Chapter 3

The Hirzebruch-Riemann-Roch Theorem

3.1 Line Bundles, Vector Bundles, Divisors

From now on, X will be a complex, irreducible, algebraic variety (not necessarily smooth). We have

(I) X with the Zariski topology and $\mathcal{O}_X$ = germs of algebraic functions. We will write X or X_{Zar}.

(II) X with the complex topology and $\mathcal{O}_X$ = germs of algebraic functions. We will write $X_{\mathbb{C}}$ for this.

(III) X with the complex topology and $\mathcal{O}_X$ = germs of holomorphic functions. We will write X^{an} for this.

(IV) X with the complex topology and $\mathcal{O}_X$ = germs of $\mathcal{C}^\infty$-functions. We will write $X_{\mathcal{C}^\infty}$ or X_{smooth} in this case.

Vector bundles come in four types: Locally trivial in the Z-topology (I); Locally trivial in the $\mathbb{C}$-topology (II, III, IV).

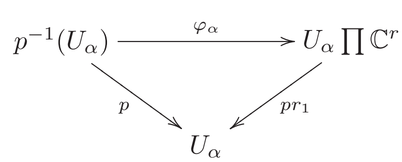
Recall that a *rank r vector bundle over X* is a space, E, together with a surjective map, $p\colon E \to X$, so that the following properties hold:

(1) There is some open covering, $\{U_\alpha \longrightarrow X\}$, of X and isomorphisms

$$\varphi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \prod \mathbb{C}^r \qquad (\textit{local triviality})$$

We also denote $p^{-1}(U_\alpha)$ by $E \restriction U_\alpha$.

(2) For every α, the following diagram commutes:



(3) Consider the diagram

$$\begin{array}{ccc}
p^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \coprod \mathbb{C}^r \\
\uparrow & & \uparrow \\
p^{-1}(U_\alpha \cap U_\beta) & \xrightarrow{\varphi_\alpha} & (U_\alpha \cap U_\beta) \coprod \mathbb{C}^r \\
\| & & \downarrow g_\alpha^\beta \\
p^{-1}(U_\beta \cap U_\alpha) & \xrightarrow{\varphi_\beta} & (U_\beta \cap U_\alpha) \coprod \mathbb{C}^r \\
\downarrow & & \downarrow \\
p^{-1}(U_\beta) & \xrightarrow{\varphi_\beta} & U_\beta \coprod \mathbb{C}^r
\end{array}$$

where $g_\alpha^\beta = \varphi_\beta \circ \varphi_\alpha^{-1} \upharpoonright p^{-1}(U_\alpha \cap U_\beta)$. Then,

$$g_\alpha^\beta \upharpoonright U_\alpha \cap U_\beta = \mathrm{id} \quad \text{and} \quad g_\alpha^\beta \upharpoonright \mathbb{C}^r \in \mathrm{GL}_r(\Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X))$$

and the functions g_α^β in the glueing give type II, III, IV.

On triple overlaps, we have

$$g_\beta^\gamma \circ g_\alpha^\beta = g_\alpha^\gamma \quad \text{and} \quad g_\beta^\alpha = (g_\alpha^\beta)^{-1}.$$

This means that the $\{g_\alpha^\beta\}$ form a 1-cocycle in $Z^1(\{U_\alpha \longrightarrow X\}, \mathbb{GL}_r)$. Here, we denote by $\mathbb{GL}_r(X)$, or simply $\mathbb{GL}_r$, the sheaf defined such that, for every open, $U \subseteq X$,

$$\Gamma(U, \mathbb{GL}_r(X)) = \mathrm{GL}_r(\Gamma(U, \mathcal{O}_X)),$$

the group of invertible linear maps of the free module $\Gamma(U, \mathcal{O}_X)^r \cong \Gamma(U, \mathcal{O}_X^r)$. When $r = 1$, we also denote the sheaf $\mathbb{GL}_1(X)$ by $\mathbb{G}_m$, or $\mathcal{O}_X^*$.

Say $\{\psi_\alpha\}$ is another trivialization. We may assume (by refining the covers) that $\{\varphi_\alpha\}$ and $\{\psi_\alpha\}$ use the same cover. Then, we have an isomorphism, $\sigma_\alpha\colon U_\alpha \coprod \mathbb{C}^r \to U_\alpha \coprod \mathbb{C}^r$:

$$\begin{array}{ccc}
 & & U_\alpha \coprod \mathbb{C}^r \\
 & \nearrow^{\varphi_\alpha} & \\
p^{-1}(U_\alpha) & & \downarrow \sigma_\alpha \\
 & \searrow_{\psi_\alpha} & \\
 & & U_\alpha \coprod \mathbb{C}^r
\end{array}$$

We see that $\{\sigma_\alpha\}$ is a 0-cochain in $C^0(\{U_\alpha \longrightarrow X\}, \mathbb{GL}_r)$. Let $\{h_\alpha^\beta\}$ be the glueing data from $\{\psi_\alpha\}$. Then, we have

$$\begin{aligned}
\varphi_\beta &= g_\alpha^\beta \circ \varphi_\alpha \\
\psi_\beta &= h_\alpha^\beta \circ \psi_\alpha \\
\psi_\alpha &= \sigma_\alpha \circ \varphi_\alpha.
\end{aligned}$$

From this, we deduce that $\sigma_\beta \circ \varphi_\beta = \psi_\beta = h_\alpha^\beta \circ \sigma_\alpha \circ \varphi_\alpha$, and then

$$\varphi_\beta = (\sigma_\beta^{-1} \circ h_\alpha^\beta \circ \sigma_\alpha) \circ \varphi_\alpha,$$

so

$$g_\alpha^\beta = \sigma_\beta^{-1} \circ h_\alpha^\beta \circ \sigma_\alpha.$$

This gives an equivalence relation, $\sim$, on $Z^1(\{U_\alpha \longrightarrow X\}, \mathbb{GL}_r)$. Set

$$H^1(\{U_\alpha \longrightarrow X\}, \mathbb{GL}_r) = Z^1/\sim.$$

This is a pointed set. If we pass to the right limit over covers by refinement and call the pointed set from the limit $\check{H}^1(X, \mathbb{GL}_r)$, we get

Theorem 3.1 *If X is an algebraic variety of one of the types $T =$ I, II, III, IV, then the set of isomorphism classes of rank r vector bundles, $\mathrm{Vect}_{T,r}(X)$, is in one-to-one correspondence with $\check{H}^1(X, \mathbb{GL}_r)$.*

Remarks:

(1) If F is some "object" and $\mathrm{Aut}(F) =$ is the group of automorphisms of F (in some catgeory), then an X-*torsor* for F is just an "object, E, over X", locally (on X) of the form $U \prod F$ and glued by the pairs (id, g), where $g \in \mathrm{Maps}(U \cap V, \mathrm{Aut}(F))$ on $U \cap V$. The theorem says: $\check{H}^1(X, \mathrm{Aut}(F))$ classifies the X-torsors for F.

Say $F = \mathbb{P}^r_{\mathbb{C}}$, we'll show that in the types I, II, III, $\mathrm{Aut}(F) = \mathbb{PGL}_r$, where

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{GL}_{r+1} \longrightarrow \mathbb{PGL}_r \longrightarrow 0 \quad \text{is exact.}$$

(2) Say $1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1$ is an exact sequence of sheaves of (not necessarily commutative) groups. Check that

$$1 \longrightarrow G'(X) \longrightarrow G(X) \longrightarrow G''(X) \xrightarrow{\delta_0} \check{H}^1(X, G') \longrightarrow \check{H}^1(X, G) \longrightarrow \check{H}^1(X, G'')$$

is an exact sequence of pointed sets. To compute $\delta_0(\sigma)$ where $\sigma \in G''(X)$, proceed as follows: Cover X by suitable U_α and pick $s_\alpha \in G(U_\alpha)$ mapping to $\sigma \upharpoonright U_\alpha$ in $G''(U_\alpha)$. Set

$$\delta_0(\sigma) = s_\alpha s_\beta^{-1} \quad \text{on } U_\alpha \cap U_\beta/\sim.$$

We find that $\delta_0(\sigma) \in \check{H}^1(X, G')$. When $G' \subseteq Z(G)$, we get the exact sequence

$$1 \longrightarrow G'(X) \longrightarrow G(X) \longrightarrow G''(X) \xrightarrow{\delta_0} \check{H}^1(X, G') \longrightarrow \check{H}^1(X, G) \longrightarrow \check{H}^1(X, G'') \xrightarrow{\delta_1} \check{H}^2(X, G')$$

(3) Apply the above to the sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{GL}_{r+1} \longrightarrow \mathbb{PGL}_r \longrightarrow 1.$$

If X is a projective variety, we get

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X^*) \longrightarrow \mathbb{GL}_{r+1}(\Gamma(X, \mathcal{O}_X)) \longrightarrow \mathbb{PGL}_r(\Gamma(X, \mathcal{O}_X)) \longrightarrow 0,$$

because $\Gamma(X, \mathcal{O}_X^*) = \mathbb{C}^*$ and $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$. Consequently, we also have

$$0 \longrightarrow \check{H}^1(X, \mathcal{O}_X^*) \longrightarrow \check{H}^1(X, \mathbb{GL}_{r+1}) \longrightarrow \check{H}^1(X, \mathbb{PGL}_r) \longrightarrow \check{H}^2(X, \mathcal{O}_X^*) = \mathrm{Br}(X),$$

where the last group, $\mathrm{Br}(X)$, is the cohomological *Brauer group* of X of type T. By our theorem, $\check{H}^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$ classifies type T line bundles, $\check{H}^1(X, \mathbb{GL}_{r+1})$ classifies type T rank $r+1$ vector bundles and $\check{H}^1(X, \mathbb{PGL}_r)$ classifies type T fibre bundles with fibre $\mathbb{P}^r_{\mathbb{C}}$ (all on X).

Let X and Y be two topological spaces and let $\pi\colon Y \to X$ be a surjective continuous map. Say we have sheaves of rings $\mathcal{O}_X$ on X and $\mathcal{O}_Y$ on Y; we have a homomorphism of sheaves of rings, $\mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_Y$. Then, each $\mathcal{O}_Y$-module (or $\mathcal{O}_Y$-algebra), $\mathcal{F}$, gives us the $\mathcal{O}_X$-module (or algebra), $\pi_*\mathcal{F}$ on X (and more generally, $R^q\pi_*\mathcal{F}$) as follows: For any open subset, $U \subseteq X$,

$$\Gamma(U, \pi_*\mathcal{F}) = \Gamma(\pi^{-1}(U), \mathcal{F}).$$

So, $\Gamma(\pi^{-1}(U), \mathcal{O}_Y)$ acts on $\Gamma(\pi^{-1}(U), \mathcal{F})$ and commutes to restriction to smaller opens. Consequently, $\pi_*\mathcal{F}$ is a $\pi_*\mathcal{O}_Y$-module (or algebra) and then $\mathcal{O}_X$ acts on it *via* $\mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_Y$. Recall also, that $R^q\pi_*\mathcal{F}$ is the sheaf on X generated by the presheaf

$$\Gamma(U, R^q\pi_*\mathcal{F}) = H^q(\pi^{-1}(U), \mathcal{F}).$$

If $\mathcal{F}$ is an algebra (not commutative), then only π_* and $R^1\pi_*$ are so-far defined.

Let's look at $\mathcal{F}$ and $\Gamma(Y, \mathcal{F}) = \Gamma(\pi^{-1}(X), \mathcal{F}) = \Gamma(X, \pi_*\mathcal{F})$. Observe that

$$\Gamma(Y, -) = \Gamma(X, -) \circ \pi_*.$$

So, if π_* maps an injective resolution to an exact sequence, then the usual homological algebra gives the spectral sequence of composed functors (Leray spectral sequence)

$$E_2^{p,q} = H^p(X, R^q\pi_*\mathcal{F}) \Longrightarrow H^\bullet(Y, \mathcal{F}).$$

We get the exact sequence of terms of low degree (also called edge sequence)

$$1 \longrightarrow H^1(X, \pi_*\mathcal{F}) \longrightarrow H^1(Y, \mathcal{F}) \longrightarrow H^0(X, R^1\pi_*\mathcal{F}) \xrightarrow{\delta_0} H^2(X, \pi_*\mathcal{F}) \longrightarrow H^2(Y, \mathcal{F}) \longrightarrow$$

In the non-commutative case, we get only

$$1 \longrightarrow H^1(X, \pi_*\mathcal{F}) \longrightarrow H^1(Y, \mathcal{F}) \longrightarrow H^0(X, R^1\pi_*\mathcal{F}).$$

Application: Let X be an algebraic variety with the Zariski topology, let $\mathcal{O}_X$ be the sheaf of germs of algebraic functions and let $Y = X_{\mathbb{C}}$ also with $\mathcal{O}_Y =$ the sheaf of germs of algebraic functions. The map $\pi\colon Y \to X$ is just the identity, which is continuous since the Zariski topology is coarser than the $\mathbb{C}$-topology. Take $\mathcal{F} =$ (possibly noncommutative) $\mathbb{GL}_r$.

Claim: $R^1\mathrm{id}_*\mathbb{GL}_r = (0)$, for all $r \geq 1$.

Proof. It suffices to prove that the stalks are zero. But these are the stalks of the corresponding presheaf

$$\varinjlim_{U \ni x} H^1_{\mathbb{C}}(U, \mathbb{GL}_r)$$

where U runs over Z-opens and H^1 is taken in the $\mathbb{C}$-topology. Pick $x \in X$ and some $\xi \in H^1_{\mathbb{C}}(U, \mathbb{GL}_r)$ for some Z-open, $U \ni x$. So, ξ consists of a vector bundle on U, locally trivial in the $\mathbb{C}$-topology. There is some open in the $\mathbb{C}$-topology, call it U_0, with $x \in U_0$ and $U_0 \subseteq U$ where $\xi \restriction U_0$ is trivial iff there exists some sections, $\sigma_1, \ldots, \sigma_r$, of ξ over U_0, and $\sigma_1, \ldots, \sigma_r$ are linearly independent everywhere on U_0. The σ_j are algebraic functions on U_0 to $\mathbb{C}^r$. Moreover, they are l.i. on U_0 iff $\sigma_1 \wedge \cdots \wedge \sigma_r$ is everywhere nonzero on U_0. But, $\sigma_1 \wedge \cdots \wedge \sigma_r$ is an algebraic function and its zero set is a Z-closed subset in X. So, its complement, V, is Z-open and $x \in U_0 \subseteq V \cap U$. It follows that $\xi \restriction V \cap U$ is trivial (since the σ_j are l.i. everywhere); so, ξ indeed becomes trivial on a Z-open, as required. $\square$

Apply our exact sequence and get

Theorem 3.2 *(Comparison Theorem) If X is an algebraic variety, then the canonical map*

$$\mathrm{Vect}^r_{\mathrm{Zar}}(X) \cong \check{H}^1(X_{\mathrm{Zar}}, \mathbb{GL}_r) \longrightarrow \check{H}^1(X_{\mathbb{C}}, \mathbb{GL}_r) \cong \mathrm{Vect}^r_{\mathbb{C}}(X)$$

is an isomorphism for all $r \geq 1$ (i.e., a bijection of pointed sets).

Thus, to give a rank r *algebraic* vector bundle in the $\mathbb{C}$-topology is the same as giving a rank r *algebraic* vector bundle in the Zariski topology.

If we use $\mathcal{O}_X$ = holomorphic (analytic) functions, then for many X, we get only an injection $\mathrm{Vect}^r_{\mathrm{Zar}}(X) \hookrightarrow \mathrm{Vect}^r_{\mathbb{C}}(X)$.

Connection with the geometry inside X:

First, assume X is smooth and irreducible (thus, connected). Let V be an irreducible subvariety of codimension 1. We know from Chapter 1 that locally on some open, U, there is some $f \in \Gamma(U, \mathcal{O}_X) = \mathcal{O}_U$ such that $f = 0$ cuts out V in U. Furthermore, f is analytic if V is, algebraic if V is. Form the free abelian group on the V's (we can also look at "locally finite" $\mathbb{Z}$-combinations in the analytic case); call these objects *Weil divisors* (W-divisors), and denote the corresponding group, $\mathrm{WDiv}(X)$.

A divisor $D \in \mathrm{WDiv}(X)$ is *effective* if $D = \sum_\alpha a_\alpha V_\alpha$, with $a_\alpha \geq 0$ for all α. This gives a cone inside $\mathrm{WDiv}(X)$ and partially orders $\mathrm{WDiv}(X)$.

Say g is a holomorphic (or algebraic) function near x. If V passes through x, in $\mathcal{O}_{X,x}$–which is a UFD (by Zariski) we can write

$$g = f^a \widetilde{g}, \quad \text{where } (\widetilde{g}, f) = 1.$$

(The equation $f = 0$ defines V near x so f is a prime of $\mathcal{O}_{X,x}$.) Notice that if $\mathfrak{p} = (f)$ in $\Gamma(U, \mathcal{O}_X) = \mathcal{O}_U$, then $g = f^a\widetilde{g}$ iff $g \in \mathfrak{p}^a$ and $g \notin \mathfrak{p}^{a+1}$ iff $g \in \mathfrak{p}^a(\mathcal{O}_U)_\mathfrak{p}$ and $g \notin \mathfrak{p}^{a+1}(\mathcal{O}_U)_\mathfrak{p}$. The ring $(\mathcal{O}_U)_\mathfrak{p}$ is a local ring of dimension 1 and is regular as X is a manifold (can be regular even if X is singular). Therefore, a is independent of x. The number a is by definition the *order of vanishing of g along V*, denoted $\mathrm{ord}_V(g)$. If g is a meromorphic function near x, we write $g = g_1/g_2$ locally in $(\mathcal{O}_U)_\mathfrak{p}$, with $(g_1, g_2) = 1$ and set

$$\mathrm{ord}_V(g) = \mathrm{ord}_V(g_1) - \mathrm{ord}_V(g_2).$$

We say that g has a zero of order a along V iff $\mathrm{ord}_V(g) = a > 0$ and a pole of order a iff $\mathrm{ord}_V(g) = -a < 0$. If $g \in \Gamma(X, \mathcal{M}\mathrm{er}(X)^*)$, set

$$(g) = \sum_{V \in \mathrm{WDiv}(X)} \mathrm{ord}_V(g) \cdot V.$$

Claim. The above sum is finite, under suitable conditions:

(a) We use algebraic functions.

(b) We use holomorphic functions and restrict X (DX).

Look at g, then $1/g$ vanishes on a Z-closed, W_0. Look at $X - W_0$. Now, $X - W_0$ is Z-open so it is a variety and $g \restriction X - W_0$ is holomorphic. Look at $V \subseteq X$ and $\mathrm{ord}_V(g) = a \neq 0$, i.e., $V \cap U \neq \emptyset$. Thus, $(g) = \mathfrak{p}^a$ in $(\mathcal{O}_U)_{\mathfrak{p}}$, which yields $(g) \subseteq \mathfrak{p}$ and then $V \cap (X - W_0) = V(\mathfrak{p}) \subseteq V((g))$. But, $V(g)$ is a union of irreducible components (algebraic case) and V is codimension 1, so V is equal to one of these components. Therefore, there are only finitely many V's arising from $X - W_0$.

The function $1/g$ vanishes on W_0, so write W_0 as a union of irreducible components. Again, there are only finitely many V arising from W_0. So, altogether, there are only finitely many V's associated with g where g has a zero or a pole. We call $(g) \in \mathrm{WDiv}(X)$ a *principal divisor*. Given any two divisors $D, E \in \mathrm{WDiv}(X)$, we define *linear* (or *rational*) *equivalence* by

$$D \sim E \quad \text{iff} \quad (\exists g \in \mathcal{M}\mathrm{er}(X))(D - E = (g)).$$

The equivalence classes of divisors modulo $\sim$ is the *Weil class group*, $\mathrm{WCl}(X)$.

Remark: All goes through for any X (of our sort) for which, for all primes, $\mathfrak{p}$, of height 1, the ring $(\mathcal{O}_U)_{\mathfrak{p}}$ is a regular local ring (of dimension 1, i.e., a P.I.D.) This is, in general, hard to check (but, OK if X is normal).

Cartier had the idea to use a general X but consider only the V's given locally as $f = 0$. For every open, $U \subseteq X$, consider $A_U = \Gamma(U, \mathcal{O}_X)$. Let S_U be the set of all non-zero divisors of A_U, a multiplicative set. We get a presheaf of rings, $U \mapsto S_U^{-1} A_U$, and the corresponding sheaf, $\mathcal{M}\mathrm{er}(X)$, is the *total fraction sheaf of* $\mathcal{O}_X$. We have an embedding $\mathcal{O}_X \longrightarrow \mathcal{M}\mathrm{er}(X)$ and we let $\mathcal{M}\mathrm{er}(X)^*$ be the sheaf of invertible elements of $\mathcal{M}\mathrm{er}(X)$. Then, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}\mathrm{er}(X)^* \longrightarrow \mathcal{D}_X \longrightarrow 0,$$

where $\mathcal{D}_X$ is the sheaf cokernel.

We claim that if we define $\mathcal{D}_X = \mathrm{Coker}\,(\mathcal{O}_X^* \longrightarrow \mathcal{M}\mathrm{er}(X)^*)$ in the $\mathbb{C}$-topology, then it is also the kernel in the Z-topology.

Take $\sigma \in \Gamma(U, \mathcal{D}_X)$ and replace X by U, so that we may assume that $U = X$. Then, as σ is liftable locally in the $\mathbb{C}$-topology, there exist a $\mathbb{C}$-open cover, U_α and some $\sigma_\alpha \in \Gamma(U, \mathcal{M}\mathrm{er}(X)^*)$ so that $\sigma_\alpha \mapsto \sigma \restriction U_\alpha$. Make the U_α small enough so that $\sigma_\alpha = f_\alpha/g_\alpha$, where f_α, g_α are holomorphic. It follows that σ_α is defined on a Z-open, $\widetilde{U}_\alpha \supseteq U_\alpha$. Look at $\widetilde{U}_\alpha \cap \widetilde{U}_\beta \supseteq U_\alpha \cap U_\beta$. We know $\sigma_\alpha/\sigma_\beta$ is invertible holomorphic on $U_\alpha \cap U_\beta$ and so,

$$\frac{\sigma_\alpha}{\sigma_\beta} \cdot \frac{\sigma_\beta}{\sigma_\alpha} \equiv 1 \quad \text{on } U_\alpha \cap U_\beta.$$

It follows that $\sigma_\alpha/\sigma_\beta$ is invertible on $\widetilde{U}_\alpha \cap \widetilde{U}_\beta$ and then, restricting slightly further we get a Z-open cover and σ_α's on it lifting σ. $\square$

Definition 3.1 A *Cartier divisor* (for short, C-divisor) on X is a global section of $\mathcal{D}_X$. Two Cartier divisors, σ, τ are *rationally equivalent*, denoted $\sigma \sim \tau$, iff $\sigma/\tau \in \Gamma(X, \mathcal{M}\mathrm{er}(X)^*)$. Of course, this means there is a $\mathbb{C}$ or Z-open cover, U_α, of X and some $\sigma_\alpha, \tau_\alpha \in \Gamma(U_\alpha, \mathcal{M}\mathrm{er}(X)^*)$ with $\sigma_\alpha/\tau_\alpha$ invertible holomorphic on $U_\alpha \cap U_\beta$. The group of Cartier divisors is denoted by $\mathrm{CDiv}(X)$ and the corresponding group of equivalence classes modulo rational equivalence by $\mathrm{Cl}(X)$ (the *class group*).

