Chapter 2

Cohomology of (Mostly) Constant Sheaves and Hodge Theory

2.1 Real and Complex

Let $X$ be a complex analytic manifold of (complex) dimension $n$. Viewed as a real manifold, $X$ is a $C^\infty$-manifold of dimension $2n$. For every $x \in X$, we know $T_{X,x}$ is a $\mathbb{C}$-vector space of complex dimension $n$, so, $T_{X,x}$ is a real vector space of dimension $2n$. Take local (complex) coordinates $z_1, \ldots, z_n$ at $x \in X$, then we get real local coordinates $x_1, y_1, \ldots, x_n, y_n$ on $X$ (as an $\mathbb{R}$-manifold), where $z_j = x_j + iy_j$. (Recall, $T_X$ is a complex holomorphic vector bundle). If we view $T_{X,x}$ as a real vector space of dimension $2n$, then we can complexify $T_{X,x}$, i.e., form

$$T_{X,x,\mathbb{C}} = T_{X,x} \otimes_{\mathbb{R}} \mathbb{C},$$

a complex vector space of dimension $2n$. A basis of $T_{X,x,\mathbb{C}}$ at $x$ (as $\mathbb{R}$-space) is just $\partial/\partial x_1, \partial/\partial y_1, \ldots, \partial/\partial x_n, \partial/\partial y_n$.

These are a $\mathbb{C}$-basis for $T_{X,x,\mathbb{C}}$, too. We can make the change of coordinates to the coordinates $z_j$ and $\bar{z}_j$, namely,

$$z_j = x_j + iy_j, \quad \bar{z}_j = x_j - iy_j,$$

and of course,

$$x_j = \frac{1}{2}(z_j + \bar{z}_j), \quad y_j = \frac{1}{2i}(z_j - \bar{z}_j).$$

So, $T_{X,x,\mathbb{C}}$ has a basis consisting of the $\partial/\partial z_j, \partial/\partial \bar{z}_j$; in fact, for $f \in C^\infty(\text{open})$, we have

$$\frac{\partial f}{\partial z_j} = \frac{\partial f}{\partial x_j} - \frac{i}{\partial y_j} \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}_j} = \frac{\partial f}{\partial x_j} + \frac{i}{\partial y_j}.$$

More abstractly, let $V$ be a $\mathbb{C}$-vector space of dimension $n$ and view $V$ as a real vector space of dimension $2n$. If $e_1, \ldots, e_n$ is a $\mathbb{C}$-basis for $V$, then $ie_1, \ldots, ie_n$ make sense. Say $e_j = f_j + ig_j$ (from $\mathbb{C}$-space to $\mathbb{R}$-space), then, $ie_j = if_j - g_j = -g_j + if_j$. Consequently, the map $(e_1, \ldots, e_n) \mapsto (ie_1, \ldots, ie_n)$ corresponds to the map

$$((f_1, g_1), \ldots, (f_n, g_n)) \mapsto ((-g_1, f_1), \ldots, (-g_n, f_n))$$

where $V$ is viewed as $\mathbb{R}$-space of dimension $2n$. The map $J$ an endomorphism of $V$ viewed as $\mathbb{R}$-space and obviously, it satisfies

$$J^2 = -\text{id}.$$
If, conversely, we have an \( R \)-space, \( V \), of even dimension, \( 2n \) and if an endomorphism \( J \in \text{End}_R(V) \) with \( J^2 = -\text{id} \) is given, then we can give \( V \) a complex structure as follows:

\[
(a + ib)v = av + bJ(v).
\]

In fact, the different complex structures on the real vector space, \( V \), of dimension \( 2n \) are in one-to-one correspondence with the homogeneous space \( \text{GL}(2n, R) / \text{GL}(n, C) \), via

\[
\text{class } A \mapsto AJA^{-1}.
\]

**Definition 2.1** An almost complex manifold is a real \( C^\infty \)-manifold together with a bundle endomorphism, \( J : T_X \to T_X \), so that \( J^2 = -\text{id} \).

**Proposition 2.1** If \( (X, O_X) \) is a complex analytic manifold, then it is almost complex.

**Proof.** We must construct \( J \) on \( T_X \). It suffices to do this locally and check that it is independent of the coordinate patch. Pick some open, \( U \), where \( T_X \rvert_U \) is trivial. By definition of a patch, we have an isomorphism \( (U, O_X \rvert_U) \cong (B_C(0, \epsilon), O_B) \) and we have local coordinates denoted \( z_1, \ldots, z_n \) in both cases. On \( T_X \rvert_U \), we have \( \partial / \partial x_1, \ldots, \partial / \partial x_n \) and \( \partial / \partial y_1, \ldots, \partial / \partial y_n \), as before. The map \( J \) is given by

\[
J = \begin{pmatrix}
\frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \\
\frac{\partial}{\partial y_1} & \cdots & \frac{\partial}{\partial y_n} \\
\frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n}
\end{pmatrix}.
\]

We need to show that this does not depend on the local trivialization. Go back for a moment to two complex manifolds, \( (X, O_X) \) and \( (Y, O_Y) \), of dimension \( 2n \) and consider a smooth map \( f : (X, O_X) \to (Y, O_Y) \). For every \( x \in X \), we have an induced map on tangent spaces, \( df : T_{X,x} \to T_{Y,y} \), where \( y = f(x) \) and if, as \( R \)-spaces, we use local coordinates \( x_1, \ldots, x_n, y_1, \ldots, y_n \) on \( T_{X,x} \) and local coordinates \( u_1, \ldots, u_n, v_1, \ldots, v_n \) on \( T_{Y,y} \), then \( df \) is given by the Jacobian

\[
J_R(f) = \begin{pmatrix}
\frac{\partial u_{\alpha}}{\partial x_j} & \frac{\partial v_{\alpha}}{\partial x_j} \\
\frac{\partial v_{\alpha}}{\partial y_j} & \frac{\partial u_{\alpha}}{\partial y_j}
\end{pmatrix}.
\]

If \( f \) is holomorphic, the Cauchy-Riemann equations imply

\[
\frac{\partial u_{\alpha}}{\partial x_j} = \frac{\partial v_{\alpha}}{\partial y_j} \quad \text{and} \quad \frac{\partial v_{\alpha}}{\partial x_j} = -\frac{\partial u_{\alpha}}{\partial y_j}.
\]

