}(r, q; \mathbb{C})\) and

\[
[\Delta]V = E(V)/\Delta(q; \mathbb{C})
\]

a bundle over \(X\) whose fibres are the \(F(q)\)'s. We have maps \(\rho_r : [r]V \rightarrow X\) and \(\rho_\Delta : [\Delta]V \rightarrow X\). Now we apply our theorems to the pairs.
3.2. CHERN CLASSES AND SEGRE CLASSES

(a) $G' = U(q)$, $G = \text{GL}(q, \mathbb{C})$.
(b) $G' = \text{U}(r) \prod U(q - r)$ and $G = \text{GL}(r, q - r, \mathbb{C})$ or $G = \text{GL}(r, \mathbb{C}) \prod \text{GL}(q - r, \mathbb{C})$.
(c) $G' = T^q$ and $G = U(q)$ or $G = \mathbb{C}^* \prod \cdots \prod \mathbb{C}^* = (\mathbb{G}_m)^q$.
(d) $G' = \Delta(q, \mathbb{C})$ and $G = \text{GL}(q, \mathbb{C})$.

and then we get, (for example) every rank $r$ vector bundle over $X$ is “actually” a rank $r$ unitary bundle over $X$ and we also have

**Theorem 3.24** If $X$ is paracompact or a differentiable manifold or a complex analytic manifold or an algebraic variety and $V$ is a rank $q$ vector bundle of the appropriate category on $X$, then

1. $V$ reduces to a rank $r$ subbundle, $V'$, and rank $q - r$ quotient bundle, $V''$, over $X$ iff $[r]V$ possesses an appropriate global section over $X$.

2. $V$ reduces to diagonal bundles over $X$ iff $[\Delta]V$ possesses an appropriate global section over $X$.

3. For the maps $\rho_r$ in case (1), resp. $\rho_\Delta$ in case (2), the bundle $\rho_r^* V$ reduces to a rank $r$ subbundle and rank $q - r$ quotient bundle over $[r]V$ (resp. reduces to diagonal bundles over $[\Delta]V$).

**Remark:** The sub, quotient, diagonal bundles are continuous, differentiable, analytic, algebraic, respectively.

Say $s : X \to [r]V$ is a global section. For every $x \in X$, we have $sx \in \text{Grass}(r, q; V_x)$; i.e., $s(x)$ is an $r$-plane in $V_x$ and so, $\bigcup_{x \in X} s(x)$ gives an “honest” rank $r$ subbundle or $V$. It is isomorphic to the subbundle, $V'$, of the reduction. Similarly, $\bigcup_{x \in X} V_x / s(x)$ is an “honest” rank $q - r$ quotient bundle of $V$; it is just $V''$.

Hence, we get

**Corollary 3.25** If the hypotheses of the previous theorem hold, then

1. $[r]V$ has a section iff there is an exact sequence

$$0 \to V' \to V \to V'' \to 0$$

of vector bundles on $X$.

2. $[\Delta]V$ has a section iff there exist exact sequences

$$0 \to L_1 \to V \to V_1'' \to 0$$
$$0 \to L_2 \to V_1'' \to V_2'' \to 0$$
$$\cdots \cdots \cdots \cdots \cdots \cdots$$
$$0 \to L_{j+1} \to V_j'' \to V_{j+1}'' \to 0$$
$$\cdots \cdots \cdots \cdots \cdots \cdots$$
$$L_q \cong V_{q-1}''$$

where the $L_j$’s are line bundles, in fact, the diagonal bundles.

**Theorem 3.26** In the continuous and differentiable categories, when $V$ has an exact sequence as in (1) of Corollary 3.25 or diagonal bundles as in (2) of Corollary 3.25, then

1. $V \cong V' \sqcup V''$.

2. $V \cong L_1 \sqcup \cdots \sqcup L_q$.

The above is false if we need splitting analytically!
All we need to prove is (1) as (2) follows by induction. We know $V$ comes from $H^1(X, \text{GL}(r, q-r; \mathbb{C}))$. By (b) above, $V$ comes from $H^1(X, U(r) \bigoplus U(q-r))$ and by (b) again, $V$ comes from $H^1(X, \text{GL}(r) \bigoplus \text{GL}(q-r)) \cong H^1(X, \text{GL}(r)) \amalg H^1(X, \text{GL}(q-r))$ and we get (1). □

**Corollary 3.27** *(Splitting Principle)* Given $V$, a continuous, differentiable, analytic, algebraic rank $q$ vector bundle over $X$, then $\rho_v^* V$ is in the continuous or differentiable category a coproduct $V = V' \amalg V''$ ($\text{rk}(V') = r$, $\text{rk}(V'') = q - r$) or $\rho^*_\Delta V$ is $V = L_1 \amalg \cdots \amalg L_q$.

Note that $[\gamma]V$ and $[\Delta]V$ are fibre bundles over $X$; consequently, there is a relation between $H^1(X, \mathbb{Z})$ and $H^1([\gamma]V, \mathbb{Z})$ (resp. $H^1([\Delta]V, \mathbb{Z})$). This is the Borel spectral sequence. Under the condition that $(E, X, F, G)$ is a fibre space over $X$ (admissible), group $G$, fibre $F$, total space $E$, there is a spectral sequence whose $E_2^{p,q}$-term is

$$H^p(X, H^q(F, \mathbb{Z}))$$

and whose ending is $\text{gr}(H^*(E, \mathbb{Z}))$,

$$H^p(X, H^q(F, \mathbb{Z})) \Longrightarrow H^*(E, \mathbb{Z}).$$

Borel proves that in our situation: The map

$$\rho^*: H^*(X, \mathbb{Z}) \to H^*([\gamma]V, \mathbb{Z})$$

(resp. $\rho^*: H^*(X, \mathbb{Z}) \to H^*([\Delta]V, \mathbb{Z})$) is an injection. From the hand-out, we also get the following: Write

$$\text{BU}(q) = \lim_{\longrightarrow} \text{Grass}(q, N; \mathbb{C}).$$

Note,

$$\text{BU}(1) = \lim_{\longrightarrow} \mathbb{P}_\mathbb{C}^{N-1} = \mathbb{P}_\mathbb{C}^\infty.$$

**Theorem 3.28** If $X$ is admissible (locally compact, $\sigma$-compact, finite dimensional) then $\text{Vect}_q(X)$ (isomorphism classes of rank $q$ vector bundles over $X$) in the continuous or differentiable category is in one-to-one correspondence with homotopy classes of maps $X \to \text{BU}(q)$. In fact, if $X$ is compact and $N \geq 2\text{dim}(X)$ then already the homotopy classes of maps $X \to \text{Grass}(q, N; \mathbb{C})$ classify rank $q$ vector bundles on $X$ (differentiably). Moreover, on $\text{BU}(q)$, there exists a bundle, the “universal quotient”, $W_q$, it has rank $q$ over $\text{BU}(q)$ (in fact, it is algebraic) so that the map is

$$f \in [X \to \text{BU}(q)] \mapsto f^*W_q.$$

We are now in the position where we can prove the uniqueness of Chern classes.

**Uniqueness of Chern Classes:**

Assume existence (Axiom (I)) and good behavior (Axioms (II)–(IV)). First, take a line bundle, $L$, on $X$. By the classification theorem there is a map

$$f: X \to \text{BU}(1)$$

so that $f^*(H) = L$ (here, $H$ is the universal quotient line bundle). By Axiom (II),

$$f^*(c(H)(t)) = c(f^*(H))(t) = c(L)(t)$$

and the left hand side is $f^*(1 + Ht)$, by Axiom (IV) (viewing $H$ as a cohomology class). It follows that the left hand side is $1 + f^*(H)t$ and so,

$$c_1(L) = f^*(H), \quad \text{and} \quad c_j(L) = 0, \quad \text{for all} \quad j \geq 2.$$
This is independent of $f$ as homotopic maps agree cohomologically.

Now, let $V$ be a rank $q$ vector bundle on $X$ and make the bundle $[\Delta]V$ whose fibre is $\mathbb{F}(q)$. Take $\rho^*(V)$, where $\rho: [\Delta]V \to X$. We know

$$\rho^* V = \prod_{j=1}^{q} L_j,$$

where the $L_j$’s are line bundles and by Axiom (II),

$$c(\rho^*(V))(t) = \prod_{j=1}^{1} (1 + c_1(L_j)(t)).$$

Now, the left hand side is $\rho^*(c(V)(t))$, by Axiom (II); then, $\rho^*$ being an injection implies $c(V)(t)$ is uniquely determined.