The idea is that if $\{(U_\alpha, \sigma_\alpha)\}_\alpha$ defines a C-divisor, then we look on U_α at

$$\sigma_\alpha^0 - \sigma_\alpha^\infty = (\text{locus}\, \sigma_\alpha = 0) - (\text{locus}\, \frac{1}{\sigma_\alpha} = 0).$$

When we have the situation where WDiv(X) exists, then the map

$$\{(U_\alpha, \sigma_\alpha)\}_\alpha \mapsto \{\sigma_\alpha^0 - \sigma_\alpha^\infty\}$$

takes C-divisors to Weil divisors. Say σ_α and σ'_α are both liftings of the same σ, then on U_α we have

$$\sigma'_\alpha = \sigma_\alpha g_\alpha \quad \text{where } g_\alpha \in \Gamma(X, \mathcal{O}_X^*).$$

Therefore,

$$\sigma_\alpha'^0 - \sigma_\alpha'^\infty = \sigma_\alpha^0 - \sigma_\alpha^\infty$$

and the Weil divisors are the same (provided they make sense). If $\sigma, \tau \in \mathrm{CDiv}(X)$ and $\sigma \sim \tau$, then there is a global meromorphic function, f, with $\sigma = f\tau$. Consequently

$$\sigma_\alpha^0 - \sigma_\alpha^\infty = (f)^0 - (f)^\infty + \tau_\alpha^0 - \tau_\alpha^\infty,$$

which shows that the corresponding Weil divisors are linearly equivalent. We get

Proposition 3.3 *If X is an algebraic variety, the sheaf $\mathcal{D}_X$ is the same in either the Zariski or $\mathbb{C}$-topology and if X allows Weil divisors (non-singular in codimension 1), then the map* $\mathrm{CDiv}(X) \longrightarrow \mathrm{WDiv}(X)$ *given by $\sigma \mapsto \sigma_\alpha^0 - \sigma_\alpha^\infty$ is well-defined and we get a commutative diagram with injective rows*

$$\begin{array}{ccc} \mathrm{CDiv}(X) & \hookrightarrow & \mathrm{WDiv}(X) \\ \downarrow & & \downarrow \\ \mathrm{Cl}(X) & \hookrightarrow & \mathrm{WCl}(X). \end{array}$$

If X is a manifold then our rows are isomorphisms.

Proof. We only need to prove the last statement. Pick $D = \sum_\alpha n_\alpha V_\alpha$, a Weil divisor, where each V_α is irreducible of codimension 1. As X is manifold, each V_α is given by $f_\alpha = 0$ on a small enough open, U; take for $\sigma \restriction U$, the product $\prod_\alpha f_\alpha^{n_\alpha}$ and this gives our C-divisor.

We can use the following in some computations.

Proposition 3.4 *Assume X is an algebraic variety and $Y \hookrightarrow X$ is a subvariety. Write $U = X - Y$, then the maps*

$$\sigma \in \mathrm{CDiv}(X) \mapsto \sigma \restriction U \in \mathrm{CDiv}(U),$$

resp.

$$\sum_\alpha n_\alpha V_\alpha \in \mathrm{WDiv}(X) \mapsto \sum_\alpha n_\alpha (V_\alpha \cap U) \in \mathrm{WDiv}(U)$$

are surjections from CDiv(X) *or* WDiv(X) *to the corresponding object in U. If $\mathrm{codim}_X(Y) \geq 2$, then our maps are isomorphisms. If $\mathrm{codim}_X(Y) = 1$ and Y is irreducible and locally principal, then the sequences*

$$\mathbb{Z} \longrightarrow \mathrm{CDiv}(X) \longrightarrow \mathrm{CDiv}(U) \longrightarrow 0 \quad \text{and} \quad \mathbb{Z} \longrightarrow \mathrm{WDiv}(X) \longrightarrow \mathrm{WDiv}(U) \longrightarrow 0$$

are exact (where the left hand map is $n \mapsto nY$).

Proof. The maps clearly exist. Given an object in U, take its closure in X, then restriction to U gives back the object. For Y of codimension at least 2, all procedures are insensitive to such Y, so we don't change anything by removing Y. A divisor $\xi \in \mathrm{CDiv}(X)$ (or WDiv(X)) goes to zero iff its "support" is contained in Y. But, Y is irreducible and so are the components of ξ. Therefore, $\xi = nY$, for some n. □

Recall that line bundles on X are in one-to-one correspondence with invertible sheaves, that is, rank 1, locally free $\mathcal{O}_X$-modules. If L is a line bundle, we associate to it, $\mathcal{O}_X(L)$, the sheaf of sections (algebraic, holomorphic, C^∞) of L.

In the other direction, if $\mathcal{L}$ is a rank 1 locally free $\mathcal{O}_X$-module, first make $\mathcal{L}^D$ and the $\mathcal{O}_X$-algebra, $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^D)$, where

$$\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^D) = \coprod_{n\geq 0} (\mathcal{L}^D)^{\otimes n}/(a\otimes b - b\otimes a).$$

On a small enough open, U,

$$\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^D) \upharpoonright U = \mathcal{O}_U[T],$$

so we form $\mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^D) \upharpoonright U) \cong U \coprod \mathbb{C}^1$, and glue using the data for $\mathcal{L}^D$. We get the line bundle, $\mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^D))$.

Given a Cartier divisor, $D = \{(U_\alpha, f_\alpha)\}$, we make the submodule, $\mathcal{O}_X(D)$, of $\mathcal{M}\mathrm{er}(X)$ given on U_α by

$$\mathcal{O}_X(D) \upharpoonright U_\alpha = \frac{1}{f_\alpha}\, \mathcal{O}_X \upharpoonright U_\alpha \subseteq \mathcal{M}\mathrm{er}(X) \upharpoonright U_\alpha.$$

If $\{(U_\alpha, g_\alpha)\}$ also defines D (we may assume the covers are the same by refining the covers if necessary), then there exist $h_\alpha \in \Gamma(U_\alpha, \mathcal{M}\mathrm{er}(X)^*)$, with

$$f_\alpha h_\alpha = g_\alpha.$$

Then, the map $\xi \mapsto \frac{1}{h_\alpha}\,\xi$ takes $\frac{1}{f_\alpha}$ to $\frac{1}{g_\alpha}$; so, $\frac{1}{f_\alpha}$ and $\frac{1}{g_\alpha}$ generate the same submodule of $\mathcal{M}\mathrm{er}(X) \upharpoonright U_\alpha$. On $U_\alpha \cap U_\beta$, we have

$$\frac{f_\alpha}{f_\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X^*),$$

and as

$$\frac{f_\alpha}{f_\beta} \cdot \frac{1}{f_\alpha} = \frac{1}{f_\beta},$$

we get

$$\frac{1}{f_\alpha}\, \mathcal{O}_{U_\alpha} \upharpoonright U_\alpha \cap U_\beta = \frac{1}{f_\beta}\, \mathcal{O}_{U_\beta} \upharpoonright U_\alpha \cap U_\beta.$$

Consequently, our modules agree on the overlaps and so, $\mathcal{O}_X(D)$ is a rank 1, locally free subsheaf of $\mathcal{M}\mathrm{er}(X)$.

Say D and E are Cartier divisors and $D \sim E$. So, there is a global meromorphic function, $f \in \Gamma(X, \mathcal{M}\mathrm{er}(X)^*)$ and on U_α,

$$f_\alpha f = g_\alpha.$$

Then, the map $\xi \mapsto \frac{1}{f}\,\xi$ is an $\mathcal{O}_X$-isomorphism

$$\mathcal{O}_X(D) \cong \mathcal{O}_X(E).$$

Therefore, we get a map from $\mathrm{Cl}(X)$ to the invertible submodules of $\mathcal{M}\mathrm{er}(X)$.

Given an invertible submodule, $\mathcal{L}$, of $\mathcal{M}\mathrm{er}(X)$, locally, on U, we have $\mathcal{L} \upharpoonright U = \frac{1}{f_U}\, \mathcal{O}_U \subseteq \mathcal{M}\mathrm{er}(X) \upharpoonright U$. Thus, $\{(U, f_U)\}$ gives a C-divisor describing $\mathcal{L}$. Suppose $\mathcal{L}$ and $\mathcal{M}$ are two invertible submodules of $\mathcal{M}\mathrm{er}(X)$ and $\mathcal{L} \cong \mathcal{M}$; say $\varphi\colon \mathcal{L} \to \mathcal{M}$ is an $\mathcal{O}_X$-isomorphism. Locally (possibly after refining covers), on U_α, we have

$$\mathcal{L} \upharpoonright U_\alpha \cong \frac{1}{f_\alpha}\, \mathcal{O}_{U_\alpha} \quad \text{and} \quad \mathcal{M} \upharpoonright U_\alpha \cong \frac{1}{g_\alpha}\, \mathcal{O}_{U_\alpha}.$$

So, $\varphi\colon \mathcal{L} \upharpoonright U_\alpha \to \mathcal{M} \upharpoonright U_\alpha$ is given by some τ_α such that

$$\varphi\Big(\frac{1}{f_\alpha}\Big) = \tau_\alpha \frac{1}{g_\alpha}.$$

Consequently, $\varphi_\alpha \restriction U_\alpha$ is multiplication by τ_α and $\varphi_\beta \restriction U_\beta$ is multiplication by τ_β. Yet $\varphi_\alpha \restriction U_\alpha$ and $\varphi_\beta \restriction U_\beta$ agree on $U_\alpha \cap U_\beta$, so $\tau_\alpha = \tau_\beta$ on $U_\alpha \cap U_\beta$. This shows that the τ_α patch and define a global τ such that

$$\tau \restriction U_\alpha = \tau_\alpha = g_\alpha \varphi\Big(\frac{1}{f_\alpha}\Big) \quad \text{and} \quad \tau \restriction U_\beta = \tau_\beta = g_\beta \varphi\Big(\frac{1}{f_\beta}\Big)$$

on overlaps. Therefore, we can define a global Φ *via*

$$\Phi = g_\alpha \varphi\Big(\frac{1}{f_\alpha}\Big) \in \mathcal{M}\mathrm{er}(X),$$

and we find $\xi \mapsto \frac{1}{\Phi}\,\xi$ gives the desired isomorphism.

Theorem 3.5 *If X is an algebraic variety (or holomorphic or C^∞ variety) then there is a canonical map,* $\mathrm{CDiv}(X) \longrightarrow$ *rank* 1*, locally free submodules of* $\mathcal{M}\mathrm{er}(X)$*. It is surjective. Two Cartier divisors D and E are rationally equivalent iff the corresponding invertible sheaves $\mathcal{O}_X(D)$ and $\mathcal{O}_X(E)$ are (abstractly) isomorphic. Hence, there is an injection of the class group,* $\mathrm{Cl}(X)$ *into the group of rank* 1*, locally free $\mathcal{O}_X$-submodules of* $\mathcal{M}\mathrm{er}(X)$ *modulo isomorphism. If X is an algebraic variety and we use algebraic functions and if X is irreducible, then every rank* 1*, locally free $\mathcal{O}_X$-module is an $\mathcal{O}_X(D)$. The map $D \mapsto \mathcal{O}_X(D)$ is just the connecting homomorphism in the cohomology sequence,*

$$H^0(X, \mathcal{D}_X) \stackrel{\delta}{\longrightarrow} H^1(X, \mathcal{O}_X^*).$$

Proof. Only the last statement needs proof. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}\mathrm{er}(X)^* \longrightarrow \mathcal{D}_X \longrightarrow 0.$$

Apply cohomology (we may use the Z-topology, by the comparison theorem): We get

$$\Gamma(X, \mathcal{M}\mathrm{er}(X)^*) \longrightarrow \mathrm{CDiv}(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow H^1(X, \mathcal{M}\mathrm{er}(X)^*).$$

But, X is irreducible and in the Z-topology $\mathcal{M}\mathrm{er}(X)$ is a constant sheaf. As constant sheaves are flasque, $\mathcal{M}\mathrm{er}(X)$ is flasque, which implies that $H^1(X, \mathcal{M}\mathrm{er}(X)^*) = (0)$. Note that this shows that there is a surjection $\mathrm{CDiv}(X) \longrightarrow \mathrm{Pic}(X)$.

How is δ defined? Given $D \in H^0(X, \mathcal{D}_X) = \mathrm{CDiv}(X)$, if $\{(U_\alpha, f_\alpha)\}$ is a local lifting of D, the map δ associates the cohomology class $[f_\beta/f_\alpha]$, where f_β/f_α is viewed as a 1-cocycle on $\mathcal{O}_X^*$. On the other hand, when we go through the construction of $\mathcal{O}_X(D)$, we have the isomorphisms

$$\mathcal{O}_X(D) \restriction U_\alpha = \frac{1}{f_\alpha}\,\mathcal{O}_{U_\alpha} \cong \mathcal{O}_{U_\alpha} \supseteq \mathcal{O}_{U_\alpha} \cap \mathcal{O}_{U_\beta} \quad (\text{mult. by } f_\alpha)$$

and

$$\mathcal{O}_X(D) \restriction U_\beta = \frac{1}{f_\beta}\,\mathcal{O}_{U_\beta} \cong \mathcal{O}_{U_\beta} \supseteq \mathcal{O}_{U_\alpha} \cap \mathcal{O}_{U_\beta} \quad (\text{mult. by } f_\beta)$$

and we see that the transition function, g_α^β, on $\mathcal{O}_{U_\alpha} \cap \mathcal{O}_{U_\beta}$ is nonother that multiplication by f_β/f_α. But then, both $\mathcal{O}_X(D)$ and $\delta(D)$ are line bundles defined by the same transition functions (multiplication by f_β/f_α) and $\delta(D) = \mathcal{O}_X(D)$. $\square$

Say $D = \{(U_\alpha, f_\alpha)\}$ is a Cartier divisor on X. Then, the intuition is that the geometric object associated to D is

$$(\text{zeros of } f_\alpha - \text{poles of } f_\alpha) \quad \text{on} \quad U_\alpha.$$

This leads to saying that the Cartier divisor D is an *effective* divisor iff each f_α is holomorphic on U_α. In this case, $f_\alpha = 0$ gives on U_α a locally principal, codimension 1 subvariety and conversely. Now each subvariety, V, has a corresponding sheaf of ideals, $\mathfrak{I}_V$. If V is locally principal, given by the f_α's, then $\mathfrak{I}_V \restriction U_\alpha = f_\alpha \mathcal{O}_X \restriction U_\alpha$. But, $f_\alpha \mathcal{O}_X \restriction U_\alpha$ is exactly $\mathcal{O}_X(-D)$ on U_α if $D = \{(U_\alpha, f_\alpha)\}$. Hence, $\mathfrak{I}_X = \mathcal{O}_X(-D)$. We get

Proposition 3.6 *If X is an algebraic variety, then the effective Cartier divisors on X are in one-to-one correspondence with the locally principal codimension 1 subvarieties of X. If V is one of the latter and if D corresponds to V, then the ideal cutting out V is exactly $\mathcal{O}_X(-D)$. Hence*

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_V \longrightarrow 0 \quad \textit{is exact.}$$

What are the global sections of $\mathcal{O}_X(D)$?

Such sections are holomorphic maps $\sigma\colon X \to \mathcal{O}_X(D)$ such that $\pi \circ \sigma = id$ (where $\pi\colon \mathcal{O}_X(D) \to X$ is the canonical projection associated with the bundle $\mathcal{O}_X(D)$). If D is given by $\{(U_\alpha, f_\alpha)\}$, the diagram

$$\begin{array}{ccccc}
\mathcal{O}_X(D) \restriction U_\alpha & = & f_\alpha \mathcal{O}_X \restriction U_\alpha & \xrightarrow{\times f_\alpha} & \mathcal{O}_X \restriction U_\alpha \\
\uparrow & & & & \uparrow \\
\mathcal{O}_X(D) \restriction U_\alpha \cap U_\beta & & & & \mathcal{O}_X \restriction U_\alpha \cap U_\beta \\
\| & & & & \downarrow g_\alpha^\beta \\
\mathcal{O}_X(D) \restriction U_\beta \cap U_\alpha & & & & \mathcal{O}_X \restriction U_\beta \cap U_\alpha \\
\downarrow & & & & \downarrow \\
\mathcal{O}_X(D) \restriction U_\beta & = & f_\beta \mathcal{O}_X \restriction U_\beta & \xrightarrow{\times f_\beta} & \mathcal{O}_X \restriction U_\beta
\end{array}$$

implies that

$$\sigma_\alpha = f_\alpha \sigma\colon U_\alpha \longrightarrow \mathcal{O}_X \restriction U_\alpha \quad \text{and} \quad \sigma_\beta = f_\beta \sigma\colon U_\beta \longrightarrow \mathcal{O}_X \restriction U_\beta.$$

However, we need

$$\sigma_\beta = g_\alpha^\beta \sigma_\alpha,$$

which means that a global section, σ, is a family of local holomorphic functions, σ_α, so that $\sigma_\beta = g_\alpha^\beta \sigma_\alpha$.But, as $g_\alpha^\beta = f_\beta / f_\alpha$, we get

$$\frac{\sigma_\alpha}{f_\alpha} = \frac{\sigma_\beta}{f_\beta} \quad \text{on } U_\alpha \cap U_\beta.$$

Therefore, the meromorphic functions, σ_α / f_α, patch and give a global meromorphic function, F_σ. We have

$$f_\alpha (F_\sigma \restriction U_\alpha) = \sigma_\alpha$$

a holomorphic function. Therefore, $(f_\alpha \restriction U_\alpha) + (F_\sigma \restriction U_\alpha) \geq 0$, for all α and as the pieces patch, we get

$$D + (F_\sigma) \geq 0.$$

Conversely, say $F \in \Gamma(X, \mathcal{M}\mathrm{er}(X))$ and $D + (F) \geq 0$. Locally on U_α, we have $D = \{(U_\alpha, f_\alpha)\}$ and $(f_\alpha F) \geq 0$. If we set $\sigma_\alpha = f_\alpha F$, we get a holomorphic function on U_α. But,

$$g_\alpha^\beta \sigma_\alpha = \frac{f_\beta}{f_\alpha} f_\alpha F = f_\beta F = \sigma_\beta,$$

so the σ_α's give a global section of $\mathcal{O}_X(D)$.

Proposition 3.7 *If X is an algebraic variety, then*

$$H^0(X, \mathcal{O}_X(D)) = \{0\} \cup \{F \in \Gamma(X, \mathcal{M}\mathrm{er}(X)) \mid (F) + D \geq 0\}.$$

in particular,

$$|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D))) = \{E \mid E \geq 0 \quad \textit{and} \quad E \sim D\},$$

the complete linear system of D, is naturally a projective space and $H^0(X, \mathcal{O}_X(D)) \neq (0)$ iff there is some Cartier divisor, $E \geq 0$, and $E \sim D$.

Recall that an $\mathcal{O}_X$-module, $\mathcal{F}$, is a Z-QC (resp. $\mathbb{C}$-QC, here QC = *quasi-coherent*) iff everywhere locally, i.e., for small (Z, resp. $\mathbb{C}$) open, U, there exist sets $I(U)$ and $J(U)$ and some exact sequence

$$(\mathcal{O}_X \restriction U)^{I(U)} \xrightarrow{\varphi_U} (\mathcal{O}_X \restriction U)^{J(U)} \longrightarrow \mathcal{F} \restriction U \longrightarrow 0.$$

Since $\mathcal{O}_X$ is coherent (usual fact that the rings $\Gamma(U_\alpha, \mathcal{O}_X) = A_\alpha$, for U_α open affine, are noetherian) or Oka's theorem in the analytic case, a sheaf, $\mathcal{F}$, is *coherent* iff it is QC and finitely generated iff it is finitely presented, i.e., everywhere locally,

$$(\mathcal{O}_X \restriction U)^q \xrightarrow{\varphi_U} (\mathcal{O}_X \restriction U)^q \longrightarrow \mathcal{F} \restriction U \longrightarrow 0 \quad \text{is exact.} \qquad (\dagger)$$

(Here, p, q are functions of U and finite).

In the case of the Zariski topology, $\mathcal{F}$ is QC iff for every affine open, U, the sheaf $\mathcal{F} \restriction U$ has the form $\widetilde{M}$, for some $\Gamma(U, \mathcal{O}_X)$-module, M. The sheaf $\widetilde{M}$ is defined so that, for every open $W \subseteq U$,

$$\Gamma(W, \widetilde{M}) = \left\{ \sigma\colon W \longrightarrow \bigcup_{\xi \in W} M_\xi \;\middle|\; \begin{array}{l} (1)\ \sigma(\xi) \in M_\xi \\ (2)\ (\forall \xi \in W)(\exists V \text{ (open)} \subseteq W, \exists f \in M, \exists g \in \Gamma(V, \mathcal{O}_X))(g \neq 0 \text{ on } V) \\ (3)\ (\forall y \in V) \left(\sigma(y) = \text{image}\left(\frac{f}{g}\right) \text{ in } M_y\right). \end{array} \right\}$$

Proposition 3.8 *Say X is an algebraic variety and $\mathcal{F}$ is an $\mathcal{O}_X$-module. Then, $\mathcal{F}$ is Z-coherent iff $\mathcal{F}$ is $\mathbb{C}$-coherent.*

Proof. Say $\mathcal{F}$ is Z-coherent, then locally Z, the sheaf $\mathcal{F}$ satisfies ($\dagger$). But, every Z-open is also $\mathbb{C}$-open, so $\mathcal{F}$ is $\mathbb{C}$-coherent.

Now, assume $\mathcal{F}$ is $\mathbb{C}$-coherent, then locally $\mathbb{C}$, we have ($\dagger$), where U is $\mathbb{C}$-open. The map φ_U is given by a $p \times q$ matrix of holomorphic functions on U. Each is algebraically defined on a Z-open containing U. The intersection of these finitely many Z-opens is a Z-open, $\widetilde{U}$ and $\widetilde{U} \supseteq U$. So, we get a sheaf

$$\widetilde{\mathcal{F}} \restriction \widetilde{U} = \text{Coker}\,((\mathcal{O}_X \restriction \widetilde{U})^q \longrightarrow (\mathcal{O}_X \restriction \widetilde{U})^p).$$

The sheaves $\widetilde{\mathcal{F}} \restriction \widetilde{U}$ patch (easy–DX) and we get a sheaf, $\widetilde{\mathcal{F}}$. On U, the sheaf $\widetilde{\mathcal{F}}$ is equal to $\mathcal{F}$, so $\widetilde{\mathcal{F}} = \mathcal{F}$. $\square$

We have the continuous map $X_{\mathbb{C}} \xrightarrow{\text{id}} X_{\text{Zar}}$ and we get (see Homework)

Theorem 3.9 *(Comparison Theorem for cohomology of coherent sheaves) If X is an algebraic variety and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then the canonical map*

$$H^q(X_{\text{Zar}}, \mathcal{F}) \longrightarrow H^q(X_{\mathbb{C}}, \mathcal{F})$$

is an isomorphism for all $q \geq 0$.