Now, this gives

\[
J_R(f) = \begin{pmatrix}
\frac{\partial v_{\alpha}}{\partial y_j} & \frac{\partial u_{\alpha}}{\partial y_j} \\
-\frac{\partial u_{\alpha}}{\partial y_j} & \frac{\partial v_{\alpha}}{\partial y_j}
\end{pmatrix} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.
\]

Going back to our problem, if we have different trivializations, on the overlap, the transition functions are holomorphic, so \( J_R(f) \) is as above. Now \( J \) in our coordinates is of the form

\[
J = \begin{pmatrix}
0_n & I_n \\
-I_n & 0
\end{pmatrix}
\]

and we have \( J J_R(f) = J_R(f) J \) when \( f \) is holomorphic (DX). \( \square \)

So, an almost complex structure is a bundle invariant.

**Question:** Does \( S^6 \) possess a complex structure?
The usual almost complex structure from $S^7 (= \text{unit Cayley numbers = unit octonions})$ is not a complex structure. Borel and Serre proved that the only spheres with an almost complex structure are: $S^0, S^2$ and $S^6.$

Say we really have complex coordinates, $z_1, \ldots, z_n$ down in $X.$ Then, on $T_X \otimes_{\mathbb{R}} \mathbb{C},$ we have the basis

$$\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n},$$

and so, in this basis, if we write $f = (w_1, \ldots, w_n), \text{where } w_\alpha = u_\alpha + i v_\alpha,$ we get

$$J_R(f) = \begin{pmatrix} \frac{\partial w_\alpha}{\partial z_j} & \frac{\partial w_\alpha}{\partial \bar{z}_j} \\ \frac{\partial w_\alpha}{\partial \bar{z}_j} & \frac{\partial w_\alpha}{\partial z_j} \end{pmatrix}$$

and, again, if $f$ is holomorphic, we get

$$\frac{\partial w_\alpha}{\partial \bar{z}_j} = \frac{\partial \bar{w}_\alpha}{\partial z_j} = 0,$$

which yields

$$J_R(f) = \begin{pmatrix} \frac{\partial w_\alpha}{\partial z_j} & 0 \\ 0 & \frac{\partial \bar{w}_\alpha}{\partial z_j} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}.$$ 

Write

$$J(f) = \frac{\partial w_\alpha}{\partial z_j}$$

and call it the holomorphic Jacobian. We get

(1) $J_R(f) = \begin{pmatrix} J(f) & 0 \\ 0 & J(f) \end{pmatrix}, \text{so, } \mathbb{R}\text{-rank } J_R(f) = 2\mathbb{C}\text{-rank } J(f).$

(2) We have $\det(J_R(f)) = |\det(J(f))|^2 \geq 0,$ and $\det(J_R(f)) > 0$ if $f$ is a holomorphic isomorphism (in this case, $m = n =$ the common dimension of $X, Y).$

Hence, we get the first statement of

**Proposition 2.2** Holomorphic maps preserve the orientation of a complex manifold and each complex manifold possesses an orientation.

**Proof.** We just proved the first statement. To prove the second statement, as orientations are preserved by holomorphic maps we need only give an orientation locally. But, locally, a patch is biholomorphic to a ball in $\mathbb{C}^n.$ Therefore, it is enough to give $\mathbb{C}^n$ an orientation, i.e., to give $\mathbb{C}$ an orientation. However, $\mathbb{C}$ is oriented as $(x, ix)$ gives the orientation. \(\square\)

Say we have a real vector space, $V,$ of dimension $2n$ and look at $V \otimes_{\mathbb{R}} \mathbb{C}.$ Say $V$ also has a complex structure, $J.$ Then, the extension of $J$ to $V \otimes_{\mathbb{R}} \mathbb{C}$ has two eigenvalues, $\pm i.$ On $V \otimes_{\mathbb{R}} \mathbb{C},$ we have the two eigenspaces, $(V \otimes_{\mathbb{R}} \mathbb{C})^{1,0} = \text{the } i\text{-eigenspace and } (V \otimes_{\mathbb{R}} \mathbb{C})^{0,1} = \text{the } -i\text{-eigenspace.}$ Of course,

$$(V \otimes_{\mathbb{R}} \mathbb{C})^{0,1} = \overline{(V \otimes_{\mathbb{R}} \mathbb{C})^{1,0}}.$$ 

Now, look at $\bigwedge^p(V \otimes_{\mathbb{R}} \mathbb{C}).$ We can examine

$$\bigwedge^p(V \otimes_{\mathbb{R}} \mathbb{C}) \overset{\text{def}}{=} \bigwedge^p[(V \otimes_{\mathbb{R}} \mathbb{C})^{1,0}] \text{ and } \bigwedge^q(V \otimes_{\mathbb{R}} \mathbb{C}) \overset{\text{def}}{=} \bigwedge^q[(V \otimes_{\mathbb{R}} \mathbb{C})^{0,1}].$$
and also

\[ \bigwedge^{p,q}(V \otimes_R \mathbb{C}) \overset{\text{def}}{=} \bigwedge^{p,0}(V \otimes_R \mathbb{C}) \otimes \bigwedge^{0,q}(V \otimes_R \mathbb{C}). \]

Note that we have

\[ \bigwedge^{l}(V \otimes_R \mathbb{C}) = \prod_{p+q=l} \bigwedge^{p,q}(V \otimes_R \mathbb{C}). \]

Now, say \( X \) is an almost complex manifold and apply the above to \( V = T_X, T^D_X \); we get bundle decompositions for \( T_X \otimes_R \mathbb{C} \) and \( T^D_X \otimes_R \mathbb{C} \). Thus,

\[ \bigwedge^{l}(T^D_X \otimes_R \mathbb{C}) = \prod_{l=1}^{2n} \prod_{p+q=l} \bigwedge^{p,q}(T^D_X \otimes_R \mathbb{C}). \]

Note that \( J \) on \( \bigwedge^{p,q} \) is multiplication by \((-1)^{q_i(p+q)}\). Therefore, \( J \) does not act by scalar multiplication in general on \( \bigwedge^{l}(V \otimes_R \mathbb{C}) \).