**Remark:** Look at $U(q) \supseteq U(1) \prod U(q-1) \supseteq \mathbb{T}^q$. Then,

$$U(1) \prod U(q-1) / \mathbb{T}^q \hookrightarrow U(q) / \mathbb{T}^q = \mathbb{F}(q)$$

and the left hand side is $U(q-1) / \mathbb{T}^{q-1} = \mathbb{F}(q-1)$. So, we have an injection $\mathbb{F}(q-1) \hookrightarrow \mathbb{F}(q)$ over the base $U(q) / U(1) \prod U(q-1)$, which is just $\mathbb{F}^{q-1}$. Thus, we can view $\mathbb{F}(q)$ as a fibre bundle over $\mathbb{F}^{q-1}$ and the fibre is $\mathbb{F}(q-1)$.

Take a principal $U(q)$-bundle, $E$, over $X$ and make $E/\mathbb{T}^q$, a fibre space whose fibre is $\mathbb{F}(q)$. Let $E_1$ be $E/U(1) \prod U(q-1)$, a fibre space whose fibre is $\mathbb{F}^{q-1}$. Then, we have a map

$$E/\mathbb{T}^q \longrightarrow E_1,$$

where the fibre is $U(1) \prod U(q-1) / \mathbb{T}^q = \mathbb{F}(q-1)$. We get

If we repeat this process, we get the tower

$$E/\mathbb{T}^q = [\Delta]E$$

$$\text{fibre } \mathbb{F}(q-1)$$

$$E_1$$

$$\text{fibre } \mathbb{F}^{q-1}$$

$$X.$$
So, to show \( \rho^* \) is injective, all we need to show is the same fact when the fibre \( \mathbb{P}^n \) and the \( \mathbb{P}^r \)-bundle comes from a vector bundle.

**Suggestion:** Look in Hartshorne in Chapter III, Section ? on projective fibre bundles and Exercise ?? about

\[
\rho^*(\mathcal{O}_{\mathbb{P}(E)}(l)) = \mathcal{O}_X(E).
\]

Sup up to tangent bundles and wedges and use Hodge:

\[
H^\bullet(X, \mathbb{C}) = \text{in term of the holomorphic cohomology of } \bigwedge^\text{top} T.
\]

We get that \( \rho^* \) is injective on \( H^\bullet(X, \mathbb{C}) \), not \( H^\bullet(X, \mathbb{Z}) \).

**Existence of Chern Classes:**

Start with \( L \), a line bundle over \( X \). Then, there is a map (continuous, diff.), \( f : X \to \mathbb{F}_C^N \), for \( N > 0 \) and \( L = f^*(H) \). Then, set \( c_1(L) = f^*(H) \), where \( H \) is the cohomology class of the hyperplane bundle in \( H^2(\mathbb{P}^N, \mathbb{Z}) \) and \( c_2(L) = 0 \) if \( j \geq 2 \). If another map, \( g \), is used, then \( f^*(H) = L = h^*(L) \) implies that \( f \) and \( g \) are homotopic, so \( f^* \) and \( g^* \) agree on cohomology and \( c_1(L) \) is independent of \( f \). It is also independent of \( N \), we we already proved. Clearly, Axiom (II) and Axiom (IV) are built in.

Now, let \( V \) be a rank \( q \) vector bundle over \( X \). Make \([\Delta]V\) and let \( \rho \) be the map \( \rho : [\Delta]V \to X \). Look at \( \rho^*V \). We know that

\[
\rho^*V = \prod_{j=1}^{q} L_j,
\]

where the \( L_j \)'s are line bundles. By the above,

\[
c_j(L_j)(t) = 1 + c_1(L_j)t = 1 + \gamma_j t.
\]

Look at the polynomial

\[
\prod_{j=1}^{q} (1 + \gamma_j t) \in H^\bullet([\Delta]V, \mathbb{Z})[t].
\]

If we show this polynomial (whose coefficients are the symmetric functions \( \sigma_i(\gamma_1, \ldots, \gamma_q) \)) is in the image of \( \rho^* : H^\bullet(X, \mathbb{Z})[t] \to H^\bullet([\Delta]V, \mathbb{Z})[t] \), then there is a unique polynomial \( c(V)(t) \) so that

\[
\rho^*(c(V)(t)) = \prod_{j=1}^{q} (1 + \gamma_j t).
\]

(Then, \( c_2(V) = \sigma_1(\gamma_1, \ldots, \gamma_q) \).) Look at the normalizer of \( \mathbb{T}^q \) in \( U(q) \). Some \( a \) belongs to this normalizer iff \( aT^qa^{-1} = T^q \). As the new diagonal matrix, \( aT^qa^{-1} \) (where \( a \in \mathbb{T}^q \) has the same characteristic polynomial as \( \theta \), it follows that \( aT^qa^{-1} \) is just \( \theta \), but with its diagonal entries permuted. This gives a map

\[
\mathcal{N}_{U(q)}(\mathbb{T}^q) \to \mathfrak{S}_q.
\]

What is the kernel of this map? We have \( a \in \text{Ker} \) iff \( aT^qa^{-1} = \theta \), i.e., \( a\theta = \theta a \), for all \( \theta \in \mathbb{T}^q \). This means (see the 2 \times 2 case) \( a \in \mathbb{T}^q \) and thus, we have an injection

\[
\mathcal{N}_{U(q)}(\mathbb{T}^q)/\mathbb{T}^q \hookrightarrow \mathfrak{S}_q.
\]

The left hand side, by definition, is the Weyl group, \( W \), of \( U(q) \). In fact (easy DX), \( W \cong \mathfrak{S}_q \).

Look at \([\Delta]V\) and write a covering of \( X \) trivializing \([\Delta]V\), call it \( \{U_\alpha\} \). We have

\[
[\Delta]V \upharpoonright U_\alpha \cong U_\alpha \prod U(q)/\mathbb{T}^q.
\]
Make the element $a$ act on the latter via
\[
a(u, \xi \mathbb{T}^q) = (u, \xi \mathbb{T}^q a^{-1}) = (u, \xi a^{-1} \mathbb{T}^q).
\]
These patch as the transition functions are left translations. This gives an automorphism of $[\Delta]V$, call it $\tilde{a}$, determined by each $a \in W$. We get a map
\[
\tilde{a}^*: H^*([\Delta]V, -) \rightarrow H^*([\Delta]V, -).
\]
Now, as $a \in W$ acts on $\mathbb{T}^q$ by permuting the diagonal elements it acts on $H^1([\Delta]V, \mathbb{T}^q)$ by permuting the diagonal bundles, say $L_j$, call this action $a^\#$. Moreover, $\rho^*V$ comes from a unique element of $H^1([\Delta]V, \mathbb{T}^q)$, which implies that $\tilde{a}$ acts on $\rho^*V$ by permuting its cofactors. But, $\tilde{a}^*$ also acts on $H^1([\Delta]V, \mathbb{T}^q)$ and one should check (by a Čech cohomology argument) that $\tilde{a}^* = a^\#$.

Now associate to the $L_j$’s their Chern classes, $\gamma_j$, and $\tilde{a}^*(\gamma_j)$ goes over to $a^\#(\gamma_j)$, i.e., permute the $\gamma_j$’s. Thus, $W$ acts on the $L_j$ and $\gamma_j$ by permuting them. Our polynomial
\[
\prod_{j=1}^{q}(1 + \gamma_j t)
\]
goes to itself via the action of $W$. But, Borel’s Theorem is that an element of $H^*([\Delta]V, \mathbb{Z})$ lies in the image of $\rho^*: H^*([\Delta]V, \mathbb{Z}) \rightarrow H^*([\Delta]V, \mathbb{Z})$ iff $W$ fixes it. So, by the above, our elementary symmetric functions lie in $\text{Im} \rho^*$; so, Chern classes exist. Furthermore, it is clear that they satisfy Axioms (I), (II), (IV).

Finally, consider Axiom (III). Suppose $V$ splits over $X$ as
\[
V = \bigoplus_{j=1}^{q} L_j.
\]
We need to show that $c(V)(t) = \prod_{j=1}^{q}(1 + c_1(L_j)t)$.