Say V is a closed subvariety of $X = \mathbb{P}^n_{\mathbb{C}}$. Then, V is given by a coherent sheaf of ideals of $\mathcal{O}_X$, say $\mathfrak{I}_V$ and we have the exact sequence

$$0 \longrightarrow \mathfrak{I}_V \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_V \longrightarrow 0,$$

where $\mathcal{O}_V$ is the sheaf of germs of holomorphic functions on V and has support on V. If V is a hypersurface, then V is given by $f = 0$, where f is a form of degree d. If D is a Cartier divisor of f, then $\mathfrak{I}_V = \mathcal{O}_X(-D)$. Similarly another hypersurface, W, is given by $g = 0$ and if $\deg(f) = \deg(g)$, then f/g is a global meromorphic function on $\mathbb{P}^n$. Therefore, $(f/g) = V - W$, which implies $V \sim W$. In particular, $g = (\text{linear form})^d$ and so, $V \sim dH$, where H is a hyperplane. Therefore the set of effective Cartier disisors of $\mathbb{P}^n$ is in one-to-one correspondence with forms of varying degrees $d \geq 0$ and

$$\text{Cl}(\mathbb{P}^n) \cong \mathbb{Z},$$

namely, $V \mapsto \deg(V) = \delta(V)$ (our old notation) $= (\deg(f)) \cdot H \in H^2(\mathbb{P}^n, \mathbb{Z})$. We deduce,

$$\mathrm{Pic}^0(\mathbb{P}^n) = (0) \quad \text{and} \quad \mathrm{Pic}(\mathbb{P}^n) = \mathrm{Cl}(\mathbb{P}^n) = \mathbb{Z}.$$

Say V is a closed subvariety of $\mathbb{P}^n_{\mathbb{C}}$, then we have the exact sequence

$$0 \longrightarrow \mathfrak{I}_V \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_V \longrightarrow 0.$$

Twist with $\mathcal{O}_{\mathbb{P}^n}(d)$, i.e., tensor with $\mathcal{O}_{\mathbb{P}^n}(d)$ (Recall that by definition, $\mathcal{O}_{\mathbb{P}^n}(d) = \mathcal{O}_{\mathbb{P}^n}(dH)$, where H is a hyperplane). We get the exact sequence

$$0 \longrightarrow \mathfrak{I}_V(d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow \mathcal{O}_V(d) \longrightarrow 0$$

(with $\mathfrak{I}_V(d) = \mathfrak{I}_V \otimes \mathcal{O}_{\mathbb{P}^n}(d)$ and $\mathcal{O}_V(d) = \mathcal{O}_V \otimes \mathcal{O}_{\mathbb{P}^n}(d)$) and we can apply cohomology, to get

$$0 \longrightarrow H^0(\mathbb{P}^n, \mathfrak{I}_V(d)) \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow H^0(V, \mathcal{O}_V(d)) \quad \text{is exact,}$$

as $\mathcal{O}_V(d)$ has support V. Now,

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \{0\} \cup \{E \geq 0,\ E \sim dH\}.$$

If $E = \sum_Q a_Q Q$, where $\dim(Q) = n-1$ and $a_Q \geq 0$, we set $\deg(E) = \sum_Q a_Q \deg(Q)$. If $E \geq 0$, then $\deg(E) \geq 0$, from which we deduce

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \begin{cases} (0) & \text{if } d < 0 \\ \mathbb{C}^{\binom{n+d}{d}} & \text{i.e., all forms of degree } d \text{ in } X_0, \ldots, X_n, \text{ if } d \geq 0. \end{cases}$$

We deduce,

$$H^0(\mathbb{P}^n, \mathfrak{I}_V(d)) = \{\text{all forms of degree } d \text{ vanishing on } V\} \cup \{0\},$$

that is, all hypersurfaces, $Z \subseteq \mathbb{P}^n$, with $V \subseteq Z$ (and 0).

Consequently, to give $\xi \in H^0(\mathbb{P}^n, \mathfrak{I}_V(d))$ is to give a hypersurface of $\mathbb{P}^n$ containing V. Therefore,

$$H^0(\mathbb{P}^n, \mathfrak{I}_V(d)) = (0) \quad \text{iff} \quad \text{no hypersurface of degree } d \text{ contains } V.$$

(In particular, V is nondegenerate iff $H^0(\mathbb{P}^n, \mathfrak{I}_V(d)) = (0)$.)

We now compute the groups $H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, for all n, q, d. First, consider $d \geq 0$ and use induction on n. For $\mathbb{P}^0$, we have

$$H^q(\mathbb{P}^0, \mathcal{O}_{\mathbb{P}^0}(d)) = \begin{cases} (0) & \text{if } q > 0 \\ \mathbb{C} & \text{if } q = 0. \end{cases}$$

Next, $\mathbb{P}^1$. The sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^0} \longrightarrow 0 \quad \text{is exact.}$$

By tensoring with $\mathcal{O}_{\mathbb{P}^1}(d)$, we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^0}(d) \longrightarrow 0 \quad \text{is exact}$$

by taking cohomology, we get

$$0 \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1)) \xrightarrow{\alpha} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \xrightarrow{\beta} H^0(\mathbb{P}^0, \mathcal{O}_{\mathbb{P}^0}(d)) \longrightarrow$$
$$\longrightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1)) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \longrightarrow 0$$

since $H^1(\mathbb{P}^0, \mathcal{O}_{\mathbb{P}^0}(d)) = (0)$, by hypothesis. Now, if we pick coordinates, the embedding $\mathbb{P}^0 \hookrightarrow \mathbb{P}^1$ corresponds to $x_0 = 0$. Consequently, the map α is multiplication by x_0 and the map β is $x_0 \mapsto 0$. Therefore,

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)), \quad \text{for all } d \geq 0,$$

and we deduce

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}^g = (0),$$

and $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = (0)$, too. We know that

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = \mathbb{C}^{d+1}; \quad d \geq 0;$$

and we just proved that

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = (0); \quad d \geq -1.$$

In order to understand the induction pattern, let us do the case of $\mathbb{P}^2$. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(d-1) \stackrel{\alpha}{\longrightarrow} \mathcal{O}_{\mathbb{P}^2}(d) \stackrel{\beta}{\longrightarrow} \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow 0$$

and by taking cohomology, we get

$$\begin{array}{l}
0 \longrightarrow H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1)) \stackrel{\alpha}{\longrightarrow} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \stackrel{\beta}{\longrightarrow} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \longrightarrow \\
\longrightarrow H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1)) \longrightarrow H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \longrightarrow \\
\longrightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1)) \longrightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \longrightarrow 0
\end{array}$$

By the induction hypothesis, $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = (0)$ if $d \geq -1$, so

$$H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-1)) \cong H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)), \quad \text{for all } d \geq -1.$$

Therefore,

$$H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \cong H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}), \quad \text{for all } d \geq -2.$$

But, the dimension of the right hand side is $h^{0,1} = 0$ (the irregularity, $h^{0,1}$, of $\mathbb{P}^2$ is zero). We conclude that

$$H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) = (0) \quad \text{for all } d \geq -2.$$

A similar reasoning applied to H^2 shows

$$H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \cong H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}), \quad \text{for all } d \geq -2.$$

The dimension of the right hand side group is $H^{0,2} = p_g(\mathbb{P}^2) = 0$, so we deduce

$$H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) = (0) \quad \text{for all } d \geq -2.$$

By induction, we get

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \begin{cases} \mathbb{C}^{\binom{n+d}{d}} & \text{if } d \geq 0 \\ (0) & \text{if } d < 0 \end{cases}$$

and

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = (0) \quad \text{if } d \geq -n, \text{ for all } q > 0.$$

For the rest of the cases, we use Serre duality and the Euler sequence. Serre duality says

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))^D \cong H^{n-q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d) \otimes \Omega^n_{\mathbb{P}^n}).$$

From the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \coprod_{n+1 \text{ times}} \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow T^{1,0}_{\mathbb{P}^n} \longrightarrow 0,$$

by taking the highest wedge, we get

$$\bigwedge^{n+1}\Big(\coprod_{n+1} \mathcal{O}_{\mathbb{P}^n}(1)\Big) \cong \bigwedge^{n} T^{1,0}_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n},$$

from which we conclude

$$(\Omega^n_{\mathbb{P}^n})^D \cong \bigwedge^{n+1}\Big(\coprod_{n+1} \mathcal{O}_{\mathbb{P}^n}(1)\Big) \cong \mathcal{O}_{\mathbb{P}^n}(n+1).$$

Therefore

$$\omega_{\mathbb{P}^n} = \Omega^n_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-(n+1)) = \mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n}),$$

where $K_{\mathbb{P}^n}$ is the canonical divisor on $\mathbb{P}^n$, by definition. Therefore, we have

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong H^{n-q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d-n-1))^D.$$

If $1 \leq q \leq n-1$ and $d \geq -n$, then we know that the left hand side is zero. As $1 \leq n-q \leq n-1$, it follows that

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d-n-1)) = (0) \quad \text{when } d \geq -n.$$

Therefore,

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = (0) \quad \text{for all } d \text{ and all } q \text{ with } 1 \leq q \leq n-1.$$

We also have

$$H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))^D \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d-n-1)),$$

and the right hand side is (0) if $-d-(n+1) < 0$, i.e., $d \geq -n$. Thus, if $d \leq -(n+1)$, then we have $\delta = -d-(n+1) \geq 0$, so

$$H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\delta))^D = \mathbb{C}^{\binom{n+\delta}{\delta}}, \quad \text{where } \delta = -(d+n+1).$$

The pairing is given by

$$\frac{1}{f} \otimes \frac{f}{x_0x_1\cdots x_n} \mapsto \int_{\mathbb{P}^n} \frac{dx_0 \wedge \cdots \wedge dx_n}{x_0\cdots x_n},$$

where $\deg(f) = -d$, with $d \leq -n-1$. Summarizing all this, we get

Theorem 3.10 *The cohomology of line bundles on $\mathbb{P}^n$ satisfies*

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = (0) \quad \textit{for all } n, d \textit{ and all } q \textit{ with } 1 \leq q \leq n-1.$$

Furthermore,

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \mathbb{C}^{\binom{n+d}{d}}, \quad \textit{if } d \geq 0, \textit{ else } (0),$$

and

$$H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \mathbb{C}^{\binom{n+\delta}{\delta}}, \quad \textit{where } \delta = -(d+n+1) \textit{ and } d \leq -n-1, \textit{ else } (0).$$

We also proved that

$$\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-(n+1)) = \mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n}).$$

3.2 Chern Classes and Segre Classes

The most important spaces (for us) are the Kähler manifolds and unless we explicitly mention otherwise, X will be Kähler. But, we can make Chern classes if X is worse.

Remark: The material in this Section is also covered in Hirzebruch [8] and under other forms in Chern [4], Milnor and Stasheff [11], Bott and Tu [3], Madsen and Tornehave [9] and Griffith and Harris [6].

Let X be *admissible* iff

(1) X is *σ-compact*, i.e.,

   (a) X is locally compact and

   (b) X is a countable union of compacts.

(2) The *combinatorial dimension* of X is finite.

Note that (1) implies that X is paracompact. Consequently, everthing we did on sheaves goes through.

Say X is an algebraic variety and $\mathcal{F}$ is a QC $\mathcal{O}_X$-module. Then, $H^0(X, \mathcal{F})$ encodes the most important geometric information contained in $\mathcal{F}$. For example, $\mathcal{F}$ = a line bundle or a vector bundle, then

$$H^0(X, \mathcal{F}) = \text{space of global sections of given type.}$$

If $\mathcal{F} = \mathfrak{I}_V(d)$, where $V \subseteq \mathbb{P}^n$, then

$$H^0(X, \mathcal{F}) = \text{hypersurfaces containing } V.$$

This leads to the Riemann-Roch (RR) problem.

Given X and a QC $\mathcal{O}_X$-module, $\mathcal{F}$,

(a) Determine when $H^0(X, \mathcal{F})$ has finite dimension and

(b) If so, compute the dimension, $\dim_{\mathbb{C}} H^0(X, \mathcal{F})$.

Some answers:

(a) Finiteness Theorem: If X is a compact, complex, analytic manifold and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $H^q(X, \mathcal{F})$ has finite dimension for every $q \geq 0$.

(b) It was noticed in the fifties (Kodaira and Spencer) that if $\{X_t\}_{t\in S}$ is a reasonable family of compact algebraic varieties ($\mathbb{C}$-analytic manifolds), (S is just a $\mathbb{R}$-differentiable smooth manifold and the X_t are a proper flat family), then

$$\chi(X_t, \mathcal{O}_{X_t}) = \sum_{i=0}^{\dim X_t} (-1)^i \dim(H^i(X_t, \mathcal{O}_{X_t}))$$

was independent of t.

The Riemann-Roch problem goes back to Riemann and the finiteness theorem goes back to Oka, Cartan-Serre, Serre, Grauert, Grothendieck, ...

Examples. (1) Riemann (1850's): If X is a compact Riemann surface, then

$$\chi(X, \mathcal{O}_X) = 1 - g$$

where g is the number of holes of X (as a real surface).

(2) Max Noether (1880's): If X is a compact, complex surface, then

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}(K_X^2 + \text{top Euler char.}(X)).$$

(Here, $K_X^2 = \mathcal{O}_X(K_X) \cup \mathcal{O}_X(K_X)$ in the cohomology ring, an element of $H^4(X, \mathbb{Z})$.)

(3) Severi, Eger-Todd (1920, 1937) conjectured:

$$\chi(X, \mathcal{O}_X) = \text{some polynomial in the Euler-Todd class of } X,$$

for X a general compact algebraic, complex manifold.

(4) In the fourties and fifties (3) was reformulated as a statement about Chern classes–no proof before Hirzebruch.

(5) September 29, 1952: Serre (letter to Kodaira and Spencer) conjectured: If $\mathcal{F}$ is a rank r vector bundle over the compact, complex algebraic manifold, X, then

$$\chi(X, \mathcal{F}) = \text{polynomial in the Chern classes of } X \text{ and those of } \mathcal{F}.$$

Serre's conjecture (5) was proved by Hirzebruch a few months later.

To see this makes sense, we'll prove

Theorem 3.11 *(Riemann-Roch for a compact Riemann Surface and for a line bundle) If X is a compact Riemann surface and if $\mathcal{L}$ is a complex analytic line bundle on X, then there is an integer,* $\deg(\mathcal{L})$*, it is* $\deg(D)$ *where $\mathcal{L} \cong \mathcal{O}_X(D)$, where D is a Cartier divisor on X, and*

$$\dim_{\mathbb{C}} H^0(X, \mathcal{L}) - \dim_{\mathbb{C}} H^0(X, \omega_X \otimes \mathcal{L}^D) = \deg(\mathcal{L}) + 1 - g$$

where $g = \dim H^0(X, \omega_X) = \dim H^1(X, \mathcal{O}_X)$ is the genus of X.

Proof. First, we know X is an algebraic variety (a curve), by Riemann's theorem (see Homework). From another Homework (from Fall 2003), X is embeddable in $\mathbb{P}^N_{\mathbb{C}}$, for some N, and by GAGA (yet to come!), $\mathcal{L}$ is an algebraic line bundle. It follows that $\mathcal{L} = \mathcal{O}_X(D)$, for some Cartier divisor, D. Now, if $f \in \mathcal{M}\mathrm{er}(X)$, we showed (again, see Homework) that $f\colon X \to \mathbb{P}^1_{\mathbb{C}} = S^2$ is a branched covering map and this implies that

$$\#(f^{-1}(\infty)) = \#(f^{-1}(0)) = \text{degree of the map},$$

so $\deg(f) = \#(f^{-1}(0)) - \#(f^{-1}(\infty)) = 0$. As a consequence, if $E \sim D$, then $\deg(E) = \deg(D)$ and the first statement is proved. Serre duality says

$$H^0(X, \omega_X \otimes \mathcal{L}^D) \cong H^1(X, \mathcal{L})^D.$$

Thus, the left hand side of the Riemann-Roch formula is just $\chi(X, \mathcal{O}_X(D))$, where $\mathcal{L} = \mathcal{O}_X(D)$. Observe that $\chi(X, \mathcal{O}_X(D))$ is an Euler function in the bundle sense (this is always true of Euler-Poincaré characteristics). Look at any point , P, on X, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow \kappa_P \longrightarrow 0,$$

where κ_P is the skyscraper sheaf at P, i.e.,

$$(\kappa_P)_x = \begin{cases} (0) & \text{if } x \neq P \\ \mathbb{C} & \text{if } x = P. \end{cases}$$

If we tensor with $\mathcal{O}_X(D)$, we get the exact sequence

$$0 \longrightarrow \mathcal{O}_X(D-P) \longrightarrow \mathcal{O}_X(D) \longrightarrow \kappa_P \otimes \mathcal{O}_X(D) \longrightarrow 0.$$

When we apply cohomology, we get

$$\chi(X, \kappa_P \otimes \mathcal{O}_X(D)) + \chi(X, \mathcal{O}_X(D-P)) = \chi(X, \mathcal{O}_X(D)).$$

There are three cases.

(a) $D = 0$. The Riemann-Roch formula is a tautology, by definition of g and the fact that $H^0(X, \mathcal{O}_X) = \mathbb{C}$.

(b) $D > 0$. Pick any P appearing in D. Then, $\deg(D-P) = \deg(D) - 1$ and we can use induction. The base case holds, by (a). Using the induction hypothesis, we get

$$1 + \deg(D-P) + 1 - g = \chi(X, \mathcal{O}_X(D)),$$

which says

$$\chi(X, \mathcal{O}_X(D)) = \deg(D) + 1 - g,$$

proving the induction step when $D > 0$.

(c) D is arbitrary. In this case, write $D = D^+ - D^-$, with $D^+, D^- \geq 0$; then

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D^-) \longrightarrow \kappa_{D^-} \longrightarrow 0 \quad \text{is exact}$$

and

$$\deg(\kappa_{D^-}) = \deg(D^-) = \chi(X, \mathcal{O}_X(D^-)).$$

If we tensor the above exact sequence with $\mathcal{O}_X(D)$, we get

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D + D^-) \longrightarrow \kappa_{D^-} \longrightarrow 0 \quad \text{is exact.}$$

When we apply cohomology, we get

$$\chi(X, \mathcal{O}_X(D)) + \deg(D^-) = \chi(X, \mathcal{O}_X(D + D^-)) = \chi(X, \mathcal{O}_X(D^+)).$$

However, by (b), we have $\chi(X, \mathcal{O}_X(D^+)) = \deg(D^+) + 1 - g$, so we deduce

$$\chi(X, \mathcal{O}_X(D)) = \deg(D^+) - \deg(D^-) + 1 - g = \deg(D) + 1 - g,$$

which finishes the proof. $\square$

We will show:

(a) $\mathcal{L}$ possesses a class, $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$.

(b) If X is a Riemann surface and $[X] \in H_2(X, \mathbb{Z}) = \mathbb{Z}$ is its fundamental class, then $\deg(\mathcal{L}) = c(\mathcal{L})[X] \in \mathbb{Z}$. Then, the Riemann-Roch formula becomes

$$\begin{aligned} \chi(X, \mathcal{L}) &= c_1(\mathcal{L})[X] + 1 - g \\ &= \left[c_1(\mathcal{L}) + \frac{1}{2}(2-2g)\right][X] \\ &= \left[c_1(\mathcal{L}) + \frac{1}{2}c_1(T_X^{1,0})\right][X]. \end{aligned}$$

This is Hirzebruch's form of the Riemann-Roch theorem for Riemann surfaces and line bundles.

What about vector bundles?

Theorem 3.12 *(Atiyah-Serre on vector bundles) Let X be either a compact, complex C^∞-manifold or an algebraic variety. If E is a rank r vector bundle on X, of class C^∞ in case X is just C^∞, algebraic if X is algebraic, in the latter case assume E is generated by its global sections (that is, the map, $\Gamma_{\mathrm{alg}}(X, \mathcal{O}_X(E)) \longrightarrow E_x$, given by $\sigma \mapsto \sigma(x)$, is surjective for all x), then, there is a trivial bundle of rank $r-d$ (where $d = \dim_{\mathbb{C}} X$) denoted $\mathbb{I}^{r-d}$, and a bundle exact sequence*

$$0 \longrightarrow \mathbb{I}^{r-d} \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

and the rank of the bundle E'' is at most d.

Proof. Observe that if $r < d$, there is nothing to prove and $\mathrm{rk}(E'') = \mathrm{rk}(E)$ and also if $r = d$ take (0) for the left hand side. So, we may assume $r > d$. In the C^∞-case, we always have E generated by its global C^∞-sections (partition of unity argument).

Pick x, note $\dim E_x = r$, so there is a finite dimensional subspace of $\Gamma(X, \mathcal{O}_X(E))$ surjecting onto E_x. By continuity (or algebraicity), this holds $\mathbb{C}$-near (resp. Z-near) x. Cover by these opens and so

(a) In the C^∞-case, finitely many of these opens cover X (recall, X is compact).

(b) In the algebraic case, again, finitely many of these opens cover X, as X is quasi-compact in the Z-topology.

Therefore, there exists a finite dimensional space, $W \subseteq \Gamma(X, \mathcal{O}_X(E))$, and the map $W \longrightarrow E_x$ given by $\sigma \mapsto \sigma(x)$ is *surjective* for all $x \in X$. Let

$$\ker(x) = \mathrm{Ker}\,(W \longrightarrow E_x).$$

Consider the projective space $\mathbb{P}(\ker(x)) \hookrightarrow \mathbb{P} = \mathbb{P}(W)$. Observe that $\dim \ker(x) = \dim W - r$ is independent of x. Now, look at $\bigcup_{x\in X} \mathbb{P}(\ker(x))$ and let Z be its Z-closure. We have

$$\dim Z = \dim X + \dim W - r - 1 = \dim W + d - r - 1,$$

so, $\mathrm{codim}(Z \hookrightarrow \mathbb{P}) = r - d$. Thus, there is some projective subspace, T, of $\mathbb{P}$ with $\dim T = r - d - 1$, so that

$$T \cap Z = \emptyset.$$

Then, $T = \mathbb{P}(S)$, for some subspace, S, of W ($\dim S = r - d$). Look at

$$X \prod S = X \prod \mathbb{C}^{r-d} = \mathbb{I}^{r-d}.$$

Send $\mathbb{I}^{r-d}$ to E *via* $(x, s) \mapsto s(x) \in E$. As $T \cap Z = \emptyset$, the value $s(x)$ is *never* zero. Therefore, for any $x \in X$, $\mathrm{Im}(\mathbb{I}^{r-d} \hookrightarrow E)$ has full rank; set $E'' = E/\mathrm{Im}((\mathbb{I}^{r-d} \hookrightarrow E)$ = a vector bundle of rank d, then

$$0 \longrightarrow \mathbb{I}^{r-d} \longrightarrow E \longrightarrow E'' \longrightarrow 0 \quad \text{is exact}$$

as a bundle sequence. $\square$

Remarks:

(a) If $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ is bundle exact, then

$$c_1(E) = c_1(E') + c_1(E'').$$

(b) If E is the trivial bundle, $\mathbb{I}^r$, then $c_j(E) = 0$, for $j = 1, \ldots, r$.