Say \( X \) is now a complex manifold and \( f: X \to Y \) is a \( C^\infty \)-map to another complex manifold, \( Y \). Then, for every \( x \in X \), we have the linear map

\[ Df: T_{X,x} \otimes_R \mathbb{C} \longrightarrow T_{Y,f(x)} \otimes_R \mathbb{C}. \]

The map \( f \) won’t in general preserve the decomposition \( T_{X,x} \otimes_R \mathbb{C} = T_{X,x}^{1,0} \sqcup T_{X,x}^{0,1} \).

However, \( f \) is holomorphic iff for every \( x \in X \), we have \( Df: T_{X,x}^{1,0} \to T_{Y,f(x)}^{1,0} \).

Let us now go back to a real manifold, \( X \). We have the usual exterior derivative

\[ d: \bigwedge^{l}(T^D_X \otimes_R \mathbb{C}) \longrightarrow \bigwedge^{l+1}(T^D_X \otimes_R \mathbb{C}), \]

namely, if \( \xi_1, \ldots, \xi_{2n} \) are real coordinates at \( x \), we have

\[ \sum_{|I|=l} a_I d\xi_I \mapsto \sum_{|I|=l} da_I \wedge d\xi_I. \]

here, the \( a_I \) are \( \mathbb{C} \)-valued function on \( X \) near \( x \) and \( d\xi_I = d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_l} \), with \( I = \{ i_1 < i_2 < \cdots < i_l \} \).

In the almost complex case, we have the \( p,q \)-decomposition of \( T^D_X \otimes_R \mathbb{C} \) and consequently

\[ \bigwedge^{p,q}(T^D_X \otimes_R \mathbb{C}) \overset{i_{p,q}}{\longrightarrow} \bigwedge^{l}(T^D_X \otimes_R \mathbb{C}) \overset{d}{\longrightarrow} \bigwedge^{l+1}(T^D_X \otimes_R \mathbb{C}) = \prod_{r+s=l+1} \bigwedge^{r,s}(T^D_X \otimes_R \mathbb{C}). \]

We let

\[ \partial = \{ \partial_{p,q} = \text{pr}_{p+1,q} \circ d \circ i_{p,q}: \bigwedge^{p,q}(T^D_X \otimes_R \mathbb{C}) \longrightarrow \bigwedge^{p+1,q}(T^D_X \otimes_R \mathbb{C}) \}_{p,q} \]

and

\[ \overline{\partial} = \{ \overline{\partial}_{p,q} = \text{pr}_{p,q+1} \circ d \circ i_{p,q}: \bigwedge^{p,q}(T^D_X \otimes_R \mathbb{C}) \longrightarrow \bigwedge^{p,q+1}(T^D_X \otimes_R \mathbb{C}) \}_{p,q}. \]
2.1. REAL AND COMPLEX

Note that \( d = \partial + \overline{\partial} + \text{other stuff} \). Let us take a closer look in local coordinates. We can pick \( \xi_1, \ldots, \xi_n \), some coordinates for \( T^{1,0}_X \), then \( \xi_1, \ldots, \xi_n \) are coordinates for \( T^{0,1}_X \) (say \( x_1, \ldots, x_{2n} \) are local coordinates in the base). Then, any \( \omega \in \Lambda^{p,q}(T^D_P \otimes \mathbb{C}) \) has the form

\[
\omega = \sum_{|I| = p \atop |\bar{I}| = q} a_{I, \bar{I}} d\xi_I \wedge d\xi_{\bar{I}},
\]

and so

\[
d\omega = \sum_{|I| = p \atop |\bar{I}| = q} da_{I, \bar{I}} \wedge d\xi_I \wedge d\xi_{\bar{I}} + \sum_{|I| = p \atop |\bar{I}| = q} a_{I, \bar{I}} d(d\xi_I \wedge d\xi_{\bar{I}}) = \partial \omega + \overline{\partial} \omega + \text{stuff}.
\]

If we are on a complex manifold, then we can choose the \( \xi_j \) so that \( \xi_j = \partial/\partial z_j \) and \( \bar{\xi}_j = \partial/\partial \bar{z}_j \), constant over our neighborhood and then,

\[
d\omega = \sum_{|I| = p \atop |\bar{I}| = q} da_{I, \bar{I}} d\xi_I \wedge d\bar{\xi}_{\bar{I}}
\]

\[
= \sum_{|I| = p \atop |\bar{I}| = q} \sum_{j=1}^n \frac{\partial a_{I, \bar{I}}}{\partial z_j} dz_j \wedge d\bar{z}_I \wedge d\bar{\xi}_{\bar{I}} + \frac{\partial a_{I, \bar{I}}}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_I \wedge d\xi_{\bar{I}}
\]

\[
= \partial \omega + \overline{\partial} \omega = (\partial + \overline{\partial})\omega.
\]

Consequently, on a complex manifold, \( d = \partial + \overline{\partial} \).

On an almost complex manifold, \( d^2 = 0 \), yet, \( \partial^2 \neq 0 \) and \( \overline{\partial}^2 \neq 0 \) in general.

However, suppose we are lucky and \( d = \partial + \overline{\partial} \). Then,

\[
0 = d^2 = \partial^2 + \overline{\partial} \partial + \partial \overline{\partial} + \overline{\partial}^2,
\]

and we deduce that \( \overline{\partial}^2 = \partial^2 = \overline{\partial} \partial + \partial \overline{\partial} = 0 \), in this case.

**Definition 2.2** The almost complex structure on \( X \) is **integrable** iff near every \( x \in X \), there exist real coordinates, \( \xi_1, \ldots, \xi_n \) in \( T^{1,0}_X \) and \( \bar{\xi}_1, \ldots, \bar{\xi}_n \) in \( T^{0,1}_X \), so that \( d = \partial + \overline{\partial} \).

By what we just did, a complex structure is integrable. A famous theorem of Newlander-Nirenberg (1957) shows that if \( X \) is an almost complex \( C^\infty \)-manifold whose almost complex structure is integrable, then there exists a unique complex structure (i.e., complex coordinates everywhere) inducing the almost complex one.