As $V$ splits over $X$, the fibre bundle $\rho: [\Delta]V \rightarrow X$ has a section; call it $s$. So, $s^*\rho^* = \text{id}$ and
\[
c(V)(t) = s^*\rho^*(c(V)(t)) = s^*(\rho^*(c(V)(t))).
\]
By Axiom (II), $s^*(\rho^*(c(V)(t))) = s^*(c(\rho^*(V))(t))$. Since $\rho^* = \bigoplus_{j=1}^{q} \rho^* L_j$ and we know that if we set $\gamma_j = c_1(\rho^*(L_j))$, then
\[
\rho^*(c(V)(t)) = c(\rho^*(V)(t)) = \prod_{j=1}^{q}(1 + \gamma_j t).
\]
But then,
\[
c(V)(t) = s^* \prod_{j=1}^{q}(1 + \gamma_j t) = \prod_{j=1}^{q}(1 + s^*(\gamma_j) t). \quad (\dagger)
\]
However, $L_j = s^*(\rho^*(L_j))$ implies
\[
c_1(L_j) = s^*(c_1(\rho^*(L_j))) = s^*(\gamma_j).
\]
The above plus $(\dagger)$ yields
\[
c(V)(t) = \prod_{j=1}^{q}(1 + c_1(L_j)t),
\]
as required. □

Eine kleine Vektorraumbündel Theorie:

Say $V$ (rank $q$) and $W$ (rank $q'$) have diagonal bundles $L_1, \ldots, L_q$ and $M_1, \ldots, M_{q'}$ over $X$. Then, the following hold:

1. $V^D$ has $L_1^D, \ldots, L_q^D$ as diagonal line bundles;
2. $V \sqcup W$ has $L_1, \ldots, L_q, M_1, \ldots, M_{q'}$ as diagonal line bundles;
3. $V \otimes W$ has $L_i \otimes M_j$ (all $i, j$) as diagonal line bundles;
4. $\Lambda^r V$ has $L_{i_1} \otimes \cdots \otimes L_{i_r}$, where $1 \leq i_1 < \cdots < i_r \leq q$, as diagonal line bundles;
4. $S^r V$ has $L_1^{m_1} \otimes \cdots \otimes L_q^{m_q}$, where $m_i \geq 0$ and $m_1 + \cdots + m_q = r$, as diagonal line bundles.

Application to the Chern Classes.

0. (Splitting Principle) Given a rank $q$ vector bundle, $V$, make believe $V$ splits as $V = \prod_{j=1}^q L_j$ (for some line bundles, $L_j$), write $\gamma_j = c_1(L_j)$, the $\gamma_j$ are the Chern roots of $V$. Then,

$$c(V)(t) = \prod_{j=1}^q (1 + \gamma_j t).$$

1. $c(V^D)(t) = \prod_{j=1}^q (1 - \gamma_j t)$ when $c(V)(t) = \prod_{j=1}^q (1 + \gamma_j t)$. That is, $c_i(V^D) = (-1)^i c_i(V)$.
2. If $0 \to V' \to V \to V'' \to 0$ is exact, then $c(V)(t) = c(V')c(V'')(t)$.
3. If $c(V)(t) = \prod_{j=1}^q (1 + \gamma_j t)$ and $c(W)(t) = \prod_{j=1}^{q'} (1 + \delta_j t)$, then $c(V \otimes W)(t) = \prod_{j,k=1}^{q,q'} (1 + (\gamma_j + \delta_k)t)$.
4. If $c(V)(t) = \prod_{j=1}^q (1 + \gamma_j t)$, then

$$c\left(\bigwedge^r V\right)(t) = \prod_{1 \leq i_1 < \cdots < i_r \leq q} (1 + (\gamma_{i_1} + \cdots + \gamma_{i_r})t).$$

In particular, when $r = q$, there is just one factor in the polynomial, it has degree 1, it is $1 + (\gamma_1 + \cdots + \gamma_q)t$. By (2), we get

$$c_1\left(\bigwedge^q V\right)(t) = c_1(V) \quad \text{and} \quad c_l\left(\bigwedge^q V\right)(t) = 0 \quad \text{if} \quad l \geq 2.$$

5. If $c(V)(t) = \prod_{j=1}^q (1 + \gamma_j t)$, then

$$c(S^r V)(t) = \prod_{\substack{m_1 \geq 0 \cr m_1 + \cdots + m_q = r}} (1 + (m_1\gamma_1 + \cdots + m_q\gamma_q)t).$$

6. If $\operatorname{rk}(V) \leq q$, then $\deg(c(V)(t)) \leq q$ (where $\deg(c(V)(t)$ is the degree of $c(V)(t)$ as a polynomial in $t$).
(7) Suppose we know \( c(V) \), for some vector bundle, \( V \), and \( L \) is a line bundle. Write \( c = c_1(L) \). Then, the Chern classes of \( V \otimes L \) are

\[
c_1(V \otimes L) = \sigma_1(\gamma_1 + c, \gamma_2 + c, \ldots, \gamma_r + c),
\]

where \( r = \text{rk}(V) \) and the \( \gamma_j \) are the Chern roots of \( V \). This is because the Chern polynomial of \( V \otimes L \) is

\[
c(V \otimes L)(t) = \prod_{i=1}^{r}(1 + (\gamma_i + c)t).
\]

Examples. (1) If \( \text{rk}(V) = 2 \), then

\[
c(V \otimes L)(t) = (1 + (\gamma_1 + c)t)(1 + (\gamma_2 + c)t) = 1 + (\gamma_1 + \gamma_2 + 2c)t + (\gamma_1 \gamma_2 + c(\gamma_1 + \gamma_2) + c^2)t^2,
\]

so

\[
c_1(V \otimes L) = c_1(V) + 2c, \quad c_2(V \otimes L) = c_2(V) + c_1(V)c + c^2.
\]

(2) If \( \text{rk}(V) = 3 \), then

\[
c(V \otimes L)(t) = (1 + (\gamma_1 + c)t)(1 + (\gamma_2 + c)t)(1 + (\gamma_3 + c)t)
\]

and so,

\[
c(V \otimes L)(t) = 1 + (\gamma_1 + \gamma_2 + \gamma_3 + 3c)t + (\sigma_2(\gamma_1, \gamma_2, \gamma_3) + 2\sigma_1(\gamma_1, \gamma_2, \gamma_3)c + 3c^2)t^2 + (\sigma_3(\gamma_1, \gamma_2, \gamma_3) + 3\sigma_2(\gamma_1, \gamma_2, \gamma_3)c^2 + 3\sigma_2(\gamma_1, \gamma_2, \gamma_3)c + 3c^3)t^3.
\]

We deduce

\[
c_1(V \otimes L) = c_1(V) + 3c_1(L), \quad c_2(V \otimes L) = c_2(V) + 2c_1(V)c_1(L) + 3c_1(L)^2,
\]

\[
c_3(V \otimes L) = c_3(V) + c_2(V)c_1(L) + c_1(V)c_1(L)^2 + c_1(L)^3.
\]

In the case of \( \mathbb{P}^n \), it is easy to compute the Chern classes. By definition,

\[
c(\mathbb{P}^n)(t) = c(T_{\mathbb{P}^n}^{1,0})(t).
\]

We can use the Euler sequence

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \bigoplus_{n+1} \mathcal{O}_{\mathbb{P}^n}(H) \longrightarrow T_{\mathbb{P}^n}^{1,0} \longrightarrow 0
\]

to deduce that

\[
c(\mathcal{O}_{\mathbb{P}^n})(t)c(T_{\mathbb{P}^n}^{1,0})(t) = c(\mathcal{O}_{\mathbb{P}^n}(H))(t)^{n+1}.
\]

It follows that

\[
c(T_{\mathbb{P}^n}^{1,0})(t) = (1 + Ht)^{n+1} \pmod{t^{n+1}} = \sum_{j=0}^{n} \binom{n+1}{j} H^j t^j
\]

and so,

\[
c_j(T_{\mathbb{P}^n}^{1,0}) = \binom{n+1}{j} H^j \in H^{2j}(\mathbb{P}^n, \mathbb{Z}).
\]
(Here $H^j = H \cdot \ldots \cdot H$, the cup-product in cohomology). In particular,
\[ c_1(T^1_{\mathbb{P}^n}) = (n+1)H = c\left(\bigwedge^n T^1_{\mathbb{P}^n}\right). \]

Now, if $\omega_{\mathbb{P}^n}$ is the canonical bundle on $\mathbb{P}^n$, i.e., $\omega_{\mathbb{P}^n} = \bigwedge^n T^0_{\mathbb{P}^n} D = \left(\bigwedge^n T^1_{\mathbb{P}^n}\right) D$, we get
\[ c_1(\omega_{\mathbb{P}^n}) = -(n+1)H. \]