(c) If $\mathrm{rk}(E) = r$, then $c_1(E) = c_1(\bigwedge^r E)$.

In view of (a)–(c), Atiyah-Serre can be reformulated as

$$c_1(E) = c_1\Big(\bigwedge^{\mathrm{rk}\,E} E\Big) = c_1(E'') = c_1\Big(\bigwedge^{\mathrm{rk}\,E''} E''\Big).$$

We now use the Atiyah-Serre theorem to prove a version of Riemann-Roch first shown by Weil.

Theorem 3.13 *(Riemann-Roch on a Riemann surface for a vector bundle) If X is a compact Riemann surface and E is a complex analytic rank r vector bundle on X, then*

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(E)) - \dim_{\mathbb{C}} H^1(X, \omega_X \otimes \mathcal{O}_X(E)^D) = \chi(X, \mathcal{O}_X) = c_1(E) + \mathrm{rk}(E)(1-g).$$

Proof. The first equality is just Serre Duality. As before, by Riemann's theorem X is projective algebraic and by GAGA, E is an algebraic vector bundle. Now, as $X \hookrightarrow \mathbb{P}^N$, it turns out (Serre) that for $\delta >> 0$, the "twisted bundle", $E \otimes \mathcal{O}_X(\delta)$ $(= E \otimes \mathcal{O}_X^{\otimes\delta})$ is generated by its global holomorphic sections. We can apply Atiyah-Serre to $E \otimes \mathcal{O}_X(\delta)$. We get

$$0 \longrightarrow \mathbb{I}^{r-1} \longrightarrow E \otimes \mathcal{O}_X(\delta) \longrightarrow E'' \longrightarrow 0 \quad \text{is exact,}$$

where $\mathrm{rk}(E'') = 1$. If we twist with $\mathcal{O}_X(-\delta)$, we get the exact sequence

$$0 \longrightarrow \coprod_{r-1} \mathcal{O}_X(-\delta) \longrightarrow E \longrightarrow E''(-\delta) \longrightarrow 0.$$

(Here, $E''(-\delta) = E'' \otimes \mathcal{O}_X(-\delta)$.) Now, use induction on r. The case $r = 1$ is ordinary Riemann-Roch for line bundles. Assume the induction hypothesis for $r-1$. As χ is an Euler function, we have

$$\chi(X, \mathcal{O}_X(E)) = \chi(X, E''(-\delta)) + \chi\Big(\coprod_{r-1} \mathcal{O}_X(-\delta)\Big).$$

The first term on the right hand side is

$$c_1(E''(-\delta)) + 1 - g,$$

by ordinary Riemann-Roch and the second term on the right hand side is

$$c_1\Big(\coprod_{r-1} \mathcal{O}_X(-\delta)\Big) + (r-1)(1-g).$$

by the induction hypothesis. We deduce that

$$\chi(X, \mathcal{O}_X(E)) = c_1(E''(-\delta)) + c_1\Big(\coprod_{r-1} \mathcal{O}_X(-\delta)\Big) + r(1-g).$$

But, we know that

$$c_1(E) = c_1(E''(-\delta)) + c_1\Big(\coprod_{r-1} \mathcal{O}_X(-\delta)\Big),$$

so we conclude that

$$\chi(X, \mathcal{O}_X(E)) = c_1(E) + r(1-g),$$

establishing the induction hypothesis and the theorem. □

Remark: We can write the above as

$$\chi(X, \mathcal{O}_X(E)) = c_1(E) + \frac{\mathrm{rk}(E)}{2} c_1(T_X^{1,0}),$$

which is Hirzebruch's form of Riemann-Roch.

We will need later some properties of $\chi(X, \mathcal{O}_X)$ and $p_g(X)$. Recall that $p_g(X) = \dim_{\mathbb{C}} H^n(X, \mathcal{O}_X) = \dim_{\mathbb{C}} H^0(X, \Omega^n_X)$, where $\Omega^l_X = \bigwedge^l T^{1,0}_X$. (The vector spaces $H^0(X, \Omega^l_X)$ were what the Italian geometers (in fact, all geometers) of the nineteenth century understood.)

Proposition 3.14 *The functions $\chi(X, \mathcal{O}_X)$ and $p_g(X)$ are multiplicative on compact, Kähler manifolds, i.e.,*

$$\begin{aligned} \chi\Big(X \prod Y, \mathcal{O}_{X\prod Y}\Big) &= \chi(X, \mathcal{O}_X)\chi(Y, \mathcal{O}_Y) \\ p_g\Big(X \prod Y\Big) &= p_g(X)p_g(Y). \end{aligned}$$

Proof. Remember that

$$\dim_{\mathbb{C}} H^l(X, \mathcal{O}_X) = \dim_{\mathbb{C}} H^0(X, \Omega^l_X) = h^{0,l} = h^{l,0}.$$

Then,

$$\chi(X, \mathcal{O}_X) = \sum_{j=0}^{n} (-1)^j \dim_{\mathbb{C}} H^0(X, \Omega^j_X) = \sum_{j=0}^{n} (-1)^j h^{j,0}.$$

Also recall the Künneth formula

$$\coprod_{\substack{p+p'=a \\ q+q'=b}} H^q(X, \Omega^p_X) \otimes H^{q'}(X, \Omega^{p'}_X) \cong H^b\Big(X \prod Y, \Omega^a_{X\prod Y}\Big).$$

Set $b = 0$, then $q = q' = 0$ and we get

$$\sum_{p+p'=a} h^{p,0}(X) h^{p',0}(Y) = h^{a,0}\Big(X \prod Y\Big).$$

Then,

$$\begin{aligned} \chi(X, \mathcal{O}_X)\chi(Y, \mathcal{O}_Y) &= \left(\sum_{r=0}^{m} (-1)^r h^{r,0}(X)\right)\left(\sum_{s=0}^{n} (-1)^s h^{s,0}(Y)\right) \\ &= \sum_{r,s=0}^{m+n} (-1)^{r+s} h^{r,0}(X) h^{s,0}(Y) \\ &= \sum_{k=0}^{m+n} (-1)^k \sum_{r+s=k} h^{r,0}(X) h^{s,0}(Y) \\ &= \sum_{k=0}^{m+n} (-1)^k h^k(X \prod Y) = \chi\Big(X \prod Y, \mathcal{O}_{X\prod Y}\Big). \end{aligned}$$

The second statement is obvious from Künneth. □

Next, we introduce Hirzebruch's axiomatic approach.

Let E be a complex vector bundle on X, where X is one of our spaces (admissible). It will turn out that E is a unitary bundle (a $\mathrm{U}(q)$-bundle, where $q = \mathrm{rk}(E)$).

Chern classes are cohomology classes, $c_l(E)$, satisfying the following axioms:

Axiom (I). (Existence and Chern polynomial). If E is a rank q unitary bundle over X and X is admissible, then there exist cohomology classes, $c_l(E) \in H^{2l}(X, \mathbb{Z})$, the *Chern classes* of E and we set

$$c(E)(t) = \sum_{l=0}^{\infty} c_l(E)t^l \in H^*(X, \mathbb{Z})[[t]],$$

with $c_0(E) = 1$.

As $\dim_{\mathbb{C}} X = d < \infty$, we get $c_l(E) = 0$ for $l > d$, so $C(E)(t)$ is in fact a polynomial in $H^*(X, \mathbb{Z})[t]$ called the *Chern polynomial* of E where $\deg(t) = 2$.

Say $\pi\colon Y \to X$ and E is a $\mathrm{U}(q)$-bundle over X, then we have two maps

$$H^*(X, \mathbb{Z}) \xrightarrow{\pi^*} H^*(Y, \mathbb{Z}) \quad \text{and} \quad H^1(X, \mathrm{U}(q)) \xrightarrow{\pi^*} H^1(Y, \mathrm{U}(q)).$$

Axiom (II). (Naturality). For every E, a $\mathrm{U}(q)$-bundle on X and map, $\pi\colon Y \to X$, (with X, Y admissible), we have

$$c(\pi^* E)(t) = \pi^*(c(E))(t),$$

as elements of $H^*(Y, \mathbb{Z})[[t]]$.

Axiom (III). (Whitney coproduct axiom). If E, a $\mathrm{U}(q)$-bundle is a coproduct (in the $\mathbb{C}$ or C^∞-sense),

$$E = \coprod_{j=1}^{\mathrm{rk}(E)} E_j$$

of $\mathrm{U}(1)$-bundles, then

$$c(E)(t) = \prod_{j=1}^{\mathrm{rk}(E)} c(E_j)(t).$$

Axiom (IV). (Normalization). If $X = \mathbb{P}^n_{\mathbb{C}}$ and $\mathcal{O}_X(1)$ is the $\mathrm{U}(1)$-bundle corresponding to the hyperplane divisor, H, on $\mathbb{P}^n_{\mathbb{C}}$, then

$$c(\mathcal{O}_X(1))(t) = 1 + Ht,$$

where H is considered in $H^2(X, \mathbb{Z})$.

Remark: If $i\colon \mathbb{P}^{n-1}_{\mathbb{C}} \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$, then

$$i^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$$

and $i^*(H)$ in $H^2(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$ is $H_{\mathbb{P}^{n-1}_{\mathbb{C}}}$. By Axiom (II) and Axiom (IV)

$$i^*(1 + H_{\mathbb{P}^n_{\mathbb{C}}} t) = i^*(c(\mathcal{O}_{\mathbb{P}^n})(t)) = c(i^*(\mathcal{O}_{\mathbb{P}^n})(t)) = 1 + H_{\mathbb{P}^{n-1}_{\mathbb{C}}}.$$

Therefore, we can use *any* n to normalize.

Some Remarks on bundles. First, on $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$: Geometric models of $\mathcal{O}_{\mathbb{P}^n}(\pm 1)$.

Consider the map

$$\mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n.$$

If we blow up 0 in $\mathbb{C}^{n+1}$, we get $B_0(\mathbb{C}^{n+1})$ as follows: In $\mathbb{C}^{n+1} \prod \mathbb{P}^n$, look at the subvariety given by

$$\{\langle\langle z\rangle; (\xi)\rangle \mid z_i\xi_j = z_j\xi_i,\ 0 \leq i, j \leq n\}.$$

By definition, this is $B_0(\mathbb{C}^{n+1})$, an algebraic variety over $\mathbb{C}$. We have the two projections

$$\begin{array}{ccc} & B_0(\mathbb{C}^{n+1}) & \\ {}^{pr_1}\swarrow & & \searrow^{pr_2} \\ \mathbb{C}^{n+1} & & \mathbb{P}^n. \end{array}$$

Look at the fibre, $pr_1^{-1}(\langle z\rangle)$ over $z \in \mathbb{C}^{n+1}$. There are two cases:

(a) $\langle z\rangle = 0$, in which case, $pr_1^{-1}(\langle z\rangle) = \mathbb{P}^n$.

(b) $\langle z\rangle \neq 0$, so, there is some j with $z_j \neq= 0$. We get $\xi_i = \frac{z_i}{z_j}\xi_j$, for all i, which implies:

(α) $\xi_j \neq 0$.

(β) All ξ_i are determined by ξ_j.

(γ) $\dfrac{\xi_i}{\xi_j} = \dfrac{z_i}{z_j}$.

This implies

$$(\xi) = \left(\frac{\xi_0}{\xi_j} \colon \frac{\xi_1}{\xi_j} \colon \cdots : 1 \colon \cdots \frac{\xi_n}{\xi_j}\right) = \left(\frac{z_0}{z_j} \colon \frac{z_1}{z_j} \colon \cdots : 1 \colon \cdots \frac{z_n}{z_j}\right).$$

Therefore, $pr_1^{-1}(\langle z\rangle) = \langle\langle z\rangle; (z)\rangle$, a single point.

Let us now look ar $pr_2^{-1}(\xi)$, for $(\xi) \in \mathbb{P}^n$. Since $(\xi) \in \mathbb{P}^n$, there is some j such that $\xi_j \neq 0$. A point $\langle\langle z\rangle; (\xi)\rangle$ above (ξ) is given by all $\langle z_0 \colon z_1 \colon \cdots : z_n\rangle$ so that

$$z_i = \frac{\xi_i}{\xi_j} z_j.$$

Let $z_j = t$, then the fibre above ξ is the complex line

$$z_0 = \frac{\xi_0}{\xi_j}t,\ z_1 = \frac{\xi_1}{\xi_j}t,\ \cdots,\ z_j = t,\ \cdots,\ z_n = \frac{\xi_n}{\xi_j}t.$$

We get a line family over $\mathbb{P}^n$. Thus, $pr_2 \colon B_0(\mathbb{C}^{n+1}) \to \mathbb{P}^n$ is a line family.

(A) What kinds of maps, $\sigma \colon \mathbb{P}^n \to B_0(\mathbb{C}^{n+1})$, exist with σ holomorphic and $pr_2 \circ \sigma = \mathrm{id}$?

If σ exists, then $pr_1 \circ \sigma \colon \mathbb{P}^n \to \mathbb{C}^{n+1}$ is holomorphic; this implies that $pr_1 \circ \sigma$ is a constant map. But, $\sigma(\xi)$ belongs to a line through $(\xi) = (\xi_0 \colon \cdots : \xi_n)$, for all (ξ), yet $pr_1 \circ \sigma = \mathrm{const}$, so this point must lie on all line. This can only happen if $\sigma(\xi) = 0$ in the line through ξ.

(B) I claim $B_0(\mathbb{C}^{n+1})$ is locally trivial, i.e., a line bundle. If so, (A) says $B_0(\mathbb{C}^{n+1})$ has *no global holomorphic* sections and we will know that $B_0(\mathbb{C}^{n+1}) = \mathcal{O}_{\mathbb{P}^n}(-q)$, for some $q > 0$.

To show that $B_0(\mathbb{C}^{n+1})$ is locally trivial over $\mathbb{P}^n$, consider the usual cover, $U_0, \ldots, U_n$, of $\mathbb{P}^n$ (recall, $U_j = \{(\xi) \in \mathbb{P}^n \mid \xi_j \neq 0\}$). If $v \in B_0(\mathbb{C}^{n+1}) \upharpoonright U_j$, then $v = \langle\langle z\rangle; (x)\rangle$, with $\xi_j \neq 0$. Define φ_j as the map

$$v \mapsto \langle(\xi); z_j\rangle \in U_j \prod \mathbb{C}$$

and the backwards map

$$\langle(\xi); t\rangle \in U_j \prod \mathbb{C} \mapsto \langle\langle z\rangle; (\xi)\rangle, \quad \text{where} \quad z_i = \frac{\xi_i}{\xi_j}t, \quad i = 0, \ldots, n.$$

The reader should check that the point of $\mathbb{C}^{n+1} \prod \mathbb{P}^n$ so constructed is in $B_0(\mathbb{C}^{n+1})$ and that the maps are inverses of one another.

We can make a section, σ_j, of $B_0(\mathbb{C}^{n+1}) \upharpoonright U_j$, *via*

$$\sigma((\xi)) = \left\langle \left\langle \frac{\xi_0}{\xi_j}, \ldots, \frac{\xi_{j-1}}{\xi_j}, 1, \ldots, \frac{\xi_n}{\xi_j} \right\rangle ; (\xi) \right\rangle,$$

and we see that $\varphi(\sigma((\xi))) = \langle((\xi); 1\rangle \in U_j \prod \mathbb{C}$, which shows that σ is a holomorphic section which is never zero. The transition function, g_i^j, renders the diagram

$$\begin{array}{ccc}
B_0 \upharpoonright U_i & \xrightarrow{\varphi_i} & U_i \prod \mathbb{C} \\
\uparrow & & \Big| \\
B_0 \upharpoonright U_i \cap U_j & & g_i^j \\
\downarrow & & \downarrow \\
B_0 \upharpoonright U_j & \xrightarrow{\varphi_j} & U_j \prod \mathbb{C}
\end{array}$$

commutative. It follows that

$$\varphi_j(v) = g_i^j(\varphi_i(v) = g_i^j(\langle(\xi); z_i\rangle) = \langle(\xi); z_j\rangle$$

and we conclude that $g_i^j(z_i) = z_j$, which means that g_i^j is multiplication by $z_j/z_i = \xi_j/\xi_i$.

We now make another bundle on $\mathbb{P}^n$, which will turn out to be $\mathcal{O}_{\mathbb{P}^n}(1)$. Embed $\mathbb{P}^n$ in $\mathbb{P}^{n+1}$ by viewing $\mathbb{P}^n$ as the hyperplane defined by $z_{n+1} = 0$ and let $P = (0\colon \cdots : 1) \in \mathbb{P}^{n+1}$. Clearly, $P \notin \mathbb{P}^n$. We have the projection, $\pi\colon (\mathbb{P}^{n+1} - \{P\}) \to \mathbb{P}^n$, from P onto $\mathbb{P}^r$, where

$$\pi(z_0\colon \cdots : z_n : z_{n+1}) = (z_0\colon \cdots : z_n).$$

We get a line family over $\mathbb{P}^n$, where the fibre over $Q \in \mathbb{P}^n$ is just the line l_{PQ} (since $P \notin \mathbb{P}^n$, this line is always well defined). The parametric equations of this line are

$$(u\colon t) \mapsto (uz_0\colon \cdots : uz_n : t),$$

where $(u\colon t) \in \mathbb{P}^1$ and $Q = (z_0\colon \cdots : z_n)$. When $t = 0$, we get Q and hen $u = 0$, we get P. Next, we prove that $\mathbb{P}^{n+1} - \{P\}$ is locally trivial. Make a section, σ_j, of π over $U_j \subseteq \mathbb{P}^n$ by setting

$$\sigma_j((\xi)) = (\xi\colon \xi_j).$$

This points corresponds to the point $(1\colon \xi_j)$ on l_{PQ} and $\xi_j \neq 0$, so it is well-defined. As Q is the point of l_{PQ} for which $t = 0$, we have $\sigma_j((\xi)) \neq Q$. We make an isomorphism, $\psi_j\colon (\mathbb{P}^{n+1} - \{P\}) \upharpoonright U_j \to U_j \prod \mathbb{C}$, *via*

$$(z_0\colon \cdots : z_{j-1} : z_j : z_{j+1} : \cdots : z_{n+1}) \mapsto \left(z_0\colon \cdots : z_n : \frac{z_{n+1}}{z_j} \right).$$

Observe that

$$s_j((\xi)) = \psi_j \circ \sigma_j((\xi)) = \psi_j(\xi\colon \xi_j) = (\xi\colon 1) \in U_j \prod \mathbb{C}.$$

For any $(z_0\colon \cdots : z_{n+1}) \in (\mathbb{P}^{n+1} - \{P\}) \upharpoonright U_i \cap U_j$, we have $z_i \neq 0$ and $z_j \neq 0$; moreover

$$\psi_i(z_0\colon \cdots : z_{n+1}) = \left(z_0\colon \cdots : z_n : \frac{z_{n+1}}{z_i} \right) \quad \text{and} \quad \psi_j(z_0\colon \cdots : z_{n+1}) = \left(z_0\colon \cdots : z_n : \frac{z_{n+1}}{z_j} \right).$$

This means that the transition function, h_i^j, on $U_i \cap U_j$, is multiplication by z_i/z_j. These are the inverses of the transition functions of our previous bundle, $B_0(\mathbb{C}^{n+1})$, which means that the bundle $\mathbb{P}^{n+1} - \{P\}$ is the dual bundle of $B_0(\mathbb{C}^{n+1})$. We will use geometry to show that the bundle $\mathbb{P}^{n+1} - \{P\}$ is in fact $\mathcal{O}_{\mathbb{P}^n}(1)$.

Look at the hyperplanes, H, of $\mathbb{P}^{n+1}$. They are given by linear forms,

$$H\colon \sum_{j=0}^{n+1} a_j Z_j = 0.$$

The hyperplanes through P form a $\mathbb{P}^n$, since $P \in H$ iff $a_{n+1} = 0$. The rest of the hyperplanes are in the affine space, $\mathbb{C}^{n+1} = \mathbb{P}^{n+1} - \mathbb{P}^n$. Indeed such hyperplanes, $H_{(\alpha)}$, are given by

$$H_{(\alpha)}\colon \sum_{j=0}^{n} \alpha_j Z_j + Z_{n+1} = 0, \quad (\alpha_0, \ldots, a_n) \in \mathbb{C}^{n+1}.$$

Given any hyperplane, $H_{(\alpha)}$ (with $\alpha \in \mathbb{C}^{n+1}$), find the intersection, $\sigma_{(\alpha)}(Q)$, of the line l_{PQ} with $H_{(\alpha)}$. Note that $\sigma_{(\alpha)}$ is a global section of $\mathbb{P}^{n+1} - \{P\}$. The affine line obtained from l_{PQ} by deleting P is given by

$$\tau \mapsto (z_0 \colon \cdots \colon z_n \colon \tau),$$

where $Q = (z_0 \colon \cdots \colon z_n)$. This lines cuts $H_{(\alpha)}$ iff

$$\sum_{j=0}^{n} \alpha_j z_j + \tau = 0,$$

so we deduce $\tau = -\sum_{j=0}^{n} \alpha_j z_j$ and

$$\sigma_{(\alpha)}(z_0 \colon \cdots \colon z_n) = \left(z_0 \colon \cdots \colon z_n \colon -\sum_{j=0}^{n} \alpha_j z_j\right),$$

which means that $\sigma_{(\alpha)}$ is a holomorphic section. Now, consider a holomorphic section, $\sigma\colon \mathbb{P}^n \to (\mathbb{P}^{n+1} - \{P\}) \hookrightarrow \mathbb{P}^{n+1}$, of $\pi\colon (\mathbb{P}^{n+1} - \{P\}) \to \mathbb{P}^n$. As σ is an algebraic map and $\mathbb{P}^r$ is proper, $\sigma(\mathbb{P}^n)$ is Z-closed, irreducible and has dimension n in $\mathbb{P}^{n+1}$. Therefore, $\sigma(\mathbb{P}^n)$ is a hypersurface. But, our map factors through $\mathbb{P}^{n+1} - \{P\}$, so $\sigma(\mathbb{P}^n) \subseteq \mathbb{P}^{n+1} - \{P\}$. This hypersurface has some degree, d, but all the lines l_{PQ} cut $\sigma(\mathbb{P}^n)$ in a single point, which implies that $d = 1$, i.e., $\sigma(\mathbb{P}^n)$ is a hyperplane *not* through P. Putting all these facts together, we have shown that space of global sections $\Gamma(\mathbb{P}^n, \mathbb{P}^{n+1} - \{P\})$ is in one-to-one correspondence with the hyperplanes $H_{(\alpha)}$, i.e., the linear forms $\sum_{j=0}^{n} \alpha_j z_j$ (a $\mathbb{C}^{n+1}$). Therefore, we conclude that $\mathbb{P}^{n+1} - \{P\}$ *is* $\mathcal{O}_{\mathbb{P}^n}(1)$. Since $B_0(\mathbb{C}^{n+1})$ is the dual of $\mathbb{P}^{n+1} - \{P\}$, we also conclude that $B_0(\mathbb{C}^{n+1}) = \mathcal{O}_{\mathbb{P}^n}(-1)$.