**Remark:** Say \( V \) has a complex structure given by \( J \). We have

\[
V = V \otimes_{\mathbb{R}} \mathbb{R} \hookrightarrow V \otimes_{\mathbb{R}} \mathbb{C} \cong V^{1,0} \oplus V^{0,1}.
\]

The vector space \( V^{1,0} \) also has a complex structure, namely, multiplication by \( i \). So, we have an isomorphism \( V \cong V^{1,0} \), as \( \mathbb{R} \)-spaces, but also an isomorphism \( V \cong V^{1,0} \), as \( \mathbb{C} \)-spaces, where the complex structure on \( V \) is \( J \) and the complex structure on \( V^{0,1} \) is multiplication by \( i \). Therefore, we also have an isomorphism \( V \cong V^{1,0} \), where the complex structure on \( V \) is \( -J \) and the complex structure on \( V^{0,1} \) is multiplication by \( -i \).

For tangent spaces, \( T^{1,0}_X \) is spanned by \( \partial/\partial z_1, \ldots, \partial/\partial z_n \), the space \( T^{0,1}_X \) is spanned by \( \partial/\partial \bar{z}_1, \ldots, \partial/\partial \bar{z}_n \); also, \( T^{D,1,0}_X \) is spanned by \( dz_1, \ldots, dz_n \) and \( T^{D,0,1}_X \) is spanned by \( d\bar{z}_1, \ldots, d\bar{z}_n \).
2.2 Cohomology, de Rham, Dolbeault

Let $X$ be a real $2n$-dimensional $C^\infty$-manifold and let $d$ be the exterior derivative, then we get the complex

$$T_X^D \xrightarrow{d} \bigwedge^2 T_X^D \xrightarrow{d} \cdots \xrightarrow{d} \bigwedge^{2n} T_X^D,$$

($d^2 = 0$). The same is true for complex-valued forms, we have the complex

$$T_X^D \otimes \mathbb{C} \xrightarrow{d} \bigwedge^2 T_X^D \otimes \mathbb{C} \xrightarrow{d} \cdots \xrightarrow{d} \bigwedge^{2n} T_X^D \otimes \mathbb{C},$$

($d^2 = 0$). Here, there is an abuse of notation: $T_X^D$ denotes a sheaf, so we should really use a notation such as $T_X^D$. To alleviate the notation, we stick to $T_X^D$, as the context makes it clear that it is a sheaf. These maps induce maps on global $X$-sections, so we get the complexes

$$\Gamma(X, T_X^D) \xrightarrow{d} \bigwedge^2 \Gamma(X, T_X^D) \xrightarrow{d} \cdots \xrightarrow{d} \bigwedge^{2n} \Gamma(X, T_X^D)$$

and

$$\Gamma(X, T_X^D \otimes \mathbb{C}) \xrightarrow{d} \bigwedge^2 \Gamma(X, T_X^D \otimes \mathbb{C}) \xrightarrow{d} \cdots \xrightarrow{d} \bigwedge^{2n} \Gamma(X, T_X^D \otimes \mathbb{C}).$$

Define

\begin{align*}
Z_{\text{DR}}^l(X) &= \text{Ker } d, \quad \text{where } d: \bigwedge^l \Gamma(X, T_X^D) \rightarrow \bigwedge^{l+1} \Gamma(X, T_X^D) \\
Z_{\text{DR}}^l(X)_{\mathbb{C}} &= \text{Ker } d, \quad \text{where } d: \bigwedge^l \Gamma(X, T_X^D \otimes \mathbb{C}) \rightarrow \bigwedge^{l+1} \Gamma(X, T_X^D \otimes \mathbb{C}) \\
B_{\text{DR}}^l(X) &= \text{Im } d, \quad \text{where } d: \bigwedge^l \Gamma(X, T_X^D) \rightarrow \bigwedge^{l} \Gamma(X, T_X^D) \\
B_{\text{DR}}^l(X)_{\mathbb{C}} &= \text{Ker } d, \quad \text{where } d: \bigwedge^l \Gamma(X, T_X^D \otimes \mathbb{C}) \rightarrow \bigwedge^{l} \Gamma(X, T_X^D \otimes \mathbb{C}) \\
H_{\text{DR}}^l(X) &= Z_{\text{DR}}^l(X)/B_{\text{DR}}^l(X) \\
H_{\text{DR}}^l(X)_{\mathbb{C}} &= Z_{\text{DR}}^l(X)_{\mathbb{C}}/B_{\text{DR}}^l(X)_{\mathbb{C}}.
\end{align*}

Note: $H_{\text{DR}}^l(X)_{\mathbb{C}} = H_{\text{DR}}^l(X) \otimes \mathbb{C}$. These are the de Rham cohomology groups. For Dolbeault cohomology, take $X$, a complex manifold of dimension $n$, view it as a real manifold of dimension $2n$, consider the complexified cotangent bundle, $T_X^D \otimes \mathbb{C}$, and decompose its wedge powers as

$$\bigwedge^l (T_X^D \otimes \mathbb{C}) = \bigoplus_{p+q=l} \bigwedge^p (T_X^D \otimes \mathbb{C}).$$

Since $X$ is a complex manifold, $d = \partial + \overline{\partial}$ and so, $\partial^2 = \overline{\partial}^2 = 0$. Therefore, we get complexes by fixing $p$ or $q$:

(a) Fix $q$: \( \bigwedge^{0,q} (T_X^D \otimes \mathbb{C}) \xrightarrow{\partial} \bigwedge^{1,q} (T_X^D \otimes \mathbb{C}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \bigwedge^{n,q} (T_X^D \otimes \mathbb{C}) \).

(b) Fix $p$: \( \bigwedge^{p,0} (T_X^D \otimes \mathbb{C}) \xrightarrow{\overline{\partial}} \bigwedge^{p,1} (T_X^D \otimes \mathbb{C}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \bigwedge^{p,n} (T_X^D \otimes \mathbb{C}) \).