Say a variety $X$ sits inside $\mathbb{P}^n$ and assume $X$ is a manifold. Let $\mathcal{J}$ be the ideal sheaf of $X$. By definition, $\mathcal{J}$ is the kernel in the exact sequence
\[ 0 \to \mathcal{J} \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0. \]

If $X$ is a hypersurface of degree $d$, we know
\[ \mathcal{J} = \mathcal{O}_{\mathbb{P}^n}(-d) = \mathcal{O}_{\mathbb{P}^n}(-dH). \]

We also have the exact sequence
\[ 0 \to T_X \to T_{\mathbb{P}^n} | X \to N_{X \hookrightarrow \mathbb{P}^n} \to 0, \]
where $N_{X \hookrightarrow \mathbb{P}^n}$ is a rank $n-q$ bundle on $X$, with $q = \dim X$ (the normal bundle). If we write $i: X \to \mathbb{P}^n$, we get
\[ \left(\bigwedge^n T_{\mathbb{P}^n}\right) | X = \bigwedge^n T_X \otimes \bigwedge^{n-q} N_{X \hookrightarrow \mathbb{P}^n}, \]
and so,
\[ i^*(1 + c_1\left(\bigwedge^n T_{\mathbb{P}^n}\right)t) = (1 + c_1\left(\bigwedge^n T_X\right)t)(1 + c_1\left(\bigwedge^{n-q} N_{X \hookrightarrow \mathbb{P}^n}\right)t), \]
which yields
\[ 1 + i^*((n+1)H)t = 1 + c_1(T_X)t + c_1(N_{X \hookrightarrow \mathbb{P}^n})t. \]

For the normal bundle, we can compute using $\mathcal{J}$. Look at a small open, then we have the usual case of $\mathbb{C}$-algebras
\[ \mathbb{C} \hookrightarrow A \twoheadrightarrow B \]
where $A$ corresponds to local functions on $\mathbb{P}^n$ restricted to $X$ and $B$ to local functions on $X$ and we have the exact sequence of relative Kähler differentials
\[ \Omega^1_{A/\mathbb{C}} \otimes_A B \to \Omega^1_{B/\mathbb{C}} \to \Omega^1_{B/A} \to 0. \]

If $A$ mapping onto $B$ is given, then $\Omega^1_{B/A} = (0)$, $B = A/\mathfrak{A}$ (globally, $\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n}/\mathcal{J}$), and we get
\[ 0 \to \text{Ker} \to \Omega^1_A \otimes_A A/\mathfrak{A} \to \Omega^1_{A/\mathfrak{A}} \to 0. \]

Now, $\mathcal{J} \to \Omega^1_A \otimes_A A/\mathfrak{A}$, via $\xi d\xi \mapsto \otimes 1$ and in fact, $\mathcal{J} \to 0$. We conclude that
\[ i^*(\mathcal{J}) = \mathcal{J}/\mathcal{J}^2 \to i^*(\Omega^1_{\mathbb{P}^n}) \to \Omega^1_X \to 0. \]

Because $X$ is a manifold, the arrow on the left is an injection. To see this we need only look locally at $x$. We can take completions and then use either the $C^1$-implicit function theorem or the holomorphic implicit function theorem or the formal implicit function theorem and get the result (DX). If we dualize, from
\[ 0 \to \mathcal{J}/\mathcal{J}^2 = i^*(\mathcal{J}) \to i^*\mathcal{O}^1_{\mathbb{P}^n} \to \Omega^1_X \to 0 \]
we get
\[ 0 \longrightarrow T_X \longrightarrow i^*T_{\mathbb{P}^n} = T_{\mathbb{P}^n} \upharpoonright X \longrightarrow (\mathcal{I}/\mathcal{I}^2)^D \longrightarrow 0 \]

Therefore,
\[ N_X \hookrightarrow \mathbb{P}^n = (\mathcal{I}/\mathcal{I}^2)^D = i^*(\mathcal{I})^D = (\mathcal{I} \upharpoonright X)^D. \]

Thus,
\[ c_1(N_X \hookrightarrow \mathbb{P}^n) = -c_1(\mathcal{I}/\mathcal{I}^2), \]

and
\[ (n + 1)i^*(H) + c_1(\mathcal{I}/\mathcal{I}^2) = c_1(T_X). \]

We obtain a version of the adjunction formula:
\[ c_1(\omega_X) = -(n + 1)i^*(H) - c_1(\mathcal{I}/\mathcal{I}^2). \]

When \( X \) is a hypersurface of degree \( d \), then \( \mathcal{I} = \mathcal{O}_{\mathbb{P}^n}(-dH) \) and
\[ \mathcal{I}/\mathcal{I}^2 = i^*(\mathcal{I}) = \mathcal{O}_X(-d \cdot i^*H). \]

We deduce that \(-c_1(\mathcal{I}/\mathcal{I}^2) = d(i^*H)\) and
\[ c_1(\omega_X) = (d - n - i)i^*H. \]

Say \( n = 2 \), and \( \dim X = 1 \), a curve in \( \mathbb{P}^2 \). When \( X \) is smooth, we have
\[ c_1(\omega_X) = (d - n - 1)i^*(H). \]

**Facts** soon to be proved:

(a) \( i^*(H) = H \cdot X \), in the sense of intersection theory (that is, \( \deg X \) points on \( X \)).

(b) \( c_1(L) \) on a curve is equal to the degree of the divisor of \( L \).

It follows from above that
\[ \deg(\omega_X) = (d - 2 - 1)d = d(d - 3). \]

However, from Riemann-Roch on a curve, we know \( \deg(\omega_X) = 2g - 2 \), so we conclude that for a smooth algebraic curve, its genus, \( g \), is given by
\[ g = \frac{1}{2}(d - 1)(d - 2). \]

In particular, observe that \( g = 2 \) is missed.

We know from the theory that if we know all \( c_1 \)'s then we can determine all \( c_n \)'s for all \( n \) by the splitting principle.

There are three general methods for determining \( c_1 \):

(I) The exponential sequence.

(II) Curvature.

(III) Degree of a divisor.
Proposition 3.29 Say $X$ is an admissible, or a differentiable manifold, or a complex analytic manifold or an algebraic manifold. In each case, write $\mathcal{O}_X$ for the sheaf of germs of appropriate functions on $X$. Then, from the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow e \mathcal{O}_X \longrightarrow 0,$$

where $e(f) = \exp(2\pi i f)$, we get in each case the connecting map

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z})$$

(†)

and all obvious diagrams commute

** Steve, what are these obvious diagrams? **

and as the group $H^1(X, \mathcal{O}_X^*)$ classifies the line bundles of appropriate type, we get $\delta(L)$, a cohomology class in $H^2(X, \mathbb{Z})$ and we have

$$c_1(L) = \delta(L).$$

In the continuous and differentiable case, $\delta$ is an isomorphism. Therefore, a continuous or differentiable line bundle is completely determined by its first Chern class.

Proof. That the diagrams commute is clear. For the isomorphism statement, we have the cohomology sequence

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X).$$

But, in the continuous or $C^\infty$-case, $\mathcal{O}_X$ is a fine sheaf, so $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = (0)$ and we get

$$H^1(X, \mathcal{O}_X^*) \cong H^2(X, \mathbb{Z}).$$

First, we show that (†) can be reduced to the case $X = \mathbb{P}^1_{\mathbb{C}} = S^2$.

** Steve, in this case, are we assuming that $X$ is projective? **

Take a line bundle, $L$ on $X$ (continuous or $C^\infty$), then, for $N >> 0$, there is a function, $f : X \rightarrow \mathbb{P}^N_{\mathbb{C}}$, so that $f^*H = L$. Now, we have the diagram

$$\begin{array}{ccc}
H^1(\mathbb{P}^N_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}^*) & \xrightarrow{\delta} & H^2(\mathbb{P}^N_{\mathbb{C}}, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^1(X, \mathcal{O}_X^*) & \xrightarrow{\delta} & H^2(X, \mathbb{Z})
\end{array}$$

which commutes by cofunctoriality of cohomology. Consequently, the existence of (†) on the top line implies the existence of (†) in general. Now, consider the inclusions

$$\mathbb{P}^1_{\mathbb{C}} \hookrightarrow \mathbb{P}^2_{\mathbb{C}} \hookrightarrow \cdots \hookrightarrow \mathbb{P}^N_{\mathbb{C}},$$

and $H$ on $\mathbb{P}^N_{\mathbb{C}}$ pulls back at each stage to $H$ and Chern classes have Axiom (II). Then, one sees that we are reduced to $\mathbb{P}^1_{\mathbb{C}}$.