In order to prove that Chern classes exist, we need to know more about bundles. The reader may wish to consult Atiyah [2], Milnor and Stasheff [11], Hirsh [7], May [10] or Morita [12] for a more detailed treatment of bundles.

Recall that if G is a group, then $H^1(X, G)$ classifies the G-torsors over X, e.g., (in our case) the fibre bundles, fibre F, over X (your favorite topology) with $\mathrm{Aut}(F) = G$. When $F = G$ and G acts by left translation to make it $\mathrm{Aut}(F)$, the fibre bundle is called a *principal bundle*. Look at $\varphi\colon G' \to G$, a homomorphism of groups. Now, we know that we get a map

$$H^1(X, G') \longrightarrow H^1(X, G).$$

We would like to see this geometrically and we may take as representations principal bundles. Say $E' \in H^1(X, G')$ a principal bundle with fibre G' and group G'. Consider $G \prod E'$ and make an equivalence relation $\sim$ *via*: For all $\sigma \in G'$, all $g \in G$, all $e' \in E'$

$$(g\varphi(\sigma), e') \sim (g, e'\sigma^{-1}).$$

Set $E'_{G'\longrightarrow G} = \varphi_*(E') = G\prod E'/\sim$.

Let us check that the fibre over $x \in X$ is G. Since E' is locally trivial, we have $E' \restriction U \cong U \prod G'$, for some small enough open, U. The action of G' is such that: For $\sigma \in G'$ and $(u, \tau) \in U \prod G'$,

$$\sigma(u, \tau) = (u, \sigma\tau).$$

Over U, we have $(G \prod E') \restriction U = G \prod U \prod G'$, so our $\varphi_*(E')$ is still locally trivial and the action is on the left on G, its fibre. It follows that

$$E' \mapsto \varphi_*(E')$$

is our map $H^1(X, G') \longrightarrow H^1(X, G)$.

Next, say $\theta\colon Y \to X$ is a map (of spaces), then we get a map

$$H^1(X, G) \xrightarrow{\theta^*} H^1(Y, G).$$

Given $E \in H^1(X, G)$, we have the commutative diagram

$$\begin{array}{ccc} E \prod_X Y & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle \pi_E} \\ Y & \xrightarrow{\theta} & X, \end{array}$$

so we get a space, $\theta^*(E) = E \prod_X Y$, over Y. Over a "small" open, U, of X, we have $E \restriction U \cong G \prod U$ and

$$\theta^*(E) \restriction \theta^{-1}(U) \cong G \prod \theta^{-1}(U),$$

and this gives

$$H^1(X, G) \xrightarrow{\theta^*} H^(Y, G).$$

Say G is a (Lie) group and we have a linear representation, $\varphi\colon G \to \mathrm{GL}(r, \mathbb{C})$. By the above, we get a map

$$E \mapsto E_{G\longrightarrow \mathrm{GL}(r,\mathbb{C})} = \varphi_*(E)$$

from principal G-bundles over X to principal $\mathrm{GL}(r, \mathbb{C})$-bundles over X. But if V is a fixed vector space of dimension r, the construction above gives a rank r vector bundle $\mathrm{GL}(r, \mathbb{C}) \prod V/\sim$. If $\mathcal{V}$ is a rank r vector bundle over $\mathbb{C}$, then look at the sheaf, $\mathcal{I}som(\mathbb{I}^r, \mathcal{V})$, whose fibre at x is the space $\mathrm{Isom}(\mathbb{C}^r, \mathcal{V}_x)$. This sheaf defines a $\mathrm{GL}(r, \mathbb{C})$-bundle.

Say $G' \subseteq G$ is a closed subgroup of the topological group, G.

If G is a real Lie group and G' is a closed subgroup, then G' is also a real Lie group (E. Cartan). But, if G is a complex Lie group and G' is a closed subgroup, then G' need *not* be a complex Lie group. For example, look at $G = \mathbb{C}^* = \mathrm{GL}(1, \mathbb{C})$ and $G' = \mathrm{U}(1) = \{z \in \mathbb{C} \mid |z| = 1\}$.

Convention: If G is a complex Lie group, when we say G' is a closed subgroup we mean a complex Lie group, closed in G.

Say G is a topological group and G' is a closed subgroup of G. Look at the space G/G' and at the continuous map, $\pi\colon G \to G/G'$. We say π has a local section iff there is some some $V \subseteq G/G'$ with $1_G \cdot G' \in V$ and a continuous map

$$s\colon V \to G, \quad \text{such that} \quad \pi \circ s = \mathrm{id}_V.$$

When we untwist this definition we find that it means $s(v) \in v$, where v is viewed as a coset. Generally, one must assume the existence of a local section–this is not true in general.

Theorem 3.15 *If G and G' are topological groups and G' is a closed subgroup of G, assume a local section exists. Then*

(1) The map $G \longrightarrow G/G'$ makes G a continuous principal bundle with fibre and group G' and base G/G'.

(2) If G is a real Lie group and G' is a closed subgroup, then a local smooth section always exists and G is a smooth principal bundle over G/G', with fibre (and group) G'.

(3) If G is a complex Lie group and G' is a closed complex Lie subgroup, then a complex analytic local section always exists and makes G is a complex holomorphic principal bundle over G/G', with fibre (and group) G'.

Proof. The proof of (1) is deferred to the next theorem.

(2) & (3). Use local coordinates, choosing coordinates trasnverse to G' after choosing coordinates in G' near $1_{G'}$. The rest is (DX)– because we get a local section and we repeat the proof for (1) to prove the bundle assertion. □

Now, say E is a fibre bundle, with group G over X (and fibre F) and say G' is a closed subgroup of G. Then, we have a new bundle, E/G'. The bundle E/G' is obtrained from E by identifying in each fibre the elements x and $x\sigma$, where $\sigma \in G'$. Then, the group of E/G' is still G and the fibre is F/G'. In particular, if E is principal, then the group of E/G' is G and its fibre is G/G'. We have a map $E \longrightarrow E/G'$ and a diagram

$$\begin{array}{ccc} E & \longrightarrow & E/G' \\ & {\scriptstyle \pi_E}\searrow \quad \swarrow & \\ & X & \end{array}$$

Theorem 3.16 *If $G \longrightarrow G/G'$ possesses a local section, then for a principal G-bundle E over X*

(1) E/G' is a fibre bundle over X, with fibre G/G'.

(2) $E \longrightarrow E/G'$ is in a natural way a principal bundle (over E/G') with group and fibre G'. If $\xi \in H^1(X, G)$ represents E, write $\xi_{G'}$ for the element of $H^1(E/G', G')$ whose bundle is just $E \longrightarrow E/G'$.

(3) From the diagram of bundles

$$\begin{array}{ccc} E & \xrightarrow{\pi_{E\to E/G'}} & E/G' \\ & {\scriptstyle \pi_E}\searrow \quad \swarrow{\scriptstyle \pi_{E/G'}} & \\ & X & \end{array}$$

we get the commutative diagram

$$\begin{array}{ccc} H^1(X, G') & \longrightarrow & H^1(X, G) \ni \xi \\ \downarrow{\scriptstyle \pi^*_{E/G'}} & & \downarrow{\scriptstyle \pi^*_{E/G'}} \\ \xi_{G'} \in H^1(E/G', G') & \xrightarrow{i_*} & H^1(E/G', G) \end{array}$$

(Here $i\colon G' \hookrightarrow G$ is the inclusion map) and $i_*(\xi_{G'}) = \pi^*_{E/G'}(\xi)$, that is, when E is pulled back to the new base E/G', it arises from a bundle whose structure group is G'.

Figure 3.1: The fibre bundle E over E/G'

Proof. (1) is already proved (there is no need for our hypothesis on local sections).

(2) Pick a cover $\{U_\alpha\}$, of C where $E \restriction U_\alpha$ is trivial so that

$$E \restriction U_\alpha \cong U_\alpha \prod G.$$

Now, consider $G \longrightarrow G/G'$ and the local section $s\colon V(\subseteq G/G') \longrightarrow G$ (with $1_{G/G'} \in V$). We know $s(v) \in v$ (as a coset) and look at $\pi^{-1}(V)$. If $x \in \pi^{-1}(V)$, set

$$\theta(x) = (x^{-1}s(\pi(x)), \pi(x)) \in G' \prod V.$$

This gives an isomorphism (in the appropriate category), $\pi^{-1}(V) \cong G' \prod V$. If we translate V around G/G', we get G as a fibre bundle over G/G' and group G' giving (1) of the previous theorem. But, $U_\alpha \prod V$ and the $U_\alpha \prod$(translate of V) give a cover of E/G' and we have

$$E \restriction U_\alpha \cong U_\alpha \prod \pi^{-1}(V) \cong U_\alpha \prod V \prod G',$$

giving E as fibre bundle over E/G' with group and fibre G'. Here, the diagrams are obvious and the picture of Figure 3.1 finishes the proof. Both sides of the last formula are "push into the board" (by definition for i_* and by elementary computation in $\pi^*_{E/G'}(\xi)$). $\square$

Definition 3.2 If E is a bundle over X with group G and if G' is a closed subgroup of G so that the cohomology representative of G, say ξ actually arises as $i_*(\eta)$ for some $\eta \in H^1(X, G')$, then E *can have its structure group reduced to* G'.

If we restate (3) of the previous theorem in this language, we get

Corollary 3.17 *Every bundle E over X with group G when pulled back to E/G' has its structure group reduced to G'.*

Theorem 3.18 *Let E be a bundle over X, with group G and let G' be a closed subgroup of G. Then, E as a bundle over X can have its structure group reduced to G' iff the bundle E/G' admits a global section over X. In this case if $s\colon X \to E/G'$ is the global section of E/G', then $s^*(E)$ where E is considered as bundle over E/G' with group G' is the element $\eta \in H^1(X, G')$ which gives the structure group reduction. In terms of cocycles, E admits a reduction to group G' iff there exists an open cover $\{U_\alpha\}$ of X so that the transition functions*

$$g_\alpha^\beta\colon U_\alpha \cap U_\beta \to G$$

map $U_\alpha \cap U_\beta$ into the subgroup G'. The section of E/G' is given in the cover by maps $s_\alpha\colon U_\alpha \to U_\alpha \prod G/G'$, where $s_\alpha(u) = (u, 1_{G/G'})$. The cocycle g_α^β represents $s^*(E)$ when its values are considered to be in G' and represents E when its values are considered to be in G.

Proof. Consider the picture of Figure 3.1 above. Suppose E can have structure group reduced to G', then there is a principal bundle, F, for G' and its transition functions give E too. This F can be embedded in E, the fibres are G'. Apply $\pi_{E \longrightarrow E/G'}$ to F, we get get a space over X whose points lie in the bundle E/G', one point for each point of X. Thus, the map $s\colon X \longrightarrow$ point of $\pi_{E \longrightarrow E/G'}(F)$ over x, is our section of E/G' over X.

Conversely, given a section, $s\colon X \to E/G'$, we have E as principal bundle over E/G', with fibre and group G'. So, $s^*(E)$ gives a bundle, F, principal for G', lying over X. Note, F is the bundle given by $s^*(\xi_{G'})$, where ξ represents E. This shows the F constructed reduces to the group G'. The rest (with cocycles) is standard. $\square$

Look at $\mathbb{C}^q$ and $\mathrm{GL}(q,\mathbb{C})$. Write $\mathbb{C}^q_r$ for the span of $e_1,\ldots,e_r$ (the first r canonical basis vectors) = Ker π_r, where π_r is projection on the last $q-r$ basis vectors, $e_{r+1},\ldots,e_q$. Let $\mathcal{G}rass(r,q;\mathbb{C})$ denote the complex Grassmannian of r-dimensional linear subspaces in $\mathbb{C}^q$. There is a natural action of $\mathrm{GL}(q,\mathbb{C})$ on $\mathcal{G}rass(r,q;\mathbb{C})$ and it is clearly transitive. Let us look for the stabilizer of $\mathbb{C}^q_r$. It is the subgroup, $\mathrm{GL}(r,q-r;\mathbb{C})$, of $\mathrm{GL}(q,\mathbb{C})$, consisting of all matrices of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A is $r\times r$. It follows that, as a homogeneous space,

$$\mathrm{GL}(q,\mathbb{C})/\mathrm{GL}(r,q-r;\mathbb{C}) \cong \mathcal{G}rass(r,q;\mathbb{C}).$$

If we restrict the action to $\mathrm{U}(q)$, the above matrices must be of the form

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$$

where $A\in\mathrm{U}(r)$ and $C\in\mathrm{U}(q-r)$, so

$$\mathrm{U}(q)/\mathrm{U}(r)\prod\mathrm{U}(q-r) \cong \mathcal{G}rass(r,q;\mathbb{C}).$$

Remark: Note, in the real case we obtain

$$\mathrm{GL}(q,\mathbb{R})/\mathrm{GL}(r,q-r;\mathbb{R}) \cong \mathrm{O}(q)/\mathrm{O}(r)\prod\mathrm{O}(q-r) \cong \mathcal{G}rass(r,q;\mathbb{R}).$$

If one looks at oriented planes, then this becomes

$$\mathrm{GL}^+(q,\mathbb{R})/\mathrm{GL}^+(r,q-r;\mathbb{R}) \cong \mathrm{SO}(q)/\mathrm{SO}(r)\prod\mathrm{SO}(q-r) \cong \mathcal{G}rass^+(r,q;\mathbb{R}).$$

Theorem 3.19 *(Theorem A) If X is paracompact, f and g are two maps $X\longrightarrow Y$ and E is a bundle over Y, then when f is homotopic to g and not for holomorphic bundles, we have $f^*E\cong g^*E$.*

Theorem 3.20 *(Theorem B) Suppose X is paracompact and E is a bundle over X whose fibre is a cell. If Z is any closed subset of X (even empty) then any section (continuous, smooth, but not holomorphic) of E over Z admits an extension to a global section (continuous or smooth) of E. That is, the sheaf $\mathcal{O}_X(E)$ is a soft sheaf. (Note this holds when E is a vector bundle and it is Tietze's Extension Theorem).*

Theorem 3.21 *(Theorem C) Say G' is a closed subgroup of G and X is paracompact. If G/G' is a cell, then the natural map*

$$H^1_{\mathrm{top}}(X,G')\longrightarrow H^1_{\mathrm{top}}(X,G) \quad \textit{or} \quad H^1_{\mathrm{diff}}(X,G')\longrightarrow H^1_{\mathrm{diff}}(X,G)$$

is a bijection. That is, every principal G-bundle can have its structure group reduced to G' and comes from a unique principal G'-bundle.

Proof. Suppose E is a principal G-bundle and look at E/G' over X. The fibre of E/G' over X is G/G', a cell. Over a small closed set, say Z, the bundle E/G' has a section; so, by Theorem B our section extends to a global section (G/G' is a cell). Then, by Theorem 3.18, the bundle E comes from $H^1(X,G')$ and surjectivity is proved.

Now, say E and F are principal G'-bundles and that they become isomorphic as G-bundles. Take a common covering $\{U_\alpha\}$, where E and F are trivialized. Then $g_\alpha^\beta(E), g_\alpha^\beta(F)$, their transition functions become cohomologous in the G-bundle theory. This means that there exist maps, $h_\alpha\colon U_\alpha \to G$ so that

$$g_\alpha^\beta(F) = h_\beta^{-1} g_\alpha^\beta(E) h_\alpha^{-1}.$$

Consider $X \prod I$ where $I = [0,1]$ and cover $X \prod I$ by the opens

$$U_\alpha^0 = U_\alpha \prod [0,1) \quad \text{and} \quad U_\alpha^1 = U_\alpha \prod (0,1].$$

Make a principal bundle on $X \prod I$ using the following transition functions:

$$g_{\alpha\,0}^{\beta\,0}\colon U_\alpha^0 \cap U_\beta^0 \longrightarrow G$$

via $g_{\alpha\,0}^{\beta\,0}(x,t) = g_\alpha^\beta(E)(x)$;

$$g_{\alpha\,1}^{\beta\,1}\colon U_\alpha^1 \cap U_\beta^1 \longrightarrow G$$

via $g_{\alpha\,1}^{\beta\,1}(x,t) = g_\alpha^\beta(F)(x)$;

$$g_{\alpha\,0}^{\beta\,1}\colon U_\alpha^0 \cap U_\beta^1 \longrightarrow G$$

via $g_{\alpha\,0}^{\beta\,1}(x,t) = h_\beta(x) g_\alpha^\beta(F)(x) = g_\alpha^\beta(E)(x) h_\alpha(x)$. Call this new bundle (E,F) and let

$$Z = X \prod \{0\} \cup X \prod \{1\} \hookrightarrow X \prod I$$

a closed subset. Note that (E,F) over Z is a G'-bundle. Thus, Theorem 3.18 says $(E,F)/G'$ has a global section over Z. But, its fibre is G/G', a cell. Therefore, by Theorem B, the bundle $(E,F)/G'$ has a global section over all of X. By Theorem 3.18, again, the bundle (E,F) comes from a G'-bundle, $\widetilde{(E,F)}$. Write $f_0\colon X \to X \prod I$ for the function given by

$$f_0(x) = (x,0)$$

and $f_1\colon X \to X \prod I$ for the function given by

$$f_1(x) = (x,1).$$

If $\widetilde{(E,F)} \restriction X \prod \{0\} = \widetilde{(E,F)}_0$, then $f_0^*(\widetilde{(E,F)}_0) = E$, i.e., $f_0^*\widetilde{(E,F)} = E$ and similarly, $f_1^*\widetilde{(E,F)} = F$; and f_0 is homotopic to f_1. By Theorem A, we get $E \cong F$ as G'-bundles. $\square$

There is a theorem of Steenrod stating: If X is a differentiable manifold and E is a fibre bundle over X, then every continuous section of E may be approximated (with arbitrary ϵ) on compact subsets of X by a smooth section. When E is a vector bundle, this is easy to prove by an argument involving a partition of unity and approximation techniques using convolution. This proves

Theorem 3.22 *(Theorem D) If X is a differentiable manifold and G is a Lie group, then the map*

$$H^1_{\text{diff}}(X,G) \longrightarrow H^1_{\text{cont}}(X,G)$$

is a bijection.

We get the

Corollary 3.23 *If X is a differentiable manifold, then in the diagram below, for the following pairs (G',G)*

(α) $G' = \mathrm{U}(q)$, $G = \mathrm{GL}(q,\mathbb{C})$.

(β) $G' = \mathrm{U}(r) \prod \mathrm{U}(q-r)$, $G = \mathrm{GL}(r,q-r;\mathbb{C})$ or $G = \mathrm{GL}(r,\mathbb{C}) \prod \mathrm{GL}(q-r,\mathbb{C})$.

(γ) *$G' = \mathbb{T}^q = S^1 \times \cdots \times S^1$ (the real q-torus), $G = \Delta(q, \mathbb{C})$ or $G = \mathbb{G}_m \prod \cdots \prod \mathbb{G}_m = \mathbb{C}^* \prod \cdots \prod \mathbb{C}^*$ ($= \mathrm{GL}(1, \mathbb{C}) \prod \cdots \prod \mathrm{GL}(1, \mathbb{C})$) (the complex q-torus)*

all the maps are bijective

$$\begin{array}{ccc} H^1_{\mathrm{cont}}(X, G') & \longrightarrow & H^1_{\mathrm{cont}}(X, G) \\ \uparrow & & \uparrow \\ H^1_{\mathrm{diff}}(X, G') & \longrightarrow & H^1_{\mathrm{diff}}(X, G). \end{array}$$

Here,

$$\Delta(q, \mathbb{C}) = \bigcap_{r=1}^{q} \mathrm{GL}(r, q-r; \mathbb{C})$$

the upper triangular matrices.

Proof. Observe that G/G' is a cell in all cases and that $\Delta(q, \mathbb{C}) \cap \mathrm{U}(q) = \mathbb{T}^q$. $\square$

Suppose ξ corresponds to a $\mathrm{GL}(q)$-bundle which has group reduced to $\mathrm{GL}(r, q-r; \mathbb{C})$. Then, the maps

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mapsto A \quad \text{and} \quad M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mapsto C$$

give surjections $\mathrm{GL}(r, q-r; \mathbb{C}) \longrightarrow \mathrm{GL}(r, \mathbb{C})$ and $\mathrm{GL}(r, q-r; \mathbb{C}) \longrightarrow \mathrm{GL}(q-r, \mathbb{C})$, so ξ comes from $\widetilde{\xi}$ and $\widetilde{\xi}$ gives rise to ξ' and ξ'' which are $\mathrm{GL}(r, \mathbb{C})$ and $\mathrm{GL}(q-r, \mathbb{C})$-bundles, respectively. In this case one says: the $\mathrm{GL}(q, \mathbb{C})$-bundle ξ *admits a reduction to a (rank r) subbundle ξ' and a (rank $q-r$) quotient bundle ξ''*. When we use $\Delta(q, \mathbb{C})$ and $\mathrm{GL}(q, \mathbb{C})$ then we get q maps, $\varphi_l \colon \Delta(q, \mathbb{C}) \to \mathbb{C}^*$, namely

$$\varphi_j \colon \begin{pmatrix} a_1 & * & \cdots & * & * \\ 0 & a_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{q-1} & * \\ 0 & 0 & \cdots & 0 & a_q \end{pmatrix} \mapsto a_l.$$

So, if $\widetilde{\xi}$ is our $\Delta(q, \mathbb{C})$-bundle, we get q *line bundles* $\xi_1, \ldots, \xi_q$ from $\widetilde{\xi}$ and one says ξ has $\xi_1, \ldots, \xi_q$ *as diagonal line bundles*.