The above are the Dolbeault complexes and we have the corresponding cohomology groups $H^\partial_p(X)$ and $H^{\overline{\partial}}_q(X)$. Actually, the $H^\partial_p(X)$ are usually called the Dolbeault cohomology groups. The reason for that is if $f: X \rightarrow Y$ is holomorphic, then $df$ and $(df)^\partial$ respect the $p,q$-decomposition. Consequently, $\partial$.

$$\left(\partial f\right)^p_{D} : \bigwedge^p (T_{Y,f(x)}^D \otimes \mathbb{C}) \rightarrow \bigwedge^p (T_{X,x}^D \otimes \mathbb{C})$$
for all \( x \in X \) and 
\[(df)^D \circ \overline{\partial} = \overline{\partial} \circ (df)^D\]

imply that \((df)^D\) induces maps \( H^{p,q}_Y \rightarrow H^{p,q}_X \).

The main local fact is the Poincaré lemma.

**Lemma 2.3 (Poincaré Lemma)** If \( X \) is a real \( C^\infty \)-manifold and is actually a star-shaped manifold (or particularly, a ball in \( \mathbb{R}^n \)), then \( H^p_{\text{DR}}(X) = (0) \), for all \( p \geq 1 \).

If \( X \) is a complex analytic manifold and is a polydisc \( (PD(0,r)) \), then

(a) \( H^p_{\overline{\partial}}(X) = (0) \), for all \( p \geq 0 \) and all \( q \geq 1 \).

(b) \( H^p_{\overline{\partial}}(X) = (0) \), for all \( q \geq 0 \) and all \( p \geq 1 \).

**Proof.** Given any form \( \omega \in \bigwedge^{0,q}(PD(0,r)) \) with \( \overline{\partial} \omega = 0 \), we need to show that there is some \( \eta \in \bigwedge^{0,q-1}(PD(0,r)) \) so that \( \overline{\partial} \eta = \omega \). There are three steps to the proof.

**Step I.** Reduction to the case \( p = 0 \).

Say the lemma holds is \( \omega \in \bigwedge^{0,q}(PD(0,r)) \). Then, our \( \omega \) is of the form 
\[ \omega = \sum_{|I|=p} a_{I,J} d z_I \wedge d \overline{z}_J. \]

Write
\[ \omega_I = \sum_{|J|=q} a_{I,J} d \overline{z}_J \in \bigwedge^{0,q}(PD(0,r)). \]

Claim: \( \overline{\partial} \omega_I = 0 \).

We have \( \omega = \sum_{|I|=p} d z_I \wedge \omega_I \) and
\[ 0 = \overline{\partial} \omega = \sum_{|I|=p} \overline{\partial}(d z_I \wedge \omega_I) = \sum_{|I|=p} \pm d z_I \wedge \overline{\partial} \omega_I. \]

These terms are in the span of
\[ d z_{i_1} \wedge \cdots \wedge d z_{i_p} \wedge d \overline{z}_j \wedge d \overline{z}_{j_1} \wedge \cdots \wedge d \overline{z}_{j_q} \]

and by linear independence of these various wedges, we must have \( \overline{\partial} \omega_I = 0 \), for all \( I \). Then, by the assumption, there is some \( \eta_I \in \bigwedge^{0,q-1}(PD(0,r)) \), so that \( \overline{\partial} \eta_I = \omega_I \). It follows that 
\[ \omega = \sum_{|I|=p} d z_I \wedge \overline{\partial} \eta_I = \sum_{|I|=p} \pm \overline{\partial}(d z_I \wedge \eta_I) = \overline{\partial}(\sum_{|I|=p} \pm d z_I \wedge \eta_I), \]

with \( \sum_{|I|=p} \pm d z_I \wedge \eta_I \in \bigwedge^{p,q-1}(PD(0,r)) \), which concludes the proof of Step I.

**Step II:** Interior part of the proof.

We will prove that for every \( \epsilon > 0 \), there is some \( \eta \in \bigwedge^{0,q-1}(PD(0,r)) \) so that \( \overline{\partial} \eta = \omega \) in \( \text{PD}(0,r - \epsilon) \).

Let us say that \( \eta \) depends on \( d \overline{z}_1, \ldots, d \overline{z}_s \) if the terms \( a_J d \overline{z}_J \) in \( \eta \) where \( J \not\subseteq \{1, \ldots, s\} \) are all zero, i.e., in \( \eta \), only terms \( a_J d \overline{z}_J \) appear for \( J \subseteq \{1, \ldots, s\} \).
Claim: If $\omega$ depends on $d\bar{z}_1, \ldots, d\bar{z}_s$, then there is some $\eta \in \bigwedge^{0, q-1}(PD(0, r))$ so that $\omega - \bar{\partial}\eta$ depends only on $d\bar{z}_1, \ldots, d\bar{z}_{s-1}$ in $PD(0, r - \epsilon)$.

Clearly, if the claim is proved, the interior part is done by a trivial induction. In $\omega$, isolate the terms depending on $d\bar{z}_1, \ldots, d\bar{z}_{s-1}$, call these $\omega_2$ and $\omega_1$ the rest. Now, $\omega_1 = \theta \wedge d\bar{z}_s$, so $\omega = \theta \wedge d\bar{z}_s + \omega_2$ and we get

$$0 = \bar{\partial}\omega = \bar{\partial}(\theta \wedge d\bar{z}_s) + \bar{\partial}\omega_2.$$ (1)

Examine the terms

$$\frac{\partial a_J}{\partial \bar{z}_l} d\bar{z}_s \wedge d\bar{z}_J,$$ where $l > s$.

Linear independence and (1) imply

$$\frac{\partial a_J}{\partial \bar{z}_l} = 0 \text{ if } J \subseteq \{1, 2, \ldots, s-1\} \text{ and } l > s.$$ (2)

If $s \in J$, write $\bar{J} = J - \{s\}$. Look at the function

$$\eta_J(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{|\xi| \leq r - \epsilon} a_J(z_1, \ldots, z_{s-1}, \xi, z_{s+1}, \ldots, z_n) \frac{d\xi \wedge d\bar{\xi}}{\xi - z_s}.$$ (3)

We have the basic complex analysis lemma:

**Lemma 2.4** Say $g(\xi) \in C^\infty(\Delta_r)$ (where $\Delta_r$ is the open disc of radius $r$), then the function

$$f(z) = \frac{1}{2\pi i} \int_{|\xi| \leq r - \epsilon} g(\xi) \frac{d\xi \wedge d\bar{\xi}}{\xi - z}$$

is in $C^\infty(\Delta_r')$ and $\frac{\partial f}{\partial z} = g$ on $\Delta_{r-\epsilon}$.