Recall how simplicial cohomology is isomorphic (naturally) to Čech cohomology: Take a triangulation of $X$ and $v$, a vertex of a simplex, $\Delta$. Write

$$U_v = \text{st}(v) = \bigcup^{\circ} \{\sigma \mid v \in \sigma\}$$

the open star of the vertex $v$. The $U_\sigma$ form an open cover and we have:

$$U_{v_0} \cap \cdots \cap U_{v_p} = \begin{cases} 
\emptyset & \text{if } (v_0, \ldots, v_p) \text{ is not a simplex;} \\
\text{a connected nonempty set} & \text{if } (v_0, \ldots, v_p) \text{ is a simplex.}
\end{cases}$$
Given a Čech $p$-cochain, $\tau$, then

$$\tau(U_{v_0} \cap \cdots \cap U_{v_p}) = \begin{cases} 0 & \text{if } (v_0, \ldots, v_p) \text{ is not a simplex;} \\
\text{some integer} & \text{if } (v_0, \ldots, v_p) \text{ is a simplex.} \end{cases}$$

Define

$$\tau(v_0, \ldots, v_p) = \tau(U_{v_0} \cap \cdots \cap U_{v_p}).$$

Take a simplex, $\Delta = (v_0, \ldots, v_p)$ and define linear functions $\theta(\tau)$ by

$$\theta(\tau)(v_0, \ldots, v_p) = \tau(v_0, \ldots, v_p) = \tau(U_{v_0} \cap \cdots \cap U_{v_p})$$

and extend by linearity. We get a map,

$$\hat{H}^p(X, \mathbb{Z}) \cong H^p_{\text{simp}}(X, \mathbb{Z})$$

via $\tau \mapsto \theta(\tau)$, which is an isomorphism.

We are down to the case of $\mathbb{P}^1_\mathbb{C} = S^2$ and we take $H$ as the North pole. The Riemann sphere $\mathbb{P}^1_\mathbb{C}$ has coordinates $(Z_0 : Z_1)$, say $Z_1 = 0$ is the north pole $(Z_0 = 0$ is the south pole) and let

$$z = \frac{Z_0}{Z_1}, \quad w = \frac{Z_1}{Z_0}.$$

We have the standard opens, $V_0 = \{(Z_0 : Z_1) \mid Z_0 \neq 0\}$ and $V_1 = \{(Z_0 : Z_1) \mid Z_1 \neq 0\}$. The local equations for $H$ are $f_0 = w = 0$ and $f_1 = 1$. The transitions functions $g^0_\beta$ are $f_\beta/f_\alpha$, i.e.,

$$g^0_0 = \frac{f_1}{f_0} = z \quad \text{and} \quad g^0_1 = \frac{f_0}{f_1} = w.$$

Now, we triangulate $S^2$ using four triangles whose vertices are: $o = z; z = 1; z = i$ and $z = -1$. Note that $H$ is represented by a point which is in the middle of a face of the simplex $(1, i, -1)$ We have $U_0, U_1, U_i, U_{-1}$, the four open stars and $U_0 \subseteq V_1; U_1 \subseteq V_0; U_i \subseteq V_0; U_{-1} \subseteq V_0$. The $U$-cover refined the $V$-cover and on it, $g^r_s = 1$ iff both $r, s \neq 0$. Also, $g^0_s = w$, for all $t \neq 0$. To lift back the exponential, $\mathcal{O}_{\mathbb{P}^1} \xrightarrow{\exp(2\pi i)} \mathcal{O}_{\mathbb{P}^1}^*$, we form $\frac{1}{2\pi i} \log(g^t_s)$, a one-cochain with values in $\mathcal{O}_{\mathbb{P}^1}$. Since the intersections $U_r \cap U_s$ are all simply-connected, on each, we can define a single-valued branch of the log. Consistently do this on these opens $\text{via}$: Start on $U_1 \cap U_i$ and pick any single-valued branch of the log. Continue analytically to $U_i \cap U_{-1}$. Then, continue analytically to $U_{-1} \cap U_{1}$, we get $2\pi i + \log$ on $U_1 \cap U_i$. Having defined the log $g^s_r$, we take the Čech $\delta$ of the 1-cochain, that is

$$c_{rst} = \frac{1}{2\pi i} [\log g^r_s - \log g^t_r + \log g^t_s] = \frac{1}{2\pi i} [\log g^r_s + \log g^t_r + \log g^t_s].$$

If none of $r, s, t$ are 0, then $c_{rst} = 0$. So, look at $c_{0-1-1}$. We have

$$c_{0-1-1} = \frac{1}{2\pi i} [\log g_0^{-1} + \log g_{-1}^0 + \log g_1^0] = \frac{1}{2\pi i} [\log w - \text{"log" } w].$$

As $w = 1/z$, the second log is $-2\pi i + \log w$, so we get

$$c_{0-1-1} = +1.$$
Proposition 3.30 Say $X$ is a complex manifold and $L$ is a $C^\infty$ line bundle on it. Let $\nabla$ be an arbitrary connection on $X$ and write $\Theta$ for the curvature of $\Delta$. Then, the 2-form $\frac{1}{2\pi}\Theta$ is real and it represents in $H^2_{DR}(X, \mathbb{R})$ the image of $c_1(L)$ under the map

$$H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{R}).$$

Proof. Pick a trivializing cover for $L$, say $\{ U_\alpha \}$. Then, $\nabla \mid L$ on $U_\alpha$ comes from its connection matrix, $\theta_\alpha$, this is a $1 \times 1$ matrix ($L$ is a line bundle). We know (gauge transformation)

$$\theta_\alpha = g_\beta^\alpha \theta_\beta (g_\beta^\alpha)^{-1} + dg_\beta^\alpha (g_\beta^\alpha)^{-1},$$

where the $g_\beta^\alpha$ are the transition functions. But, we have scalars here, so

$$\theta_\alpha = \theta_\beta + d\log(g_\beta^\alpha),$$

that is

$$\theta_\beta - \theta_\alpha = -d\log(g_\beta^\alpha). \tag{†}$$

By Cartan-Maurer, the curvature, $\Theta$, (a 2-form) is given locally by

$$\Theta = d\theta - \theta \wedge \theta = d\theta_\alpha = d\theta_\beta.$$

We get the de Rham isomorphism in the usual way by splicing exact sequences. We begin with

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty \overset{d}{\longrightarrow} \text{cok}_1 \longrightarrow 0 \tag{*}$$

and

$$0 \longrightarrow \text{cok}_1 \overset{1}{\longrightarrow} \bigwedge^1 \overset{d}{\longrightarrow} \text{cok}_2 \longrightarrow 0 \tag{**}$$

It follows that

$$\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & C^\infty & \overset{d}{\longrightarrow} & \bigwedge^1 & \overset{d}{\longrightarrow} & \bigwedge^2 & \longrightarrow & \cdots \\
& & \downarrow \text{cok}_1 & & \downarrow \text{cok}_1 & & \downarrow \text{cok}_2 & & \downarrow \text{cok}_2 & & \\
& & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Apply cohomology to $(*)$ and $(**)$ and get

$$H^0(X, \bigwedge^1) \overset{d}{\longrightarrow} H^0(X, \text{cok}_1) \overset{\delta'}{\longrightarrow} H^1(X, \text{cok}_1) \longrightarrow H^1(X, \bigwedge^1) = (0)$$

and

$$H^1(X, C^\infty) \longrightarrow H^1(X, \text{cok}_1) \overset{\delta''}{\longrightarrow} H^2(X, \mathbb{R}) \longrightarrow H^2(X, C^\infty) = (0)$$

because $\bigwedge^1$ and $C^\infty$ are fine. We get

$$H^1(X, \text{cok}_1) \cong H^2(X, \mathbb{R}) \quad \text{and} \quad H^0(X, \text{cok}_2)/dH^0(X, \bigwedge^1) \cong H^1(X, \text{cok}_1).$$
Therefore,\[
\delta' \circ \delta' : H^0(X, \text{cok}_2) \longrightarrow H^2(X, \mathbb{R}) \longrightarrow 0.
\]
We know from the previous proof that\[
c_{\alpha, \beta, \gamma} = \frac{1}{2\pi i}[\log g^\beta_\alpha + \log g^\gamma_\alpha + \log g_\gamma^\alpha]
\]
represents \(c_1(L)\) via the \(\delta\) from the exponential sequence. So,\[
c_{\alpha, \beta, \gamma} = \frac{1}{2\pi i}[\log g^\beta_\alpha + \log g^\gamma_\alpha + \log g_\gamma^\alpha]
\]
and\[
\delta'[\Theta] = \text{cohomology class of } \Theta = \text{class of cocycle } (\theta_\beta - \theta_\alpha).
\]
Now, \(\frac{1}{2\pi i}(\theta_\beta - \theta_\alpha)\) can be lifted back to \(-\frac{1}{2\pi i}\log g_\beta^\alpha\) under \(\delta''\) and we deduce that\[
\delta'' \delta' \left(\frac{1}{2\pi i} \Theta\right) = \text{class of } -\frac{1}{2\pi i}[\log g_\beta^\alpha + \log g_\gamma^\alpha + \log g_\gamma^\alpha] = -\text{class of } c_{\alpha, \beta, \gamma} = -c_1(L).
\]
** There may be a problem with the sign! **