Set

$$\mathbb{F}_q = \mathrm{GL}(q; \mathbb{C})/\Delta(q; \mathbb{C}) = \mathrm{GL}(q; \mathbb{C})/\bigcap_{r=1}^{q} \mathrm{GL}(r, q-r; \mathbb{C}),$$

the *flag manifold*, i.e., the set of all flags

$$\{0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_q = V \mid \dim(V_j) = j\}.$$

Since $\mathbb{F}_q = \mathrm{GL}(q; \mathbb{C})/\bigcap_{r=1}^{q} \mathrm{GL}(r, q-r; \mathbb{C})$, we see that $\mathbb{F}_q$ is embedded in $\prod_{r=1}^{1} \mathcal{G}rass(r, q; \mathbb{C})$. Thus, as the above is a closed immersion, $\mathbb{F}_q$ is an algebraic variety, even a projective variety (by Segre). If V is a rank q vector bundle over X, say $E(V)$ ($\cong \mathrm{Isom}(\mathbb{C}^q, V)$) is the associated principal bundle, then write

$$[r]V = E(V)/\mathrm{GL}(r, q-r; \mathbb{C}),$$

a bundle over X whose fibres are $\mathcal{G}rass(r, q; \mathbb{C})$ and

$$[\Delta]V = E(V)/\Delta(q; \mathbb{C})$$

a bundle over X whose fibres are the $\mathbb{F}(q)$'s. We have maps $\rho_r \colon [r]V \to X$ and $\rho_\Delta \colon [\Delta]V \to X$. Now we apply our theorems to the pairs

(a) $G' = \mathrm{U}(q)$, $G = \mathrm{GL}(q, \mathbb{C})$.

(b) $G' = \mathrm{U}(r) \prod \mathrm{U}(q-r)$ and $G = \mathrm{GL}(r, q-r, \mathbb{C})$ or $G = \mathrm{GL}(r, \mathbb{C}) \prod \mathrm{GL}(q-r, \mathbb{C})$.

(c) $G' = \mathbb{T}^q$ and $G = \mathrm{U}(q)$ or $G = \mathbb{C}^* \prod \cdots \prod \mathbb{C}^* = (\mathbb{G}_m)^q$.

(d) $G' = \Delta(q, \mathbb{C})$ and $G = \mathrm{GL}(q, \mathbb{C})$

and then we get, (for example) every rank r vector bundle over X is "actually" a rank r unitary bundle over X and we also have

Theorem 3.24 *If X is paracompact or a differentiable manifold or a complex analytic manifold or an algebraic variety and V is a rank q vector bundle of the appropriate category on X, then*

(1) V reduces to a rank r subbundle, V', and rank $q-r$ quotient bundle, V'', over X iff $[r]V$ possesses an appropriate global section over X.

(2) V reduces to diagonal bundles over X iff $[\Delta]V$ possesses an appropriate global section over X.

(3) For the maps ρ_r in case (1), resp. ρ_Δ in case (2), the bundle $\rho_r^* V$ reduces to a rank r subbundle and rank $q-r$ quotient bundle over $[r]V$ (resp. reduces to diagonal bundles over $[\Delta]V$).

Remark: The sub, quotient, diagonal bundles are continuous, differentiable, analytic, algebraic, respectively.

Say $s\colon X \to [r]V$ is a global section. For every $x \in X$, we have $sx \in \mathcal{G}rass(r, q; V_x)$; i.e., $s(x)$ is an r-plane in V_v and so, $\bigcup_{x\in X} s(x)$ gives an "honest" rank r subbundle or V. It is isomorphic to the subbundle, V', of the reduction. Similarly, $\bigcup_{x\in X} V_x/s(x)$ is an "honest" rank $q-r$ quotient bundle of V; it is just V''.

Hence, we get

Corollary 3.25 *If the hypotheses of the previous theorem hold, then*

(1) $[r]V$ has a section iff there is an exact sequence

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

of vector bundles on X.

(2) $[\Delta]V]$ has a section iff there exist exact sequences

$$\begin{array}{l}
0 \longrightarrow L_1 \longrightarrow V \longrightarrow V_1'' \longrightarrow 0 \\
0 \longrightarrow L_2 \longrightarrow V_1'' \longrightarrow V_2'' \longrightarrow 0 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
0 \longrightarrow L_{j+1} \longrightarrow V_j'' \longrightarrow V_{j+1}'' \longrightarrow 0 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
\qquad\qquad L_q \cong V_{q-1}''
\end{array}$$

where the L_j's are line bundles, in fact, the diagonal bundles.

Theorem 3.26 *In the continuous and differentiable categories, when V has an exact sequence as in (1) of Corollary 3.25 or diagonal bundles as in (2) of Corollary 3.25, then*

(1) $V \cong V' \amalg V''$.

(2) $V \cong L_1 \amalg \cdots \amalg L_q$.

The above is false if we need splitting analytically!

All we need to prove is (1) as (2) follows by induction. We know V comes from $H^1(X, \mathrm{GL}(r, q-r; \mathbb{C})$. By (b) above, V comes from $H^1(X, \mathrm{U}(r) \prod \mathrm{U}(q-r))$ and by (b) again, V comes from $H^1(X, \mathrm{GL}(r) \prod \mathrm{GL}(q-r)) \cong H^1(X, \mathrm{GL}(r)) \amalg H^1(X, \mathrm{GL}(q-r))$ and we get (1). $\square$

Corollary 3.27 *(Splitting Principle) Given V, a continuous, differentiable, analytic, algebraic rank q vector bundle over X, then $\rho_r^* V$ is in the continuous or differentiable category a coproduct $V = V' \amalg V''$ ($\mathrm{rk}(V') = r$, $\mathrm{rk}(V'') = q - r$) or $\rho_\Delta^* V$ is $V = L_1 \amalg \cdots \amalg L_q$.*

Note that $[r]V$ and $[\Delta]V$ are fibre bundles over X; consequently, there is a relation between $H^j(X, \mathbb{Z})$ and $H^j([r]V, \mathbb{Z})$ (resp. $H^j([\Delta]V, \mathbb{Z})$. This is the *Borel spectral sequence*. Under the condition that (E, X, F, G) is a fibre space over X (admissible), group G, fibre F, total space E, there is a spectral sequence whose $E_2^{p,q}$-term is

$$H^p(X, H^q(F, \mathbb{Z}))$$

and whose ending is $\mathrm{gr}(H^\bullet(E, \mathbb{Z}))$,

$$H^p(X, H^q(F, \mathbb{Z})) \underset{p}{\Longrightarrow} H^\bullet(E, \mathbb{Z}).$$

Borel proves that in our situation: The map

$$\rho^* \colon H^\bullet(X, \mathbb{Z}) \to H^\bullet([r]V, \mathbb{Z})$$

(resp. $\rho^* \colon H^\bullet(X, \mathbb{Z}) \to H^\bullet([\Delta]V, \mathbb{Z})$) is an injection. From the hand-out, we also get the following: Write

$$\mathrm{BU}(q) = \varinjlim_N \mathcal{G}rass(q, N; \mathbb{C}).$$

Note,

$$\mathrm{BU}(1) = \varinjlim_N \mathbb{P}_{\mathbb{C}}^{N-1} = \mathbb{P}_{\mathbb{C}}^{\infty}.$$

Theorem 3.28 *If X is admissible (locally compact, σ-compact, finite dimensional) then $\mathrm{Vect}_q(X)$ (isomorphism classes of rank q vector bundles over X) in the continuous or differentiable category is in one-to-one correspondence with homotopy classes of maps $X \longrightarrow \mathrm{BU}(q)$. In fact, if X is compact and $N \geq 2\mathrm{dim}(X)$ then already the homotopy classes of maps $X \longrightarrow \mathcal{G}rass(q, N; \mathbb{C})$ classify rank q vector bundles on X (differentiably). Moreover, on $\mathrm{BU}(q)$, there exists a bundle, the "universal quotient", W_q, it has rank q over $\mathrm{BU}(q)$ (in fact, it is algebraic) so that the map is*

$$f \in [X \longrightarrow \mathrm{BU}(q)] \mapsto f^* W_q.$$

We are now in the position where we can prove the uniqueness of Chern classes.

Uniqueness of Chern Classes:

Assume existence (Axiom (I)) and good behavior (Axioms (II)–(IV)). First, take a line bundle, L, on X. By the classification theorem there is a map

$$f \colon X \to \mathrm{BU}(1)$$

so that $f^*(H) = L$ (here, H is the universal quotient line bundle). By Axiom (II),

$$f^*(c(H)(t)) = c(f^*(H))(t) = c(L)(t)$$

and the left hand side is $f^*(1 + Ht)$, by Axiom (IV) (viewing H as a cohomology class). It follows that the left hand side is $1 + f^*(H)t$ and so,

$$c_1(L) = f^*(H), \quad \text{and} \quad c_j(L) = 0, \quad \text{for all} \quad j \geq 2.$$

This is independent of f as homotopic maps agree cohomologically.

Now, let V be a rank q vector bundle on X and make the bundle $[\Delta]V$ whose fibre is $\mathbb{F}(q)$. Take $\rho^*(V)$, where $\rho\colon [\Delta]V \to X$. We know

$$\rho^* V = \prod_{j=1}^{q} L_j,$$

where the L_j's are line bundes and by Axiom (II),

$$c(\rho^*(V))(t) = \prod_{j=1}^{1}(1 + c_1(L_j)(t)).$$

Now, the left hand side is $\rho^*(c(V)(t))$, by Axiom (II); then, ρ^* being an injection implies $c(V)(t)$ is uniquely determined.

Remark: Look at $\mathrm{U}(q) \supseteq \mathrm{U}(1) \prod \mathrm{U}(q-1) \supseteq \mathbb{T}^q$. Then,

$$\mathrm{U}(1) \prod \mathrm{U}(q-1)/\mathbb{T}^q \hookrightarrow \mathrm{U}(q)/\mathbb{T}^q = \mathbb{F}(q)$$

and the left hand side is $\mathrm{U}(q-1)/\mathbb{T}^{q-1} = \mathbb{F}(q-1)$. So, we have an injection $\mathbb{F}(q-1) \hookrightarrow \mathbb{F}(q)$ over the base $\mathrm{U}(q)/\mathrm{U}(1) \prod \mathrm{U}(q-1)$, which is just $\mathbb{P}^{q-1}$. Thus, we can view $\mathbb{F}(q)$ as a fibre bundle over $\mathbb{P}^{q-1}$ and the fibre is $\mathbb{F}(q-1)$.

Take a principal $\mathrm{U}(q)$-bundle, E, over X and make $E/\mathbb{T}^q$, a fibre space whose fibre is $\mathbb{F}(q)$. Let E_1 be $E/\mathrm{U}(1) \prod \mathrm{U}(q-1)$, a fibre space whose fibre is $\mathbb{P}^{q-1}$. Then, we have a map

$$E/\mathbb{T}^q \longrightarrow E_1,$$

where the fibre is $\mathrm{U}(1) \prod \mathrm{U}(q-1)/\mathbb{T}^q = \mathbb{F}(q-1)$. We get

$$\begin{array}{rc} & E/\mathbb{T}^q = [\Delta]E \\ \text{fibre } \mathbb{F}(q-1) & \big\downarrow \\ & E_1 \\ \text{fibre } \mathbb{P}^{q-1} & \big\downarrow \\ & X. \end{array}$$

If we repeat this process, we get the tower

$$\begin{array}{rcl} & E/\mathbb{T}^q = [\Delta]E & \\ \text{fibre } \mathbb{P}^1 & \big\downarrow & \\ & E_{q-1} & \\ \text{fibre } \mathbb{P}^2 & \big\downarrow & \\ & E_{q-2} & \rho \\ & \vdots & \\ & E_1 & \\ \text{fibre } \mathbb{P}^{q-1} & \big\downarrow & \\ & X. & \end{array}$$

So, to show ρ^* is injective, all we need to show is the same fact when the fibre i $\mathbb{P}^n$ and the $\mathbb{P}^r$-bundle comes from a vector bundle.

Suggestion: Look in Hartshorne in Chapter III, Section ? on projective fibre bundles and Exercise ?? about

$$\rho^*(\mathcal{O}_{\mathbb{P}(E)}(l)) = \mathcal{S}^l(\mathcal{O}_X(E)).$$

Sup up to tangent bundles and wedges and use Hodge:

$$H^\bullet(X, \mathbb{C}) = \text{in term of the holomorphic cohomology of } \bigwedge^{\text{top}} T.$$

We get that ρ^* is injective on $H^\bullet(X, \mathbb{C})$, not $H^\bullet(X, \mathbb{Z})$.

Existence of Chern Classes:

Start with L, a line bundle over X. Then, there is a map (continuous, diff.), $f\colon X \to \mathbb{P}^N_{\mathbb{C}}$, for $N >> 0$ and $L = f^*(H)$. Then, set $c_1(L) = f^*(H)$, where H is the cohomology class of the hyperplane bundle in $H^2(\mathbb{P}^N, \mathbb{Z})$ and $c_j(L) = 0$ if $j \geq 2$. If another map, g, is used, then $f^*(H) = L = h^*(L)$ implies that f and g are homotopic, so f^* and g^* agree on cohomology and $c_1(L)$ is independent of f. It is also independent of N, we we already proved. Clearly, Axiom (II) and Axiom (IV) are built in.

Now, let V be a rank q vector bundle over X. Make $[\Delta]V$ and let ρ be the map $\rho\colon [\Delta]V \to X$. Look at ρ^*V. We know that

$$\rho^*V = \coprod_{j=1}^{q} L_j,$$

where the L_j's are line bundles. By the above,

$$c_j(L_j)(t) = 1 + c_1(L_j)t = 1 + \gamma_j t.$$

Look at the polynomial

$$\prod_{j=1}^{q}(1 + \gamma_j t) \in H^\bullet([\Delta]V, \mathbb{Z})[t].$$

If we show this polynomial (whose coefficients are the symmetric functions $\sigma_l(\gamma_1, \ldots, \gamma_q)$) is in the image of $\rho^*\colon H^\bullet(X, \mathbb{Z})[t] \longrightarrow H^\bullet([\Delta]V, \mathbb{Z})[t]$, then there is a unique polynomial $c(V)(t)$ so that

$$\rho^*(c(V)(t)) = \prod_{j=1}^{q}(1 + \gamma_j t).$$

(Then, $c_l(V) = \sigma_l(\gamma_1, \ldots, \gamma_q)$.) Look at the normalizer of $\mathbb{T}^q$ in $\mathrm{U}(q)$. Some a belongs to this normalizer iff $a\mathbb{T}^q a^{-1} = \mathbb{T}^q$. As the new diagonal matrix, $a\theta a^{-1}$ (where $a \in \mathbb{T}^q$ has the same characteristic polynomial as θ, it follows that $a\theta a^{-1}$ is just θ, but with its diagonal entries permuted. This gives a map

$$\mathcal{N}_{\mathrm{U}(q)}(\mathbb{T}^q) \longrightarrow \mathfrak{S}_q.$$

What is the kernel of this map? We have $a \in \mathrm{Ker}$ iff $a\theta a^{-1} = \theta$, i.e., $a\theta = \theta a$, for all $\theta \in \mathbb{T}^q$. This means (see the 2×2 case) $a \in \mathbb{T}^q$ and thus, we have an injection

$$\mathcal{N}_{\mathrm{U}(q)}(\mathbb{T}^q)/\mathbb{T}^q \hookrightarrow \mathfrak{S}_q.$$

The left hand side, by definition, is the *Weyl group*, W, of $\mathrm{U}(q)$. In fact (easy DX), $W \cong \mathfrak{S}_q$.

Look at $[\Delta]V$ and write a covering of X trivializing $[\Delta]V$, call it $\{U_\alpha\}$. We have

$$[\Delta]V \restriction U_\alpha \cong U_\alpha \coprod \mathrm{U}(q)/\mathbb{T}^q.$$

Make the element a act on the latter *via*

$$a(u, \xi\mathbb{T}^q) = (u, \xi\mathbb{T}^q a^{-1}) = (u, \xi a^{-1}\mathbb{T}^q).$$

These patch as the transition functions are *left* translations. This gives an automorphism of $[\Delta]V$, call it $\widetilde{a}$, determined by each $a \in W$. We get a map

$$\widetilde{a}^* \colon H^\bullet([\Delta]V, -) \to H^\bullet([\Delta]V, -).$$

Now, as $a \in W$ acts on $\mathbb{T}^q$ by permuting the diagonal elements it acts on $H^1([\Delta]V, \mathbb{T}^q)$ by permuting the diagonal bundles, say L_j, call this action $a^{\#}$. Moreover, $\rho^* V$ comes from a unique element of $H^1([\Delta]V, \mathbb{T}^q)$, which implies that $\widetilde{a}$ acts on $\rho^* V$ by permuting its cofactors. But, $\widetilde{a}^*$ also acts on $H^1([\Delta]V, \mathbb{T}^q)$ and one should check (by a Čech cohomology argument) that

$$\widetilde{a}^* = a^{\#}.$$

Now associate to the L_j's their Chern classes, γ_j, and $\widetilde{a}^*(\gamma_j)$ goes over to $a^{\#}(\gamma_j)$, i.e., permute the $|gamma_j$s's. Thus, W acts on the L_j and γ_j by permuting them. Our polynomial

$$\prod_{j=1}(1 + \gamma_j t)$$

goes to itself *via* the action of W. But, Borel's Theorem is that an element of $H^\bullet([\Delta]V, \mathbb{Z})$ lies in the image of $\rho^* \colon H^\bullet(X, \mathbb{Z}) \to H^\bullet([\Delta]V, \mathbb{Z})$ iff W fixes it. So, by the above, our elementary symmetric functions lie in $\mathrm{Im}\, \rho^*$; so, Chern classes exist. Furthermore, it is clear that they satisfy Axioms (I), (II), (IV).

Finally, consider Axiom (III). Suppose V splits over X as

$$V = \coprod_{j=1}^{q} L_j.$$

We need to show that $c(V)(t) = \prod_{j=1}^{1}(1 + c_1(L_j)t)$.

As V splits over X, the fibre bundle $\rho \colon [\Delta]V \longrightarrow X$ has a section; call it s. So, $s^*\rho^* = \mathrm{id}$ and

$$c(V)(t) = s^*\rho^*(c(V)(t)) = s^*(\rho^*(c(V)(t))).$$

By Axiom (II), $s^*(\rho^*(c(V)(t))) = s^*(c(\rho^*(V))(t))$. Since $\rho^* = \coprod_{j=1}^{q} \rho^* L_j$ and we know that if we set $\gamma_j = c_1(\rho^*(L_j))$, then

$$\rho^*(c(V)(t)) = c(\rho^*(V)(t)) = \prod_{j=1}^{q}(1 + \gamma_j t).$$

But then,

$$c(V)(t) = s^* \prod_{j=1}^{q}(1 + \gamma_j t) = \prod_{j=1}^{q}(1 + s^*(\gamma_j)t). \tag{†}$$

However, $L_j = s^*(\rho^*(L_j))$ implies

$$c_1(L_j) = s^*(c_1(\rho^*(L_j))) = s^*(\gamma_j).$$

The above plus (†) yields

$$c(V)(t) = \prod_{j=1}^{q}(1 + c_1(L_j)t),$$

as required. $\square$

Eine kleine Vektorraumbündel Theorie:

Say V (rank q) and W (rank q') have diagonal bundles $L_1, \ldots, L_q$ and $M_1, \ldots, M_{q'}$ over X. Then, the following hold:

(1) V^D has $L_1^D, \ldots, L^D$ as diagonal line bundles;

(2) $V \amalg W$ has $L_1, \ldots, L_q, M_1, \ldots, M_{q'}$ as diagonal line bundles;

(3) $V \otimes W$ has $L_i \otimes M_j$ (all i, j) as diagonal line bundles;

(4) $\bigwedge^r V$ has $L_{i_1} \otimes \cdots \otimes L_{i_r}$, where $1 \leq i_1 < \cdots < i_r \leq q$, as diagonal line bundles;

(4) $\mathcal{S}^r V$ has $L_1^{m_1} \otimes \cdots \otimes L_q^{m_q}$, where $m_i \geq 0$ and $m_1 + \cdots + m_q = r$, as diagonal line bundles.

Application to the Chern Classes.

(0) (Splitting Principle) Given a rank q vector bundle, V, make believe V splits as $V = \coprod_{j=1}^q L_j$ (for some line bundles, L_j), write $\gamma_j = c_1(L_j)$, the γ_j are the *Chern roots* of V. Then,

$$c(V)(t) = \prod_{j=1}^q (1 + \gamma_j t).$$

(1) $c(V^D)(t) = \prod_{j=1}^q (1 - \gamma_j t)$ when $c(V)(t) = \prod_{j=1}^q (1 + \gamma_j t)$. That is, $c_i(V^D) = (-1)^i c_i(V)$.

(2) If $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ is exact, then $c(V)(t) = c(V')(t)c(V'')(t)$.

(3) If $c(V)(t) = \prod_{j=1}^q (1 + \gamma_j t)$ and $c(W)(t) = \prod_{j=1}^{q'} (1 + \delta_j t)$, then $c(V \otimes W)(t) = \prod_{j,k=1}^{q,q'} (1 + (\gamma_j + \delta_k)t)$.

(4) If $c(V)(t) = \prod_{j=1}^q (1 + \gamma_j t)$, then

$$c\Big(\bigwedge^r V\Big)(t) = \prod_{1 \leq i_1 < \cdots < i_r \leq q} (1 + (\gamma_{i_1} + \cdots + \gamma_{i_r})t).$$

In particular, when $r = q$, there is just one factor in the polynomial, it has degree 1, it is $1 + (\gamma_1 + \cdots + \gamma_q)t$. By (2). we get

$$c_1\Big(\bigwedge^q V\Big)(t) = c_1(V) \quad \text{and} \quad c_l\Big(\bigwedge^q V\Big)(t) = 0 \quad \text{if} \quad l \geq 2.$$

(5) If $c(V)(t) = \prod_{j=1}^q (1 + \gamma_j t)$, then

$$c(\mathcal{S}^r V)(t) = \prod_{\substack{m_j \geq 0 \\ m_1 + \cdots + m_q = r}} (1 + (m_1\gamma_1 + \cdots + m_q\gamma_q)t).$$

(6) If $\mathrm{rk}(V) \leq q$, then $\deg(c(V)(t)) \leq q$ (where $\deg(c(V)(t)$ is the degree of $c(V)(t)$ as a polynomial in t).