By this lemma, we have

$$a_J(z_1, \ldots, z_n) = \frac{\partial \eta_J}{\partial \bar{z}_s} \text{ on } \Delta_{r-\epsilon}(z's)$$

and if $l > s$,

$$\frac{\partial \eta_J}{\partial \bar{z}_l} = \frac{1}{2\pi i} \int_{|\xi| \leq r - \epsilon} \frac{\partial a_J}{\partial \bar{z}_l} \frac{d\xi \wedge d\bar{\xi}}{\xi - z_s} = 0,$$

by the above. So, if we set $\eta = \sum_J \eta_J d\bar{z}_J$, then $\omega - \bar{\partial}\eta$ depends only on $d\bar{z}_1, \ldots, d\bar{z}_{s-1}$ in $PD(0, r - \epsilon)$.

**Step III**: Exhaustion.

Pick a sequence, $\{\epsilon_t\}$, with $\epsilon_t$ monotonically decreasing to 0 and examine $PD(0, r - \epsilon_t)$. Write $r_t = r - \epsilon_t$, then the sequence $\{r_t\}$ monotonically increases to $r$.

Claim. We can find a sequence, $\eta_t \in \bigwedge^{0, q-1}(PD(0, r))$, such that

1. $\eta_t$ has compact support in $PD(0, r_{t+1})$.
2. $\eta_t = \eta_{t-1}$ on $PD(0, r_{t-1})$.
3. $\bar{\partial}\eta_t = \omega$ on $PD(0, r_t)$.

We proceed by induction on $q$, here is the induction step. Pick a sequence of cutoff $C^\infty$-functions, $\gamma_t$, so that
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(i) $\gamma_t$ has compact support in $PD(0, r_{t+1})$.
(ii) $\gamma_t \equiv 1$ on $\overline{PD(0, r_t)}$.

Having chosen $\eta_t$, we will find $\eta_{t+1}$. First, by the interior part of the proof, there is some $\tilde{\eta}_{t+1} \in \Lambda^{0,a-1}(PD(0, r))$ with $\partial\tilde{\eta}_{t+1} = \omega$ in $PD(0, r_{t+1})$. Examine $\tilde{\eta}_{t+1} - \eta_t$ on $PD(0, r_t)$, then

$$\overline{\partial}(\tilde{\eta}_{t+1} - \eta_t) = \overline{\partial}\tilde{\eta}_{t+1} - \overline{\partial}\eta_t = \omega - \omega = 0.$$

By the induction hypothesis, there is some $\beta \in \Lambda^{0,a-2}(PD(0, r))$ with

$$\overline{\partial}\beta = \tilde{\eta}_{t+1} - \eta_t \text{ on } PD(0, r_t).$$

Let $\eta_{t+1} = \gamma_{t+1}(\tilde{\eta}_{t+1} - \overline{\partial}\beta) = \gamma_{t+1} \eta_t$. We have

1. $\eta_{t+1} \in C^\infty(\Lambda^{0,a-1}(PD(0, r_{t+2})))$.
2. As $\gamma_{t+1} \equiv 1$ on $PD(0, r_{t+1})$, we have $\eta_{t+1} = \tilde{\eta}_{t+1} - \overline{\partial}\beta$ and so, $\partial\eta_{t+1} = \partial\tilde{\eta}_{t+1} = \omega$ on $PD(0, r_{t+1})$.
3. $\eta_{t+1} - \eta_t = \tilde{\eta}_{t+1} - \overline{\partial}\beta - \eta_t = 0$ on $PD(0, r_t)$.

Now, for any compact subset, $K$, in $PD(0, r)$, there is some $t$ so that $K \subseteq PD(0, r_t)$. It follows that the $\eta_t$'s stabilize on $K$ and our sequence converges uniformly on compacta. Therefore,

$$\eta = \lim_{t \to \infty} \eta_t = \overline{\partial}\eta \text{ and } \partial\eta = \lim_{t \to \infty} \partial\eta_t = \omega.$$

Finally, we have to deal with the case $\gamma = 1$. Let $\omega \in \Lambda^{0,1}(PD(0, r))$, with $\partial\omega = 0$. Again, we need to find some functions, $\eta_t$, with compact support on $PD(0, r_{t+1})$, so that

(a) $\overline{\partial}\eta_t = \omega$ on $PD(0, r_t)$.

(b) $\eta_t$ converges uniformly on compacta to $\eta$, with $\overline{\partial}\eta = \omega$. Here, $\eta_t, \eta \in C^\infty(PD(0, r))$.

Say we found $\eta_t$ with

$$\|\eta_t - \eta_{t-1}\|_{\infty, PD(0, r_{t-1})} \leq \frac{1}{2^{t-1}}.$$

Pick $\eta_{t+1} \in C^\infty(PD(0, r))$, with $\overline{\partial}\eta_{t+1} = \omega$ on $PD(0, r_{t+1})$. Then, on $PD(0, r_t)$, we have

$$\overline{\partial}(\eta_{t+1} - \eta_t) = \overline{\partial}\eta_{t+1} - \overline{\partial}\eta_t = \omega - \omega = 0.$$ 

So, $\eta_{t+1} - \eta_t$ is holomorphic in $PD(0, r_t)$. Take the MacLaurin series for it and truncate it to the polynomial $\theta$ so that on the compact $PD(0, r_{t-1})$, we have

$$\|\eta_{t+1} - \eta_t - \theta\|_{\infty, PD(0, r_{t-1})} \leq \frac{1}{2^t}.$$