The next way of looking at \(c_1(L)\) works when \(L\) comes from a divisor. Say \(X\) is a complex algebraic manifold and \(L = \mathcal{O}_X(D)\), where \(D\) is a divisor,
\[
D = \sum_j a_j W_j
\]
on \(X\). Then, \(D\) gives a cycle in homology, so \([D] \in H_{2n-2}(X, \mathbb{Z})\) (here \(n = \dim_{\mathbb{C}} X\)). By Poincaré duality, our \([D]\) is in \(H^2(X, \mathbb{Z})\) and it is \(\sum a_j [W_j]\).

**Theorem 3.31** If \(X\) is a compact, complex algebraic manifold and \(D\) is a divisor on \(X\), then\[
c_1(\mathcal{O}_X(D)) = [D] \quad \text{in } H^2(X, \mathbb{Z}),
\]
that is, \(c_1(\mathcal{O}_X(D))\) is carried by the \((2n-2)\)-cycle, \(D\).

**Proof.** Recall that Poincaré duality is given by: For \(\xi \in H^r(X, \mathbb{R})\) and \(\eta \in H^s(X, \mathbb{R})\) (where \(r + s = 2n\)), then \[(\xi, \eta) = \int_X \xi \wedge \eta.\]
The homology/cohomology duality is given by: For \(\omega \in H^s(X, \mathbb{R})\) and \(Z \in H_s(X, \mathbb{R})\), then \[(Z, \omega) = \int_Z \omega.\]
We know that the cocyle (= 2-form) representing \(c_1(L)\) is \([\frac{i}{2\pi} \Theta]\), for any connection on \(X\). We must show that for every closed, real \((2n-2)\)-form, \(\omega\),
\[
\frac{i}{2\pi} \int_X \Theta \wedge \omega = \int_D \omega.
\]
We compute \(\Theta\) for a convenient connection, namely, the uniholo connection. Pick a local holomorphic frame, \(e(z)\), for \(L\), then if \(L\) has a section, \(s\), we know \(s(z) = \lambda(z)e(z)\), locally. For \(\theta\), the connection matrix in this frame, we have
(a) $\theta = \theta^{1,0}$ (holomorphic)

(b) $d(|s|^2) = (\nabla s, s) + (s, \nabla s)$ (unitary)

We have

$$\nabla s = \nabla \lambda e = (d\lambda + \theta \lambda)e.$$  

Thus, the right hand side of (b) is

$$d(|s|^2) = (d\lambda + \theta \lambda)e, \lambda e + \lambda \nabla \lambda (e, e) + \lambda \nabla \lambda (e, e) + \overline{\theta} |\lambda|^2 (e, e).$$

Write $h(z) = |e(z)|^2 = (e, e) > 0$; So, the right hand side of (b) is $\lambda dh + h(d\lambda + \overline{\theta})h$. Now, $|s|^2 = \lambda \lambda h$, so

$$d(|s|^2) = \lambda \lambda dh + h(d\lambda + \overline{\theta} \lambda)h.$$

Using (a) and the decomposition by type, we get

$$\theta = \partial\log h = \partial \log |e|^2.$$  

As $\Theta = d\theta - \theta \wedge \theta$, we get

$$\Theta = d\theta = (\partial + \overline{\theta})(\partial \log |e|^2),$$

i.e.,

$$\Theta = \overline{\partial} \partial \log |e|^2.$$  

Now, recall

$$d\bar{e} = \frac{i}{4\pi} (\overline{\partial} - \partial),$$

so that

$$dd\bar{e} = -d\bar{e} d = \frac{i}{2\pi} \partial \overline{\partial} = \frac{i}{2\pi} \overline{\partial} \partial,$$

and $2\pi id\bar{e} = \overline{\partial} \partial$. Consequently,

$$\Theta = \pi i dd\bar{e} \log |e|^2.$$  

This holds for any local frame, $e$, and has nothing to do with the fact that $L$ comes from a divisor.

Now, $L = \mathcal{O}_{X}(D)$ and assume that the local equations for $D$ are $f_{\alpha} = 0$ (on $U_{\alpha}$, some open in the trivializing cover for $L$ on $X$). We know the transition functions are

$$g_{\alpha}^{\beta} = \frac{f_{\beta}}{f_{\alpha}};$$

Therefore, the local vectors $s_{\alpha} = f_{\alpha} e_{\alpha}$ form a global section, $s$, of $\mathcal{O}_{X}(D)$. The zero locus of this section is exactly $D$. As the bundle $L$ is unitary, $g_{\alpha}^{\beta} \in U(1)$, which implies $|f_{\beta}| = |f_{\alpha}|$ and so, $|f_{\alpha} e_{\alpha}|$ is well defined. Thus for small $\epsilon > 0$,

$$D(\epsilon) = \{ z \in X \mid |s(z)|^2 < \epsilon \}$$

is a tubular neighborhood of $D$.

Look at $X - D(\epsilon)$, then $\mathcal{O}_{X}(D) \mid X - D(\epsilon)$ is trivial as the section $s$ is never zero there. Therefore, $s_{\alpha}$ will also do as a local frame for $\mathcal{O}_{X}(D)$ on $X - D(\epsilon)$. 
We need to compute $\int_X \Theta \wedge \omega$. By linearity, we may assume $D$ is one of the $W$'s. Then, by definition,

$$\int_X \Theta \wedge \omega = \lim_{\epsilon \downarrow 0} \int_{X-D(\epsilon)} 2\pi i d^c \log |s|^2 \wedge \omega$$

If we apply Stokes, we find

$$\int_X \Theta \wedge \omega = -\lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} 2\pi i d^c \log |s|^2 \wedge \omega$$

that is,

$$\int_X \Theta \wedge \omega = 2\pi i \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} d^c \log |s|^2 \wedge \omega. \quad (\dagger)$$

Now $\text{Vol}(D(\epsilon)) \rightarrow 0$ as $\epsilon \downarrow 0$, as we can see by using the Zariski stratification to reduce to the case where $D$ is non-singular. Also,

$$|s|^2 = |f_\alpha|^2 |e_\alpha|^2 = f_\alpha \overline{f}_\alpha h,$$

where $h = |e_\alpha|^2$ is positive bounded. We have

$$\log |s|^2 = \log f_\alpha + \log \overline{f}_\alpha + \log h$$

and as $d^c = \frac{i}{4\pi}(\overline{\partial} - \partial)$,

$$d^c \log |s|^2 = \frac{i}{4\pi} [-\partial \log f_\alpha + \overline{\partial} \log \overline{f}_\alpha + (\overline{\partial} - \partial) \log h].$$

It follows that

$$\frac{2\pi}{i} d^c \log |s|^2 \wedge \omega = \frac{1}{2} [-\partial \log f_\alpha \wedge \omega + \overline{\partial} \log \overline{f}_\alpha \wedge \omega + (\overline{\partial} - \partial) \log h \wedge \omega].$$

In the right hand side of $(\dagger)$, the third term is

$$\frac{1}{2} \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} (\overline{\partial} - \partial) \log h \wedge \omega.$$ 