(7) Suppose we know $c(V)$, for some vector bundle, V, and L is a line bundle. Write $c = c_1(L)$. Then, the Chern classes of $V \otimes L$ are

$$c_l(V \otimes L) = \sigma_l(\gamma_1 + c, \gamma_2 + c, \cdots, \gamma_r + c),$$

where $r = \mathrm{rk}(V)$ and the γ_j are the Chern roots of V. This is because the Chern polynomial of $V \otimes L$ is

$$c(V \otimes L)(t) = \prod_{i=1}^{r}(1 + (\gamma_i + c)t).$$

Examples. (1) If $\mathrm{rk}(V) = 2$, then

$$c(V \otimes L)(t) = (1 + (\gamma_1 + c)t)(1 + (\gamma_2 + c)t) = 1 + (\gamma_1 + \gamma_2 + 2c)t + (\gamma_1\gamma_2 + c(\gamma_1 + \gamma_2) + c^2)t^2,$$

so

$$\begin{aligned} c_1(V \otimes L) &= c_1(V) + 2c \\ c_2(V \otimes L) &= c_2(V) + c_1(V)c + c^2. \end{aligned}$$

(2) If $\mathrm{rk}(V) = 3$, then

$$c(V \otimes L)(t) = (1 + (\gamma_1 + c)t)(1 + (\gamma_2 + c)t)(1 + (\gamma_3 + c)t)$$

and so,

$$\begin{aligned} c(V \otimes L)(t) &= 1 + (\gamma_1 + \gamma_2 + \gamma_3 + 3c)t \\ &\quad + (\sigma_2(\gamma_1, \gamma_2, \gamma_3) + 2\sigma_1(\gamma_1, \gamma_2, \gamma_3)c + 3c^2)t^2 \\ &\quad + (\sigma_3(\gamma_1, \gamma_2, \gamma_3) + \sigma_1(\gamma_1, \gamma_2, \gamma_3)c^2 + \sigma_2(\gamma_1, \gamma_2, \gamma_3)c + c^3)t^3. \end{aligned}$$

We deduce

$$\begin{aligned} c_1(V \otimes L) &= c_1(V) + 3c_1(L) \\ c_2(V \otimes L) &= c_2(V) + 2c_1(V)c_1(L) + 3c_1(L)^2 \\ c_3(V \otimes L) &= c_3(V) + c_2(V)c_1(L) + c_1(V)c_1(L)^2 + c_1(L)^3. \end{aligned}$$

In the case of $\mathbb{P}^n$, it is easy to compute the Chern classes. By definition,

$$c(\mathbb{P}^n)(t) = c(T^{1,0}_{\mathbb{P}^n})(t).$$

We can use the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \coprod_{n+1} \mathcal{O}_{\mathbb{P}^n}(H) \longrightarrow \mathcal{T}^{1,0}_{\mathbb{P}^n} \longrightarrow 0$$

to deduce that

$$c(\mathcal{O}_{\mathbb{P}^n})(t)c(T^{1,0}_{\mathbb{P}^n})(t) = c(\mathcal{O}_{\mathbb{P}^n}(H)(t))^{n+1}.$$

It follows that

$$c(T^{1,0}_{\mathbb{P}^n})(t) = (1 + Ht)^{n+1} \pmod{t^{n+1}} = \sum_{j=0}^{n} \binom{n+1}{j} H^j t^j$$

and so,

$$c_j(T^{1,0}_{\mathbb{P}^n}) = \binom{n+1}{j} H^j \in H^{2j}(\mathbb{P}^n, \mathbb{Z}).$$

(Here $H^j = H \cdot \ldots \cdot H$, the cup-product in cohomology). In particular,

$$c_1(T^{1,0}_{\mathbb{P}^n}) = (n+1)H = c\Big(\bigwedge^n T^{1,0}_{\mathbb{P}^n}\Big).$$

Now, if $\omega_{\mathbb{P}^n}$ is the canonical bundle on $\mathbb{P}^n$, i.e., $\omega_{\mathbb{P}^n} = \bigwedge^n T^{0,1\,D}_{\mathbb{P}^n} = \Big(\bigwedge^n T^{1,0}_{\mathbb{P}^n}\Big)^D$, we get

$$c_1(\omega_{\mathbb{P}^n}) = -(n+1)H.$$

Say a variety X sits inside $\mathbb{P}^n_{\mathbb{C}}$ and assume X is a manifold. Let $\mathfrak{I}$ be the ideal sheaf of X. By definition, $\mathfrak{I}$ is the kernel in the exact sequence

$$0 \longrightarrow \mathfrak{I} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

If X is a hypersurface of degree d, we know

$$\mathfrak{I} = \mathcal{O}_{\mathbb{P}^n}(-d) = \mathcal{O}_{\mathbb{P}^n}(-dH).$$

We also have the exact sequence

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^n} \restriction X \longrightarrow \mathcal{N}_{X \hookrightarrow \mathbb{P}^n} \longrightarrow 0,$$

where $\mathcal{N}_{X \hookrightarrow \mathbb{P}^n}$ is a rank $n-q$ bundle on X, with $q = \dim X$ (the *normal bundle*). If we write $i\colon X \to \mathbb{P}^n$, we get

$$\Big(\bigwedge^n T_{\mathbb{P}^n}\Big) \restriction X = \bigwedge^n T_X \otimes \bigwedge^{n-q} \mathcal{N}_{X \hookrightarrow \mathbb{P}^n},$$

and so,

$$i^*(1 + c_1\Big(\bigwedge^n T_{\mathbb{P}^n}\Big)t) = (1 + c_1\Big(\bigwedge^n T_X\Big)t)(1 + c_1\Big(\bigwedge^{n-q} \mathcal{N}_{X \hookrightarrow \mathbb{P}^n}\Big)t),$$

which yields

$$1 + i^*((n+1)H)t = 1 + c_1(T_X)t + c_1(\mathcal{N}_{X \hookrightarrow \mathbb{P}^n})t.$$

For the normal bundle, we can compute using $\mathfrak{I}$. Look at a small open, then we have the usual case of $\mathbb{C}$-algebras

$$\mathbb{C} \hookrightarrow A \longrightarrow B$$

where A corresponds to local functions on $\mathbb{P}^n$ restricted to X and B to local functions on X and we have the exact sequence of relative Kähler differentials

$$\Omega^1_{A/C} \otimes_A B \longrightarrow \Omega^1_{B/C} \longrightarrow \Omega^1_{B/A} \longrightarrow 0.$$

If A mapping onto B is given, then $\Omega^1_{B/A} = (0)$, $B = A/\mathfrak{A}$ (globally, $\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n}/\mathfrak{I}$), and we get

$$0 \longrightarrow \mathrm{Ker} \longrightarrow \Omega^1_A \otimes_A A/\mathfrak{A} \longrightarrow \Omega^1_{A/\mathfrak{A}} \longrightarrow 0.$$

Now, $\mathfrak{I} \longrightarrow \Omega^1_A \otimes_A A/\mathfrak{A}$, *via* $\xi d\xi \mapsto \otimes 1$ and in fact, $\mathfrak{I} \longrightarrow 0$. We conclude that

$$i^*(\mathfrak{I}) = \mathfrak{I}/\mathfrak{I}^2 \longrightarrow i^*(\Omega^1_{\mathbb{P}^n}) \longrightarrow \Omega^1_X \longrightarrow 0.$$

Because X is a manifold, the arrow on the left is an injection. To see this we need only look locally at x. We can take completions and then use either the C^1-implicit function theorem or the holomorphic implicit function theorem or the formal implicit function theorem and get the result (DX). If we dualize, from

$$0 \longrightarrow \mathfrak{I}/\mathfrak{I}^2 = i^*(\mathfrak{I}) \longrightarrow i^*\Omega^1_{\mathbb{P}^n} \longrightarrow \Omega^1_X \longrightarrow 0$$

we get

$$0 \longrightarrow T_X \longrightarrow i^* T_{\mathbb{P}^n} = T_{\mathbb{P}^n} \upharpoonright X \longrightarrow (\mathfrak{I}/\mathfrak{I}^2)^D \longrightarrow 0$$

Therefore,

$$\mathcal{N}_{X \hookrightarrow \mathbb{P}^n} = (\mathfrak{I}/\mathfrak{I}^2)^D = i^*(\mathfrak{I})^D = (\mathfrak{I} \upharpoonright X)^D.$$

Thus,

$$c_1(\mathcal{N}_{X \hookrightarrow \mathbb{P}^n}) = -c_1(\mathfrak{I}/\mathfrak{I}^2),$$

and

$$(n+1)i^*(H) + c_1(\mathfrak{I}/\mathfrak{I}^2) = c_1(T_X).$$

We obtain a version of the *adjunction formula*:

$$c_1(\omega_X) = -(n+1)i^*(H) - c_1(\mathfrak{I}/\mathfrak{I}^2).$$

When X is a hypersurface of degree d, then $\mathfrak{I} = \mathcal{O}_{\mathbb{P}^n}(-dH)$ and

$$\mathfrak{I}/\mathfrak{I}^2 = i^*(\mathfrak{I}) = \mathcal{O}_X(-d \cdot i^* H).$$

We deduce that $-c_1(\mathfrak{I}/\mathfrak{I}^2) = d(i^*H)$ and

$$c_1(\omega_X) = (d - n - i)i^* H,$$

Say $n = 2$, and $\dim X = 1$, a curve in $\mathbb{P}^2$. When X is smooth, we have

$$c_1(\omega_X) = (d - n - 1)i^*(H).$$

Facts soon to be proved:

(a) $i^*(H) = H \cdot X$, in the sense of intersection theory (that is, $\deg X$ points on X).

(b) $c_1(L)$ on a curve is equal to the degree of the divisor of L.

It follows from above that

$$\deg(\omega_X) = (d - 2 - 1)d = d(d - 3).$$

However, from Riemann-Roch on a curve, we know $\deg(\omega_X) = 2g - 2$, so we conclude that for a smooth algebraic curve, its genus, g, is given by

$$g = \frac{1}{2}(d-1)(d-2).$$

In particular, observe that $g = 2$ is missed.

We know from the theory that if we know all c_1's then we can determine all c_n's for all n by the splitting principle.

There are three general methods for determining c_1;

(I) The exponential sequence.

(II) Curvature.

(III) Degree of a divisor.

Proposition 3.29 *Say X is an admissible, or a differentiable manifold, or a complex analytic manifold or an algebraic manifold. In each case, write $\mathcal{O}_X$ for the sheaf of germs of appropriate functions on X. Then, from the exponential sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \stackrel{e}{\longrightarrow} \mathcal{O}_X^* \longrightarrow 0,$$

where $e(f) = \exp(2\pi i f)$, we get in each case the connecting map

$$H^1(X, \mathcal{O}_X^*) \stackrel{\delta}{\longrightarrow} H^2(X, \mathbb{Z}) \qquad (\dagger)$$

and all obvious diagrams commute

*** Steve, what are these obvious diagrams? ***

and as the group $H^1(X, \mathcal{O}_X^*)$ classifies the line bundles of appropriate type, we get $\delta(L)$, a cohomology class in $H^2(X, \mathbb{Z})$ and we have

$$c_1(L) = \delta(L).$$

In the continuous and differentiable case, δ is an isomorphism. Therefore, a continuous or differentiable line bundle is completely determined by its first Chern class.

Proof. That the diagrams commute is clear. For the isomorphism statement, we have the cohomology sequence

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \stackrel{\delta}{\longrightarrow} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X).$$

But, in the continuous or C^∞-case, $\mathcal{O}_X$ is a fine sheaf, so $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = (0)$ and we get

$$H^1(X, \mathcal{O}_X^*) \cong H^2(X, \mathbb{Z}).$$

First, we show that $(\dagger)$ can be reduced to the case $X = \mathbb{P}^1_{\mathbb{C}} = S^2$.

** Steve, in this case, are we assuming that X is projective? **

Take a line bundle, L on X (continuous or C^∞), then, for $N >> 0$, there is a function, $f\colon X \to \mathbb{P}^N_{\mathbb{C}}$, so that $f^*H = L$. Now, we have the diagram

$$\begin{array}{ccc} H^1(\mathbb{P}^N_{\mathbb{C}}, \mathcal{O}^*_{\mathbb{P}^N}) & \stackrel{\delta}{\longrightarrow} & H^2(\mathbb{P}^N_{\mathbb{C}}, \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^1(X, \mathcal{O}_X^*) & \underset{\delta}{\longrightarrow} & H^2(X, \mathbb{Z}) \end{array}$$

which commutes by cofunctoriality of cohomology. Consequently, the existence of $(\dagger)$ on the top line implies the existence of $(\dagger)$ in general. Now, consider the inclusions

$$\mathbb{P}^1_{\mathbb{C}} \hookrightarrow \mathbb{P}^2_{\mathbb{C}} \hookrightarrow \cdots \hookrightarrow \mathbb{P}^N_{\mathbb{C}},$$

and H on $\mathbb{P}^N_{\mathbb{C}}$ pulls back at each stage to H and Chern classes have Axiom (II). Then, one sees that we are reduced to $\mathbb{P}^1_{\mathbb{C}}$.

Recall how simplicial cohomology is isomorphic (naturaly) to Čech cohomology: Take a triangulation of X and v, a vertex of a simplex, Δ. Write

$$U_v = \mathrm{st}(v) = \overset{\circ}{\bigcup} \{\sigma \mid v \in \sigma\}$$

the open star of the vertex v. The U_σ form an open cover and we have:

$$U_{v_0} \cap \cdots \cap U_{v_p} = \begin{cases} \emptyset & \text{if } (v_0, \ldots, v_p) \text{ is not a simplex;} \\ \text{a connected nonempty set} & \text{if } (v_0, \ldots, v_p) \text{ is a simplex.} \end{cases}$$

Given a Čech p-cochain, τ, then

$$\tau(U_{v_0}\cap\cdots\cap U_{v_p}) = \begin{cases} 0 & \text{if } (v_0,\ldots,v_p) \text{ is not a simplex;}\\ \text{some integer} & \text{if } (v_0,\ldots,v_p) \text{ is a simplex.}\end{cases}$$

Define

$$\tau(v_0,\ldots,v_p) = \tau(U_{v_0}\cap\cdots\cap U_{v_p}).$$

Take a simplex, $\Delta = (v_0,\ldots,v_p)$ and define linear functions $\theta(\tau)$ by

$$\theta(\tau)(v_0,\ldots,v_p) = \tau(v_0,\ldots,v_p) = \tau(U_{v_0}\cap\cdots\cap U_{v_p})$$

and extend by linearity. We get a map,

$$\check{H}^p(X,\mathbb{Z}) \cong H^p_{\mathrm{simp}}(X,\mathbb{Z})$$

via $\tau\mapsto\theta(\tau)$, which is an isomorphism.

We are down to the case of $\mathbb{P}^1_{\mathbb{C}} = S^2$ and we take H as the North pole. The Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ has coordinates $(Z_0\colon Z_1)$, say $Z_1 = 0$ is the north pole ($Z_0 = 0$ is the south pole) and let

$$z = \frac{Z_0}{Z_1},\quad w = \frac{Z_1}{Z_0}.$$

We have the standard opens, $V_0 = \{(Z_0\colon Z_1)\mid Z_0\neq 0\}$ and $V_1 = \{(Z_0\colon Z_1)\mid Z_1\neq 0\}$. The local equations for H are $f_0 = w = 0$ and $f_1 = 1$. The transitions functions g^β_α are f_β/f_α, i.e.,

$$g^1_0 = \frac{f_1}{f_0} = z \quad\text{and}\quad g^0_1 = \frac{f_0}{f_1} = w.$$

Now, we triangulate S^2 using four triangles whose vertices are: $o = z$; $z = 1$; $z = i$ and $z = -1$. Note that H is represented by a point which is in the middle of a face of the simplex $(1, i, -1)$ We have U_0, U_1, U_i, U_{-1}, the four open stars and $U_0\subseteq V_1$; $U_1\subseteq V_0$; $U_i\subseteq V_0$; $U_{-1}\subseteq V_0$. The U-cover refined the V-cover and on it, $g^s_r\equiv 1$ iff both $r,s\neq 0$. Also, $g^t_0 = w$, for all $t\neq 0$. To lift back the exponential, $\mathcal{O}_{\mathbb{P}^1}\overset{\exp(2\pi i-)}{\longrightarrow}\mathcal{O}^*_{\mathbb{P}^1}$, we form $\frac{1}{2\pi i}\log(g^s_r)$, a one-cochain with values in $\mathcal{O}_{\mathbb{P}^1}$. Since the intersections $U_r\cap U_s$ are all simply-connected, on each, we can define a single-valued branch of the log. Consistently do this on these opens *via*: Start on $U_1\cap U_i$ and pick any single-valued branch of the log. Continue analytically to $U_i\cap U_{-1}$. Then, continue analytically to $U_{-1}\cap U_1$, we get $2\pi i + \log$ on $U_1\cap U_i$. Having defined the $\log g^s_r$, we take the Čech δ of the 1-cochain, that is

$$c_{rst} = \frac{1}{2\pi i}[\log g^t_s - \log g^t_r + \log g^s_r] = \frac{1}{2\pi i}[\log g^s_r + \log g^t_s + \log g^r_t].$$

If none of r,s,t are 0, then $c_{rst} = 0$. So, look at $c_{0\,-1\,1}$. We have

$$c_{0\,-1\,1} = \frac{1}{2\pi i}[\log g^{-1}_0 + \log g^1_{-1} + \log g^0_1] = \frac{1}{2\pi i}[\log w - \text{“}\log\text{''} w].$$

As $w = 1/z$, the second log is $-2\pi i + \log w$, so we get

$$c_{0\,-1\,1} = +1.$$

For every even permutation σ of $(0,-1,1)$, we have $c_{\sigma(0),\sigma(-1),\sigma(1)} = +1$ and for every odd permutation σ of $(0,-1,1)$, we have $c_{\sigma(0),\sigma(-1),\sigma(1)} = -1$. Yet, the orientation of the simplex $(0,-1,1)$ is positive, so we get $\delta(H) =$ the class represented by the cocycle on one simplex (positively oriented) by 1, i.e, $c_1(H)$. $\square$

Proposition 3.30 *Say X is a complex manifold and L is a C^∞ line bundle on it. Let ∇ be an arbitrary connection on X and write Θ for the curvature of Δ. Then, the 2-form $\frac{i}{2\pi}\Theta$ is real and it represents in $H^2_{\mathrm{DR}}(X, \mathbb{R})$ the image of $c_1(L)$ under the map*

$$H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{R}).$$

Proof. Pick a trivializing cover for L, say $\{U_\alpha\}$. Then, $\nabla \upharpoonright L$ on U_α comes from its connection matrix, θ_α, this is a 1×1 matrix (L is a line bundle). We know (gauge transformation)

$$\theta_\alpha = g^\alpha_\beta \theta_\beta (g^\alpha_\beta)^{-1} + dg^\alpha_\beta (g^\alpha_\beta)^{-1},$$

where the g^α_β are the transition functions. But, we have scalars here, so

$$\theta_\alpha = \theta_\beta + d\log(g^\alpha_\beta),$$

that is

$$\theta_\beta - \theta_\alpha = -d\log(g^\alpha_\beta). \qquad (\dagger)$$

By Cartan-Maurer, the curvature, Θ, (a 2-form) is given locally by

$$\Theta = d\theta - \theta \wedge \theta = d\theta_\alpha = d\theta_\beta.$$

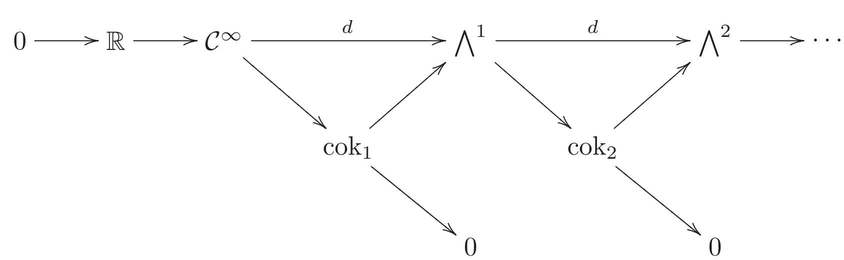
We get the de Rham isomorphism in the usual way by splicing exact sequences. We begin with

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{C}^\infty \stackrel{d}{\longrightarrow} \mathrm{cok}_1 \longrightarrow 0 \qquad (*)$$

and

$$0 \longrightarrow \mathrm{cok}_1 \longrightarrow \bigwedge^1 \stackrel{d}{\longrightarrow} \mathrm{cok}_2 \longrightarrow 0 \qquad (**)$$

It follows that



Apply cohomology to $(*)$ and $(**)$ and get

$$H^0(X, \bigwedge^1) \stackrel{d}{\longrightarrow} H^0(X, \mathrm{cok}_2) \stackrel{\delta'}{\longrightarrow} H^1(X, \mathrm{cok}_1) \longrightarrow H^1(X, \bigwedge^1) = (0)$$

and

$$H^1(X, \mathcal{C}^\infty) \longrightarrow H^1(X, \mathrm{cok}_1) \stackrel{\delta''}{\longrightarrow} H^2(X, \mathbb{R}) \longrightarrow H^2(X, C^\infty) = (0)$$

because $\bigwedge^1$ and $\mathcal{C}^\infty$ are fine. We get

$$H^1(X, \mathrm{cok}_1) \cong H^2(X, \mathbb{R}) \quad \text{and} \quad H^0(X, \mathrm{cok}_2)/dH^0(X, \bigwedge^1) \cong H^1(X, \mathrm{cok}_1).$$

Therefore,

$$\delta' \circ \delta' \colon H^0(X, \mathrm{cok}_2) \longrightarrow H^2(X, \mathbb{R}) \longrightarrow 0.$$

We know from the previous proof that

$$c_{\alpha,\beta,\gamma} = \frac{1}{2\pi i}[\log g_\alpha^\beta + \log g_\beta^\gamma + \log g_\gamma^\alpha]$$

represents $c_1(L)$ *via* the δ from the exponential sequence. So,

$$c_{\alpha,\beta,\gamma} = \frac{1}{2\pi i}[\log g_\beta^\alpha + \log g_\alpha^\gamma + \log g_\gamma^\beta]$$

and

$$\delta'[\Theta] = \text{cohomology class of } \Theta = \text{class of cocycle } (\theta_\beta - \theta_\alpha).$$

Now, $\frac{1}{2\pi i}(\theta_\beta - \theta_\alpha)$ can be lifted back to $-\frac{1}{2\pi i}\log g_\beta^\alpha$ under δ'' and we deduce that

$$\delta''\delta'\left(\frac{1}{2\pi i}\Theta\right) = \text{class of } -\frac{1}{2\pi i}[\log g_\beta^\alpha + \log g_\alpha^\gamma + \log g_\gamma^\beta] = -\text{class of } c_{\alpha\beta\gamma} = -c_1(L).$$

** There may be a problem with the sign! **

The next way of looking at $c_1(L)$ works when L comes from a divisor. Say X is a complex algebraic manifold and $L = \mathcal{O}_X(D)$, where D is a divisor,

$$D = \sum_j a_j W_j$$

on X. Then, D gives a cycle in homology, so $[D] \in H_{2n-2}(X, \mathbb{Z})$ (here $n = \dim_{\mathbb{C}} X$). By Poincaré duality, our $[D]$ is in $H^2(X, \mathbb{Z})$ and it is $\sum a_j[W_j]$.