Take $\eta_{t+1} = \gamma_{t+1}(\eta_{t+1} - \theta)$. Now, $\eta_{t+1}$ has compact support on $PD(0, r_{t+2})$ and on $PD(0, r_{t+1})$, we have $\gamma_{t+1} \equiv 1$. This implies that $\eta_{t+1} = \eta_{t+1} - \theta$, so

$$\|\eta_{t+1} - \eta_t\|_{\infty, PD(0, r_{t-1})} \leq \frac{1}{2^t}$$

and

$$\partial\eta_{t+1} = \overline{\partial}\eta_{t+1} + \overline{\partial}\theta = \overline{\partial}\eta_{t+1} = \omega \text{ on } PD(0, r_{t+1}),$$

as $\theta$ is a polynomial. Therefore, the $\eta_t$'s converge uniformly on compacta and if $\eta = \lim_{t \to \infty} \eta_t$, we get $\overline{\partial}\eta = \omega$. \(\square\)
Corollary 2.5 (\(\partial\overline{\partial}\text{ Poincaré}\)) Say \(\omega \in \bigwedge^{p,q}(U)\), where \(U \subseteq X\) is an open subset of a complex manifold, \(X\), and assume \(d\omega = 0\). Then, for all \(x \in U\), there is a neighborhood, \(V \ni x\), so that \(\omega = \partial\overline{\partial}\eta\) on \(V\), for some \(\eta \in \bigwedge^{p-1,q-1}(V)\).

**Proof.** The statement is local on \(X\), therefore we may assume \(X = \mathbb{C}^n\). By ordinary \(d\text{-Poincaré}\), for every \(x \in X\), there is some open, \(V_1 \ni x\), and some \(\zeta \in \bigwedge^{p+q-1}(V_1)\), so that \(\omega = d\zeta\). Now,

\[
\bigwedge^{p+q-1}(V_1) = \bigwedge_{r+s=p+q-1}^{r,s}(V_1),
\]

so, \(\zeta = (\zeta_{r,s})\), where \(\zeta_{r,s} \in \bigwedge^{r,s}(V_1)\). We have

\[
\omega = d\zeta = \sum_{r,s} d\zeta_{r,s} = \sum_{r,s} (\partial + \overline{\partial})\zeta_{r,s}.
\]

Observe that if \((r,s) \neq (p-1,q)\) or \((r,s) \neq (p,q-1)\), then the \(\zeta_{r,s}\)'s have \(d\zeta_{r,s} \notin \bigwedge^{p,q}(V_1)\). It follows that \(\zeta_{r,s} = 0\) and we can delete these terms from \(\zeta\); we get \(\zeta = \zeta_{p-1,q} + \zeta_{p,q-1}\) with \(d\zeta = 0\). We also have

\[
\omega = d\zeta = (\partial + \overline{\partial})\zeta = \partial\zeta_{p-1,q} + \overline{\partial}\zeta_{p,q-1} + \partial\zeta_{p-1,q} + \partial\zeta_{p,q-1} = \omega + \overline{\partial}\zeta_{p-1,q} + \partial\zeta_{p,q-1},
\]

that is, \(\overline{\partial}\zeta_{p-1,q} + \partial\zeta_{p,q-1} = 0\). Yet, \(\overline{\partial}\zeta_{p-1,q}\) and \(\partial\zeta_{p,q-1}\) belong to different bigraded components, so \(\overline{\partial}\zeta_{p-1,q} = \partial\zeta_{p,q-1} = 0\). We now use the \(\overline{\partial}\) and \(\partial\text{-Poincaré}\) lemma to get a polydisc, \(V \subseteq V_1\) and some forms \(\eta_1\) and \(\eta_2\) in \(\bigwedge^{p-1,q-1}(V)\), so that \(\zeta_{p-1,q} = \overline{\partial}\eta_1\) and \(\zeta_{p,q-1} = \partial\eta_2\). We get

\[
\overline{\partial}\partial(\eta_1) = \partial\zeta_{p-1,q} \quad \text{and} \quad \partial\overline{\partial}(\eta_2) = -\partial\zeta_{p,q-1}
\]

and so,

\[
\overline{\partial}\partial(\eta_1 - \eta_2) = \partial\zeta_{p-1,q} + \overline{\partial}\zeta_{p,q-1} = \omega,
\]

which concludes the proof. \(\square\)

**Remark:** Take \(\mathcal{C}^\infty = \) the sheaf of germs of real-valued \(\mathcal{C}^\infty\)-functions on \(X\), then

\[
\mathcal{H} = \text{Ker} \left( \overline{\partial}\partial : \mathcal{C}^\infty \longrightarrow \bigwedge^1(X) \right)
\]

is called the sheaf of germs of pluri-harmonic functions.

**Corollary 2.6** With \(X\) as in Corollary 2.5, the sequences

\[
0 \longrightarrow \Omega^p_X \overset{\partial}{\longrightarrow} \bigwedge^p X \overset{\overline{\partial}}{\longrightarrow} \bigwedge^p X \overset{\partial}{\longrightarrow} \bigwedge^{p-1} X \overset{\overline{\partial}}{\longrightarrow} \bigwedge^{p-1} X \overset{\partial}{\longrightarrow} \cdots
\]

(when \(p = 0\), it is \(0 \longrightarrow \mathcal{O}_X \overset{\partial}{\longrightarrow} \bigwedge^0 X \overset{\overline{\partial}}{\longrightarrow} \bigwedge^0 X \overset{\partial}{\longrightarrow} \bigwedge^1 X \overset{\overline{\partial}}{\longrightarrow} \bigwedge^1 X \overset{\partial}{\longrightarrow} \cdots\)),

\[
0 \longrightarrow \Omega^q_X \overset{\partial\overline{\partial}}{\longrightarrow} \bigwedge^q X \overset{\overline{\partial}}{\longrightarrow} \bigwedge^q X \overset{\partial\overline{\partial}}{\longrightarrow} \bigwedge^{q-1} X \overset{\overline{\partial}}{\longrightarrow} \cdots
\]

and

\[
0 \longrightarrow \mathcal{H} \overset{\partial\overline{\partial}}{\longrightarrow} \mathcal{C}^\infty X \overset{\overline{\partial}\partial}{\longrightarrow} \bigwedge^1 X \overset{\partial}{\longrightarrow} \bigwedge^1 X \overset{\overline{\partial}}{\longrightarrow} \bigwedge^2 X \overset{\partial}{\longrightarrow} \cdots
\]

are resolutions (i.e., exact sequences of sheaves) of \(\Omega^p_X, \Omega^q_X, \mathcal{H}\), respectively.
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Proof. These are immediate consequences of $\bar{\partial}, \partial, \partial \bar{\partial}$ and d-Poincaré. 