Now, $(\overline{\partial} - \partial) \log h$ is bounded ($X$ is compact) and $\text{Vol}(\partial D(\epsilon)) \rightarrow 0$ as $\epsilon \downarrow 0$. So, this third term vanishes in the limit. But, $\overline{\partial} \log \overline{f}_\alpha = \partial \log f_\alpha$ and $\omega = \overline{\omega}$, as $\omega$ is real. Consequently,

$$\overline{\partial} \log \overline{f}_\alpha \wedge \omega = \partial \log f_\alpha \wedge \omega.$$ 

From $(\dagger)$, we get

$$\int_X \Theta \wedge \omega = \frac{1}{2} \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} -\partial \log f_\alpha \wedge \omega + \overline{\partial} \log f_\alpha \wedge \omega$$

$$= -\frac{1}{2} \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \omega - \overline{\partial} \log f_\alpha \wedge \omega$$

$$= -i \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \omega.$$ 

Now, $f_\alpha = 0$ is the local equation of $D$ and we can compute the integral on the right hand side away from the singularities of $D$ as the latter have measure 0. The divisor $D$ is compact, so we can cover it by polydiscs centered at nonsingular points of $D$, say $\zeta_0$ is a such a point. By the local complete intersection then, there exist local coordinates for $X$ near $\zeta_0$, of the form

$$z_1 = f_\alpha, \quad z_2, \ldots, z_n, \quad \text{rest}$$
on $\Delta \cap U_\alpha$ (where $\Delta$ is a polydisc). Break up $\omega$ as
\[
\omega = g(z_1, \ldots, z_n) \frac{dz_2 \wedge \cdots \wedge d\overline{z}_2 \wedge \cdots}{\text{rest}} + \kappa,
\]
where $\kappa$ is a form involving $dz_1$ and $d\overline{z}_1$ in each summand. Also, as
\[
\partial \log f_\alpha = (\partial + \overline{\partial}) \log f_\alpha = \frac{df_\alpha}{f_\alpha} = \frac{dz_1}{z_1},
\]
we get
\[
\partial \log f_\alpha \wedge \omega = \frac{dz_1}{z_1} g(z_1, \ldots, z_n) \frac{dz_2 \wedge \cdots \wedge d\overline{z}_2 \wedge \cdots}{\text{rest}} + \text{terms involving } \frac{dz_1 \wedge d\overline{z}_1}{z_1} \text{stuff}.
\]
Furthermore, $dz_1 \wedge d\overline{z}_1 = 2 dx \wedge dy = 2i dr_1 d\theta$ (in polar coordinates), so
\[
\left| \frac{dz_1 \wedge d\overline{z}_1}{z_1} \right| = 2|dr_1||d\theta_1|,
\]
and when $\epsilon \downarrow 0$, this term goes to 0. Therefore
\[
\lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon) \cap \Delta} \frac{dz_1}{z_1} g(z_1, \ldots, z_n) d(\text{rest}) d(\overline{\text{rest}}) = \lim_{\epsilon \downarrow 0} \int_{|z_1| = C\epsilon} \prod \text{rest of polydisc} \frac{dz_1}{z_1} g(z_1, \ldots, z_n) d(\text{rest}) d(\overline{\text{rest}})
\]
and by Cauchy's integral formula, this is
\[
\lim_{\epsilon \downarrow 0} \int_{\text{rest of polydisc} \cap \partial D(\epsilon)} 2\pi i g(0, z_2, \ldots, z_n) d(\text{rest}) d(\overline{\text{rest}}) = 2\pi i \int_{D \cap \Delta} \omega.
\]
Adding up the contributions from the finite cover of polydiscs, we get
\[
\exists \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \omega = \exists 2\pi i \int_D \omega = 2\pi \int_D \omega,
\]
as $\omega$ is real. But then,
\[
-i \exists \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} \log f_\alpha \wedge \omega = -2\pi i \int_X \omega
\]
from which we finally deduce $\int_X \Theta \wedge \omega = -2\pi i \int_D \omega$, that is,
\[
\int_X \frac{i}{2\pi} \Theta \wedge \omega = \int_D \omega,
\]
as required. \(\square\)

**Corollary 3.32** Suppose $V$ is a $U(q)$-bundle on our compact $X$ (so that differentiably, $V$ is generated by its sections). Or, if $V$ is a holomorphic bundle, assume it is generated by its holomorphic sections. Take a generic section, $s$, of $V$ and say $V$ has rank $r$. Then, the set $s = 0$ has complex codimension $r$ (in homology) and is the carrier of $c_r(V)$.

**Proof.** The case $r = 1$ is exactly the theorem above. Differentiably,
\[
V = L_1 \prod L_2 \prod \cdots \prod L_r,
\]
for the diagonal line bundles of $V$. Holomorphically, this is also OK but over the space $[\Delta]V$. So, the transition matrix is a diagonal matrix
\[
\text{diag}(g_{1\alpha}^\beta, \ldots, g_{r\alpha}^\beta) \text{ on } U_\alpha \cap U_\beta
\]
and \( s_\alpha = (s_1, \ldots, s_r) \). So, 
\[
\text{diag}(g^\beta_\alpha) s_\alpha = (g^\beta_1 s_1, \ldots, g^\beta_r s_r) = s_\beta
\]
which shows that each \( s_j \alpha \) is a section of \( L_j \). Note that \( s = 0 \) iff all \( s_j = 0 \). But, the locus \( s_j = 0 \) carries \( c_1(L_j) \), by the previous theorem. Therefore, \( s = 0 \) corresponds to the intersection in homology of the carriers of \( c_1(L_1), \ldots, c_1(L_r) \). But, intersection in homology is equivalent to product in cohomology, so the cohomology class for \( s = 0 \) is 
\[
c_1(L_1)c_1(L_2) \cdots c_1(L_r) = c(V)
\]
as desired. \( \square \)

**General Principle for Computing \( c_q(V) \), for a rank \( r \) vector bundle, \( V \).**

1. Let \( L \) be an ample line bundle, then \( V \otimes L^\otimes m \) is generated by its sections for \( m >> 0 \).

2. Pick \( r \) generic sections, \( s_1, \ldots, s_r \), of \( V \otimes L^\otimes m \). Form \( s_1 \wedge \cdots \wedge s_{r-q+1} \), a section of \( \wedge^{r-q+1}(V \otimes L^\otimes m) \). Then, the zero locus of \( s_1 \wedge \cdots \wedge s_{r-q+1} \) carries the Chern class, \( c_q(V \otimes L^\otimes m) \), of \( V \otimes L^\otimes m \).

\[
\begin{align*}
\text{When } q = r, \text{ this is the corollary. When } q = 1, \text{ we have } s_1 \wedge \cdots \wedge s_r, \text{ a section of } \wedge^r V \otimes L^\otimes m, \text{ and it is generic (as the fibre dimension is 1). We get } c_1(\wedge^r V \otimes L^\otimes m) \text{ and we know that it is equal to } c_1(V \otimes L^\otimes m). \\
\end{align*}
\]

3. Use the relation from the Chern polynomial 
\[
c_q(V \otimes L^\otimes m)(t) = \prod (1 + (\gamma_j + mc_1(L))t)
\]
to get the elementary symmetric functions of the \( \gamma_j \)'s, i.e., \( c_q(V) \).

**Remark:** if \( 1 < q < r \), our section \( s_1 \wedge \cdots \wedge s_{r-q+1} \) is not generic but it works.

**Theorem 3.33** Say \( X \) is a complex analytic or algebraic, compact, smooth, manifold and \( j: W \hookrightarrow X \) is a smooth, complex, codimension \( q \) submanifold. Write \( \mathcal{N} \) for the normal bundle of \( W \) in \( X \); this is rank \( q \) \((\mathbb{U}(q))\) vector bundle on \( W \). The subspace \( W \) corresponds to a cohomology class, \( \xi \), in \( H^{2q}(X, \mathbb{Z}) \) (in fact, in \( H^{q,q}(X, \mathbb{C}) \)) and so we get \( j^*\xi \in H^{2q}(W, \mathbb{Z}) \). Then, we have 
\[
c_q(\mathcal{N}) = j^*W.
\]