Theorem 3.31 *If X is a compact, complex algebraic manifold and D is a divisor on X, then*

$$c_1(\mathcal{O}_X(D)) = [D] \quad \text{in } H^2(X, \mathbb{Z}),$$

that is, $c_1(\mathcal{O}_X(D))$ is carried by the $(2n-2)$-cycle, D.

Proof. Recall that Poincaré duality is given by: For $\xi \in H^r(X, \mathbb{R})$ and $\eta \in H^s(X, \mathbb{R})$ (where $r + s = 2n$), then

$$(\xi, \eta) = \int_X \xi \wedge \eta.$$

The homology/cohomology duality is given by: For $\omega \in H^s(X, \mathbb{R})$ and $Z \in H_s(X, \mathbb{R})$, then

$$(Z, \omega) = \int_Z \omega.$$

We know that the cocyle (= 2-form) representing $c_1(L)$ is $\left[\frac{i}{2\pi}\Theta\right]$, for any connection on X. We must show that for every closed, real $(2n-2)$-form, ω,

$$\frac{i}{2\pi}\int_X \Theta \wedge \omega = \int_D \omega.$$

We compute Θ for a convenient connection, namely, the uniholo connection. Pick a local holomorphic frame, $e(z)$, for L, then if L has a section, s, we know $s(z) = \lambda(z)e(z)$, locally. For θ, the connection matrix in this frame, we have

(a) $\theta = \theta^{1,0}$ (holomorphic)

(b) $d(|s|^2) = (\nabla s, s) + (s, \nabla s)$ (unitary)

We have
$$\nabla s = \nabla \lambda e = (d\lambda + \theta\lambda)e.$$
Thus, the right hand side of (b) is
$$\begin{aligned} d(|s|^2) &= ((d\lambda + \theta\lambda)e, \lambda e) + (\lambda e, (d\lambda + \theta\lambda)e) \\ &= \overline{\lambda} d\lambda(e,e) + \theta|\lambda|^2(e,e) + \lambda d\overline{\lambda}(e,e) + \overline{\theta}|\lambda|^2(e,e). \end{aligned}$$

Write $h(z) = |e(z)|^2 = (e,e) > 0$; So, the right hand side of (b) is $\overline{\lambda} h d\lambda + \lambda h d\overline{\lambda} + (\theta + \overline{\theta})|\lambda|^2 h$. Now, $|s|^2 = \lambda\overline{\lambda} h$, so
$$d(|s|^2) = \lambda\overline{\lambda} dh + h(\lambda d\overline{\lambda} + \overline{\lambda} d\lambda).$$
From (b), we deduce $dh = (\theta + \overline{\theta})h$, and so,
$$\theta + \overline{\theta} = \frac{dh}{h} = d(\log h) = \partial(\log h) + \overline{\partial}(\log h).$$
Using (a) and the decomposition by type, we get
$$\theta = \partial(\log h) = \partial \log(|e|^2).$$
As $\Theta = d\theta - \theta \wedge \theta$, we get
$$\Theta = d\theta = (\partial + \overline{\partial})(\partial \log(|e|^2)),$$
i.e.,
$$\Theta = \overline{\partial}\partial \log(|e|^2).$$
Now, recall
$$d^c = \frac{i}{4\pi}(\overline{\partial} - \partial),$$
so that
$$dd^c = -d^c d = \frac{i}{2\pi}\partial\overline{\partial} = -\frac{i}{2\pi}\overline{\partial}\partial,$$
and $2\pi i dd^c = \overline{\partial}\,\partial$. Consequently,
$$\Theta = \pi i dd^c \log(|e|^2).$$
This holds for any local frame, e, and has nothing to do with the fact that L comes from a divisor.

Now, $L = \mathcal{O}_X(D)$ and assume that the local equations for D are $f_\alpha = 0$ (on U_α, some open in the trivializing cover for L on X). We know the transition functions are
$$g_\alpha^\beta = \frac{f_\beta}{f_\alpha};$$
Therefore, the local vectors $s_\alpha = f_\alpha e_\alpha$ form a global section, s, of $\mathcal{O}_X(D)$. The zero locus of this section is exactly D. As the bundle L is unitary, $g_\alpha^\beta \in \mathrm{U}(1)$, which implies $|f_\beta| = |f_\alpha|$ and so, $|f_\alpha e_\alpha|$ is well defined. Thus for small $\epsilon > 0$,
$$D(\epsilon) = \{z \in X \mid |s(z)|^2 < \epsilon\}$$
is a tubular neighborhood of D.

Look at $X - D(\epsilon)$, then $\mathcal{O}_X(D) \upharpoonright X - D(\epsilon)$ is trivial as the section s is never zero there. Therefore, s_α will also do as a local frame for $\mathcal{O}_X(D)$ on $X - D(\epsilon)$.

We need to compute $\int_X \Theta \wedge \omega$. By linearity, we may assume D is one of the W's. Then, by definition,

$$\int_X \Theta \wedge \omega = \lim_{\epsilon \downarrow 0} \int_{X - D(\epsilon)} 2\pi i dd^c \log |s|^2 \wedge \omega$$

If we apply Stokes, we find

$$\int_X \Theta \wedge \omega = -\lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} 2\pi i d^c \log |s|^2 \wedge \omega$$

that is,

$$\int_X \Theta \wedge \omega = \frac{2\pi}{i} \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} d^c \log |s|^2 \wedge \omega. \tag{$\dagger$}$$

Now $\mathrm{Vol}(D(\epsilon)) \longrightarrow 0$ as $\epsilon \downarrow 0$, as we can see by using the Zariski stratification to reduce to the case where D is non-singular. Also,

$$|s|^2 = |f_\alpha|^2 |e_\alpha|^2 = f_\alpha \overline{f}_\alpha h,$$

where $h = |e_\alpha|^2$ is positive bounded. We have

$$\log |s|^2 = \log f_\alpha + \log \overline{f}_\alpha + \log h$$

and as $d^c = \frac{i}{4\pi}(\overline{\partial} - \partial)$,

$$d^c \log |s|^2 = \frac{i}{4\pi}[-\partial \log f_\alpha + \overline{\partial} \log \overline{f}_\alpha + (\overline{\partial} - \partial) \log h].$$

It follows that

$$\frac{2\pi}{i} d^c \log |s|^2 \wedge \omega = \frac{1}{2}[-\partial \log f_\alpha \wedge \omega + \overline{\partial} \log \overline{f}_\alpha \wedge \omega + (\overline{\partial} - \partial) \log h \wedge \omega].$$

In the right hand side of ($\dagger$), the third term is

$$\frac{1}{2} \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} (\overline{\partial} - \partial) \log h \wedge \omega.$$

Now, $(\overline{\partial} - \partial) \log h$ is bounded (X is compact) and $\mathrm{Vol}(\partial D(\epsilon)) \longrightarrow 0$ as $\epsilon \downarrow 0$. So, this third term vanishes in the limit. But, $\overline{\partial} \log \overline{f}_\alpha = \overline{\partial \log f_\alpha}$ and $\omega = \overline{\omega}$, as ω is real. Consequently,

$$\overline{\partial} \log \overline{f}_\alpha \wedge \omega = \overline{\partial \log f_\alpha \wedge \omega}.$$

From ($\dagger$), we get

$$\begin{aligned}
\int_X \Theta \wedge \omega &= \frac{1}{2} \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} -\partial \log f_\alpha \wedge \omega + \overline{\partial \log f_\alpha \wedge \omega} \\
&= -\frac{1}{2} \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \omega - \overline{\partial \log f_\alpha \wedge \omega} \\
&= -i \lim_{\epsilon \downarrow 0} \Im \int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \omega.
\end{aligned}$$

Now, $f_\alpha = 0$ is the local equation of D and we can compute the integral on the right hand side away from the singularities of D as the latter have measure 0. The divisor D is compact, so we can cover it by polydics centered at nonsingular points of D, say ζ_0 is a such a point. By the local complete intersection then, there exist local coordinates for X near ζ_0, of the form

$$z_1 = f_\alpha, \quad \underbrace{z_2, \ldots, z_n}_{\text{rest}},$$

on $\Delta \cap U_\alpha$ (where Δ is a polydisc). Break up ω as

$$\omega = g(z_1, \ldots, z_n) \underbrace{dz_2 \wedge \cdots \wedge d\overline{z}_2 \wedge \cdots}_{\text{rest}} + \kappa,$$

where κ is a form involving dz_1 and $d\overline{z}_1$ in each summand. Also, as

$$\partial \log f_\alpha = (\partial + \overline{\partial}) \log f_\alpha = d \log f_\alpha = \frac{df_\alpha}{f_\alpha} = \frac{dz_1}{z_1},$$

we get

$$\partial \log f_\alpha \wedge \omega = \frac{dz_1}{z_1} g(z_1, \ldots, z_n) \underbrace{dz_2 \wedge \cdots \wedge d\overline{z}_2 \wedge \cdots}_{\text{rest}} + \text{ terms } \frac{dz_1 \wedge d\overline{z}_1}{z_1}\text{stuff}.$$

Furthermore, $dz_1 \wedge d\overline{z}_1 = 2idx \wedge dy = 2irdr_1 d\theta$ (in polar coordinates), so

$$\left| \frac{dz_1 \wedge d\overline{z}_1}{z_1} \right| = 2|dr_1||d\theta_1|,$$

and when $\epsilon \downarrow 0$, this term goes to 0. Therefore

$$\lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon) \cap \Delta} \frac{dz_1}{z_1} g(z_1, \ldots, z_n) d(\text{rest}) d(\overline{\text{rest}}) = \lim_{\epsilon \downarrow 0} \int_{(|z_1| = C\epsilon) \prod \text{rest of polydisc}} \frac{dz_1}{z_1} g(z_1, \ldots, z_n) d(\text{rest}) d(\overline{\text{rest}})$$

and by Cauchy's integral formula, this is

$$\lim_{\epsilon \downarrow 0} \int_{\text{rest of poly} \cap \partial D(\epsilon)} 2\pi i g(0, z_2, \ldots, z_n) d(\text{rest}) d(\overline{\text{rest}}) = 2\pi i \int_{D \cap \Delta} \omega.$$

Adding up the contributions from the finite cover of polydics, we get

$$\Im \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \omega = \Im\, 2\pi i \int_D \omega = 2\pi \int_D \omega,$$

as ω is real. But then,

$$-i \Im \lim_{\epsilon \downarrow 0} \int_{\partial D(\epsilon)} \log f_\alpha \wedge \omega = -2\pi i \int_X \omega$$

from which we finally deduce $\int_X \Theta \wedge \omega = -2\pi i \int_D \omega$, that is,

$$\int_X \frac{i}{2\pi} \Theta \wedge \omega = \int_D \omega,$$

as required. $\square$

Corollary 3.32 *Suppose V is a $\mathrm{U}(q)$-bundle on our compact X (so that differentiably, V is generated by its sections). Or, if V is a holomorphic bundle, assume it is generated by its holomorphic sections. Take a generic section, s, of V and say V has rank r. Then, the set $s = 0$ has complex codimension r (in homology) and is the carrier of $c_r(V)$.*

Proof. The case $r = 1$ is exactly the theorem above. Differentiably,

$$V = L_1 \coprod L_2 \coprod \cdots \coprod L_r,$$

for the diagonal line bundles of V. Holomorphically, this is also OK but over the space $[\Delta]V$. So, the transition matrix is a diagonal matrix

$$\mathrm{diag}(g_{1\,\alpha}^{\beta}, \ldots, g_{r\,\alpha}^{\beta}) \quad \text{on } U_\alpha \cap U_\beta$$

and $s_\alpha = (s_{1\,\alpha}, \ldots, s_{r\,\alpha})$. So,

$$\mathrm{diag}(g^\beta_\alpha)s_\alpha = (g^\beta_{1\,\alpha}s_{1\,\alpha}, \ldots, g^\beta_{r\,\alpha}s_{r\,\alpha}) = s_\beta$$

which shows that each $s_{j\,\alpha}$ *is* a section of L_j. Note that $s = 0$ iff all $s_j = 0$. But, the locus $s_j = 0$ carries $c_1(L_j)$, by the previous theorem. Therefore, $s = 0$ corresponds to the intersection in homology of the carriers of $c_1(L), \ldots, c_1(L_r)$. But, intersection in homology is equivalent to product in cohomology, so the cohomology class for $s = 0$ is

$$c_1(L_1)c_1(L_2)\cdots c_1(L_r) = c_r(V)$$

as desired. □

General Principle for Computing $c_q(V)$, for a rank r vector bundle, V.

(1) Let L be an ample line bundle, then $V \otimes L^{\otimes m}$ is generated by its sections for $m >> 0$.

(2) Pick r generic sections, $s_1, \ldots, s_r$, of $V \otimes L^{\otimes m}$. Form $s_1 \wedge \cdots \wedge s_{r-q+1}$, a section of $\bigwedge^{r-q+1}(V \otimes L^{\otimes m})$. Then, the zero locus of $s_1 \wedge \cdots \wedge s_{r-q+1}$ carries the Chern class, $c_q(V \otimes L^{\otimes m})$, of $V \otimes L^{\otimes m}$.

$\Big[$ When $q = r$, this is the corollary. When $q = 1$, we have $s_1 \wedge \cdots \wedge s_r$, a section of $\bigwedge^r V \otimes L^{\otimes m}$, and it is generic (as the fibre dimension is 1). We get $c_1(\bigwedge^r V \otimes L^{\otimes m})$ and we know that it is equal to $c_1(V \otimes L^{\otimes m})$.$\Big]$

(3) Use the relation from the Chern polynomial

$$c(V \otimes L^{\otimes m})(t) = \prod(1 + (\gamma_j + mc_1(L))t)$$

to get the elementary symmetric functions of the γ_j's, i.e., $c_q(V)$.

Remark: if $1 < q < r$, our section $s_1 \wedge \cdots \wedge s_{r-q+1}$ is *not* generic but it works.

Theorem 3.33 *Say X is a complex analytic or algebraic, compact, smooth, manifold and $j\colon W \hookrightarrow X$ is a smooth, complex, codimension q submanifold. Write $\mathcal{N}$ for the normal bundle of W in X; this is rank q ($\mathrm{U}(q)$) vector bundle on W. The subspace W corresponds to a cohomology class, ξ, in $H^{2q}(X, \mathbb{Z})$ (in fact, in $H^{q,q}(X, \mathbb{C})$) and so we get $j^*\xi \in H^{2q}(W, \mathbb{Z})$. Then, we have*

$$c_q(\mathcal{N}) = j^*W.$$

Proof. We begin with the case $q = 1$. In this case, W is a divisor and we know $\mathcal{N} = \mathcal{O}_X(W) \restriction W$. By Corollary 3.32, the Chern class $c_1(\mathcal{N})$ is carried by the zero locus of a section, s,of $\mathcal{N}$. Now, $W \cdot W$ in X as a cycle is just a moving of W by a small amount and then an ordinary intersection of W and the new moved cycle. We see that $W \cdot W = c_1(\mathcal{N})$ as cycle on W. But, j^*W is just $W \cdot W$ as cycle (by Poincaré duality). So, the result holds when $q = 1$. If $q > 1$ and if W is a complete intersection in X, then since $c_q(\mathcal{N})$ is computed by repeated pullbacks and each pullback gives the correct formula (by the case $q = 1$), we get the result. In the general case, we have two classes j^*W and $c_q(\mathcal{N})$. If there exists an open cover, $\{U_\alpha\}$, of W so that

$$j^*W \restriction U_\alpha = c_q(\mathcal{N}) \restriction U_\alpha \quad \text{for all } \alpha,$$

then we are done. But, W is smooth so it is a local complete intersection (LCIT). Therefore, we get the result by the previous case. □

Corollary 3.34 *If X is a compact, complex analytic manifold and if T_X = holomorphic tangent bundle has rank $q = \dim_{\mathbb{C}} X$, then*

$$c_q(T_X) = \chi_{\text{top}} = \sum_{i=0}^{2q} (-1)^i b_i$$

(Here, $b_i = \dim_{\mathbb{R}} H^i(X, \mathbb{Z})$.)

Proof. (Essentially due to Lefschetz). Look at $X \prod X$ and the diagonal embedding, $\Delta\colon X \to X \prod X$. So, $X \hookrightarrow X \prod X$ is a smooth codimension q submanifold. An easy argument shows that

$$T_X \cong \mathcal{N}_{X \hookrightarrow X \prod X} = \mathcal{N}$$

and the previous theorem implies

$$c_q(T_X) = c_q(\mathcal{N}) = X \cdot X$$

in $X \prod X$. Now, look at the map $f\colon X \to X$ given by

$$pr_2 \circ \epsilon\sigma,$$

where ϵ is small and σ is a section of $\mathcal{N}$. The fixed points of our map give the cocycle $X \cdot X$. The Lefschetz fixed point Theorem says the cycle of fixed points is given by

$$\sum_{i=0}^{2q} (-1)^i \mathrm{tr}\, f^* \text{ on } H^i(X, \mathbb{Z}).$$

But, for ϵ small, the map f is homotopic to id, so $f^* = \mathrm{id}^*$. Now, tr id^* = dimension of space = $b_i(X)$ if we are on $H^i(X)$. So the right hand side of the Lefschetz formula is χ_{top}, as claimed. $\square$

Segre Classes.

Let V be a vector bundle on X, then we have classes $s_j(V)$, and they are defined by

$$1 + \sum_{j=1}^{\infty} s_j(V) t^j = \frac{1}{c(V)(t)}.$$

As $c(V)(t)$ is nilpotent, we have

$$\frac{1}{c(V)(t)} = 1 - (c_1(V)t + c_2(V)t^2 + \cdots) + (c_1(V)t + c_2(V)t^2 + \cdots)^2 + \cdots$$

and so,

$$\begin{aligned} s_1(V) &= -c_1(V) \\ s_2(V) &= c_1^2(V) - c_2(V), \end{aligned}$$

etc.

Pontrjagin Classes.

Pontrjagin classes are defined for real O(q)-bundles over real manifolds. We have the commutative diagrams

$$\begin{array}{ccc} \mathrm{U}(q) & \stackrel{\zeta}{\hookrightarrow} & \mathrm{O}(2q) \\ \downarrow & & \downarrow \\ \mathrm{GL}(q, \mathbb{C}) & \hookrightarrow & \mathrm{GL}(2q, \mathbb{R}) \end{array}$$

where $\zeta(z_1, \ldots, z_q) = (x_1, y_1, \ldots, x_q, y_q)$, with $z_j = x_j + iy_j$ and

$$\begin{array}{ccc} \mathrm{O}(q) & \overset{\psi}{\hookrightarrow} & \mathrm{U}(q) \\ \big\downarrow & & \big\downarrow \\ \mathrm{GL}(q, \mathbb{R}) & \hookrightarrow & \mathrm{GL}(q, \mathbb{C}) \end{array}$$

where $\psi(A)$ is the real matrix now viewed as a complex matrix. Given ξ, an $\mathrm{O}(q)$-bundle, we have $\psi(q)$, a $\mathrm{U}(q)$-bundle. Define

The *Pontrjagin classes,* $p_i(\xi)$, are defined by

$$p_i(\xi) = (-1)^i c_{2i}(\psi(\xi)) \in H^{4i}(X, \mathbb{Z}).$$

The *generalized Pontrjagin classes,* $P_i(\xi)$ and the *generalized Pontrjagin polynomial,* $P(\xi)(t)$, are defined by

$$P(\xi)(t) = c(\psi(\xi))(t), \quad \text{and} \quad P_j(\xi) = c_j(\psi(xi)).$$

(Observe: $P_{2l}(\xi) = (-1)^p p_l(\xi)$.)

Now, ξ corresponds to map, $X \longrightarrow \mathrm{BO}(q)$. Then, for i even, $P_{i/2}(\xi)$ is the pullback of something in $H^i(\mathrm{BO}(q), \mathbb{Z})$. It is known that for $i \equiv 2(4)$, the cohomology ring $H^i(\mathrm{BO}(q), \mathbb{Z})$ is 2-torsion, so $2P_{\mathrm{odd}}(\xi) = 0$. So, with rational coefficients, we get

$$P_{\mathrm{odd}}(\xi) = 0 \quad \text{and} \quad P_{\mathrm{even}}(\xi) = \pm P_{\mathrm{even}/2}(\xi).$$

We have the following properties:

(0) $P(\xi)(t) = 1 + \text{stuff}$.

(1) $f^*P(\xi)(t) = P(f^*\xi)(t)$, so $f^*P_i(\xi) = P_i(f^*\xi)$.

(2) Suppose ξ, η are bundle of rank q', q'', respectively, then

$$P(\xi \amalg \eta)(t) = P(\xi)(t)P(\eta)(t)$$

and if we set $p(\xi)(t) = \sum_{j=0}^{\infty} p_j(\xi)t^{2j}$, then

$$p(\xi \amalg \eta)(t) = p(\xi)(t)p(\eta)(t), \quad \text{mod} \quad \text{elements of order 2 in } H^\bullet(X, \mathbb{Z}).$$

(3) Suppose $c(\psi(\xi))(t)$ has Chern roots γ_i. Then, the polynomial $\sum_{j=0}^{\infty}(-1)^j p_j(\xi)t^{2j}$ has roots γ_i^2; in fact,

$$\sum_{j=0}^{\infty}(-1)^j p_j(\xi)t^{2j} = \Big(\sum_l c_j(\xi)t^l\Big)\Big(\sum_m (-1)^m c_m(\xi)t^m\Big).$$

Proposition 3.35 *Say ξ is a $\mathrm{U}(q)$-bundle and make $\zeta(\xi)$, an $\mathrm{O}(2q)$-bundle. Then*

$$\sum_{j=0}^{\infty}(-1)^j p_j(\zeta(\xi))t^{2j} = \Big(\sum_l c_j(\xi)t^l\Big)\Big(\sum_m c_m(\xi^D)t^m\Big).$$

Proof. Consider the maps $\mathrm{U}(q) \hookrightarrow \mathrm{O}(2q) \hookrightarrow \mathrm{U}(2q)$. By linear algebra, if $A \in \mathrm{U}(q)$, its image in $\mathrm{U}(2q)$ by this map is

$$\begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix}$$

after an automorphism of $\mathrm{U}(2q)$, which automorphism is independent of A. By Skolem-Noether, the automorphism is of the form

$$H^{-1}(\psi\zeta(A))H,$$

for some $H \in \mathrm{GL}(2q, \mathbb{C})$. For an inner automorphism, the cohomology class of the vector bundle stays the same. Thus, this cohomology class is the class of

$$\begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix}.$$

Now, we know the transition matrix of ξ^D is the transpose inverse of that for ξ. But, A is unitary, so

$$\overline{A} = (A^{-1})^\top = A^D$$

and we deduce that $\psi\zeta(A)$ has as transition matrix

$$\begin{pmatrix} A & 0 \\ 0 & A^D \end{pmatrix}.$$

Consequently, the right hand side of our equation is

$$\Big(\sum_l c_j(\xi)t^l\Big)\Big(\sum_m c_m(\xi^D)t^m\Big),$$

as required. □