**Proof.** We begin with the case \( q = 1 \). In this case, \( W \) is a divisor and we know \( \mathcal{N} = \mathcal{O}_X(W) \mid W \). By Corollary 3.32, the Chern class \( c_1(\mathcal{N}) \) is carried by the zero locus of a section, \( s \), of \( \mathcal{N} \). Now, \( W \cdot W \) in \( X \) as a cycle is just a moving of \( W \) by a small amount and then an ordinary intersection of \( W \) and the new moved cycle. We see that \( W \cdot W = c_1(\mathcal{N}) \) as cycle on \( W \). But, \( j^*W \) is just \( W \cdot W \) as cycle (by Poincaré duality). So, the result holds when \( q = 1 \). If \( q > 1 \) and if \( W \) is a complete intersection in \( X \), then since \( c_q(\mathcal{N}) \) is computed by repeated pullbacks and each pullback gives the correct formula (by the case \( q = 1 \)), we get the result. In the general case, we have two classes \( j^*W \) and \( c_q(\mathcal{N}) \). If there exists an open cover, \( \{U_\alpha\} \), of \( W \) so that 
\[
j^*W \mid U_\alpha = c_q(\mathcal{N}) \mid U_\alpha \text{ for all } \alpha,
\]
then we are done. But, \( W \) is smooth so it is a local complete intersection (LCIT). Therefore, we get the result by the previous case. \( \square \)
Corollary 3.34 If $X$ is a compact, complex analytic manifold and if $T_X$ = holomorphic tangent bundle has rank $q = \dim \mathbb{C} X$, then
\[ c_q(T_X) = \chi_{\text{top}} = \sum_{i=0}^{2q} (-1)^i b_i \]
(Here, $b_i = \dim \mathbb{R} H^i(X, \mathbb{Z})$.)

Proof. (Essentially due to Lefschetz). Look at $X \prod X$ and the diagonal embedding, $\Delta: X \to X \prod X$. So, $X \hookrightarrow X \prod X$ is a smooth codimension $q$ submanifold. An easy argument shows that $T_X \cong N_{X \hookrightarrow X \prod X} = \mathcal{N}$ and the previous theorem implies
\[ c_q(T_X) = c_q(\mathcal{N}) = X \cdot X \]
in $X \prod X$. Now, look at the map $f: X \to X$ given by
\[ pr_2 \circ \epsilon \sigma, \]
where $\epsilon$ is small and $\sigma$ is a section of $\mathcal{N}$. The fixed points of our map give the cocycle $X \cdot X$. The Lefschetz fixed point Theorem says the cycle of fixed points is given by
\[ \sum_{i=0}^{2q} (-1)^i \text{tr} f^* \text{ on } H^i(X, \mathbb{Z}). \]
But, for $\epsilon$ small, the map $f$ is homotopic to id, so $f^* = \text{id}^*$. Now, $\text{tr} \text{id}^* = \text{dimension of space} = b_i(X)$ if we are on $H^i(X)$. So the right hand side of the Lefschetz formula is $\chi_{\text{top}}$, as claimed. □

Segre Classes.

Let $V$ be a vector bundle on $X$, then we have classes $s_j(V)$, and they are defined by
\[ 1 + \sum_{j=1}^{\infty} s_j(V) t^j = \frac{1}{c(V)(t)}. \]
As $c(V)(t)$ is nilpotent, we have
\[ \frac{1}{c(V)(t)} = 1 - (c_1(V) t + c_2(V) t^2 + \cdots) + (c_1(V) t + c_2(V) t^2 + \cdots)^2 + \cdots \]
and so,
\[ s_1(V) = -c_1(V) \]
\[ s_2(V) = c_1^2(V) - c_2(V), \]
etc.

Pontrjagin Classes.

Pontrjagin classes are defined for real $O(q)$-bundles over real manifolds. We have the commutative diagrams
\[
\begin{array}{ccc}
U(q) & \xrightarrow{\zeta} & O(2q) \\
\downarrow & & \downarrow \\
\text{GL}(q, \mathbb{C}) & \xrightarrow{\psi} & \text{GL}(2q, \mathbb{R})
\end{array}
\]
where \( \zeta(z_1, \ldots, z_q) = (x_1, y_1, \ldots, x_q, y_q) \), with \( z_j = x_j + iy_j \) and

\[
\begin{array}{ccc}
O(q) & \xrightarrow{\psi} & U(q) \\
\downarrow & & \downarrow \\
GL(q, \mathbb{R}) & \xrightarrow{\quad} & GL(q, \mathbb{C})
\end{array}
\]

where \( \psi(A) \) is the real matrix now viewed as a complex matrix. Given \( \xi \), an \( O(q) \)-bundle, we have \( \psi(q) \), a \( U(q) \)-bundle. Define

The Pontrjagin classes, \( p_i(\xi) \), are defined by

\[
p_i(\xi) = (-1)^i c_{2i}(\psi(\xi)) \in H^{4i}(X, \mathbb{Z}).
\]

The generalized Pontrjagin classes, \( P_i(\xi) \) and the generalized Pontrjagin polynomial, \( P(\xi)(t) \), are defined by

\[
P(\xi)(t) = c(\psi(\xi))(t), \quad \text{and} \quad P_j(\xi) = c_j(\psi(x)).
\]

(Observe: \( P_{2i}(\xi) = (-1)^i p_i(\xi) \).)

Now, \( \xi \) corresponds to map, \( X \longrightarrow BO(q) \). Then, for \( i \) even, \( P_{i/2}(\xi) \) is the pullback of something in \( H^i(BO(q), \mathbb{Z}) \). It is known that for \( i \equiv 2(4) \), the cohomology ring \( H^i(BO(q), \mathbb{Z}) \) is 2-torsion, so \( 2P_{odd}(\xi) = 0 \). So, with rational coefficients, we get

\[
P_{odd}(\xi) = 0 \quad \text{and} \quad P_{even}(\xi) = \pm P_{even/2}(\xi).
\]

We have the following properties:

1. \( P(\xi)(t) = 1 + \text{stuff} \).
2. \( f^*P(\xi)(t) = P(f^*\xi)(t) \), so \( f^*P_i(\xi) = P_i(f^*\xi) \).
3. Suppose \( \xi, \eta \) are bundle of rank \( q', q'' \), respectively, then

\[
P(\xi \oplus \eta)(t) = P(\xi)(t)P(\eta)(t)
\]

and if we set \( p(\xi)(t) = \sum_{j=0}^\infty p_j(\xi)t^{2j} \), then

\[
p(\xi \oplus \eta)(t) = p(\xi)(t)p(\eta)(t), \mod \text{ elements of order 2 in } H^\bullet(X, \mathbb{Z}).
\]

4. Suppose \( c(\psi(\xi))(t) \) has Chern roots \( \gamma_i \). Then, the polynomial \( \sum_{j=0}^\infty (-1)^j p_j(\xi)t^{2j} \) has roots \( \gamma_i^2 \); in fact,

\[
\sum_{j=0}^\infty (-1)^j p_j(\xi)t^{2j} = \left( \sum_{l} c_j(\xi)t^l \right) \left( \sum_{m} (-1)^m c_m(\xi)t^m \right).
\]

**Proposition 3.35** Say \( \xi \) is a \( U(q) \)-bundle and make \( \zeta(\xi) \), an \( O(2q) \)-bundle. Then

\[
\sum_{j=0}^\infty (-1)^j p_j(\zeta(\xi))t^{2j} = \left( \sum_{l} c_j(\xi)t^l \right) \left( \sum_{m} c_m(\xi^D)t^m \right).
\]

**Proof.** Consider the maps \( U(q) \hookrightarrow O(2q) \hookrightarrow U(2q) \). By linear algebra, if \( A \in U(q) \), its image in \( U(2q) \) by this map is

\[
\begin{pmatrix}
A & 0 \\
0 & \overline{A}
\end{pmatrix}
\]
after an automorphism of \( U(2q) \), which automorphism is independent of \( A \). By Skolem-Noether, the automorphism is of the form 

\[
H^{-1}(\psi \zeta(A))H,
\]

for some \( H \in \text{GL}(2q, \mathbb{C}) \). For an inner automorphism, the cohomology class of the vector bundle stays the same. Thus, this cohomology class is the class of

\[
\begin{pmatrix}
A & 0 \\
0 & \overline{A}
\end{pmatrix}.
\]

Now, we know the transition matrix of \( \xi^D \) is the transpose inverse of that for \( \xi \). But, \( A \) is unitary, so

\[
\overline{A} = (A^{-1})^\top = A^D
\]

and we deduce that \( \psi \zeta(A) \) has as transition matrix

\[
\begin{pmatrix}
A & 0 \\
0 & A^D
\end{pmatrix}.
\]

Consequently, the right hand side of our equation is

\[
\left( \sum_l c_j(\xi)^l \right) \left( \sum_m c_m(\xi^D)^m \right),
\]

as required. \( \square \)