In Corollary 2.6, the sheaf $\Omega^p_X$ is the sheaf of holomorphic $p$-forms (locally, $\omega = \sum a_idz_I$, where the $a_I$ are holomorphic functions), $\Omega^{\bar{\partial}}_X$ is the sheaf of anti-holomorphic $q$-forms ($\omega = \sum a_id\bar{z}_I$, where the $a_I$ are anti-holomorphic functions) and $\mathcal{H}$ is the sheaf of pluri-harmonic functions.

If $\mathcal{F}$ is a sheaf of abelian groups, by cohomology, we mean derived functor cohomology, i.e., we have

$$\Gamma: \mathcal{F} \mapsto \mathcal{F}(X) = \Gamma(X, \mathcal{F}),$$

a left-exact functor and

$$H^p(X, \mathcal{F}) = (R^p\Gamma)(\mathcal{F}) \in \text{Ab}. $$

We know that this cohomology can be computed using flasque (= flabby) resolutions

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G}_0 \rightarrow \mathcal{G}_1 \rightarrow \cdots \rightarrow \mathcal{G}_n \rightarrow \cdots,$$

where the $\mathcal{G}_i$'s are flasque, i.e., for every open, $U \subseteq X$, for every section $\sigma \in \mathcal{G}(U)$, there is a global section, $\tau \in \mathcal{G}(X)$, so that $\sigma = \tau \upharpoonright U$. If we apply $\Gamma$, we get a complex of (abelian) groups

$$0 \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{G}_0) \rightarrow \Gamma(X, \mathcal{G}_1) \rightarrow \cdots \rightarrow \Gamma(X, \mathcal{G}_n) \rightarrow \cdots, \tag{*}$$

and then $H^p(X, \mathcal{F}) = \text{the } p\text{th cohomology group of } (*)$.

Unfortunately, the sheaves arising naturally (from forms, etc.) are not flasque; they satisfy a weaker condition. In order to describe this condition, given a sheaf, $\mathcal{F}$, we need to make sense of $\mathcal{F}(S)$, where $S \subseteq X$ is a closed subset. Now, remember (see Appendix A on sheaves, Section A4) that for any subspace, $Y$ of $X$, if $j: Y \hookrightarrow X$ is the inclusion map, then for any sheaf, $\mathcal{F}$, on $X$, the sheaf $j^*\mathcal{F} = \mathcal{F} \upharpoonright Y$ is the restriction of $\mathcal{F}$ to $Y$. For every $x \in Y$, the stalk of $\mathcal{F} \upharpoonright Y$ at $x$ is equal to $\mathcal{F}_x$. Consequently, if $S$ is any subset of $X$, we have $\sigma \in \mathcal{F}(S)$ iff there is an open cover, $\{U_\alpha\}$, of $S$ and a family of sections, $\sigma_\alpha \in \mathcal{F}(U_\alpha)$, so that for every $\alpha$, we have

$$\sigma \upharpoonright S \cap U_\alpha = \sigma_\alpha \upharpoonright S \cap U_\alpha.$$ 

Remark: (Inserted by J.G.) If $X$ is paracompact, then for any closed subset, $S \subseteq X$, we have

$$\mathcal{F}(S) = \lim_{U \supseteq S} \mathcal{F}(U),$$

where $U$ ranges over all open subsets of $S$ (see Godement[5], Chapter 3, Section 3.3, Corollary 1). [Recall that for any cover, $\{U_\alpha\}_\alpha$ of $X$, we say that that $\{U_\alpha\}_\alpha$ is locally finite iff for every $x \in X$, there is some open subset, $U_x \ni x$, so that $U_x$ meets only finitely many $U_\alpha$. A topological space, $X$, is paracompact iff it is Hausdorff and if every open cover possesses a locally finite refinement.]

Now, we want to consider sheaves, $\mathcal{F}$, such that for every closed subset, $S$, the restriction map $\mathcal{F}(X) \rightarrow \mathcal{F}(S)$ is onto.

Definition 2.3 Let $X$ be a paracompact topological space. A sheaf, $\mathcal{F}$, is soft (mou) iff for every closed subset, $S \subseteq X$, the restriction map $\mathcal{F}(X) \rightarrow \mathcal{F} \upharpoonright S(S)$ is onto. A sheaf, $\mathcal{F}$, is fine iff for all locally finite open covers, $\{U_\alpha \rightarrow X\}$, there exists a family, $\{\eta_\alpha\}$, with $\eta_\alpha \in \text{End}(\mathcal{F})$, so that

1. $\eta_\alpha \upharpoonright \mathcal{F}_x = 0$, for all $x$ in some neighborhood of $U_\alpha^c$, i.e., $\text{supp}(\eta_\alpha) \subseteq U_\alpha$.
2. $\sum_\alpha \eta_\alpha = \text{id}.$

We say that the family $\{\eta_\alpha\}$ is a sheaf partition of unity subordinate to the given cover $\{U_\alpha \rightarrow X\}$ for $\mathcal{F}$. 

Remark: The following sheaves are fine on any complex or real $C^\infty$-manifold:

1. $C^\infty$
2. $\bigwedge^p$
3. $\bigwedge^{p,q}$
4. Any locally-free $C^\infty$-bundle ($= C^\infty$-vector bundle).

For, any open cover of our manifold has a locally finite refinement, so we may assume that our open cover is locally finite (recall, a manifold is locally compact and second-countable, which implies paracompactness). Then, take a $C^\infty$-partition of unity subordinate to our cover, $\{U_\alpha \to X\}$, i.e., a family of $C^\infty$-functions, $\varphi_\alpha$, so that

1. $\varphi_\alpha \geq 0$.
2. $\text{supp}(\varphi) < U_\alpha$ (this means $\text{supp}(\varphi)$ is compact and contained in $U_\alpha$).
3. $\sum_\alpha \varphi_\alpha = 1$.

Then, for $\eta_\alpha$, use multiplication by $\varphi_\alpha$.

Remark: If we know a sheaf of rings, $\mathcal{A}$, on $X$ is fine, then every $\mathcal{A}$-module is also fine and the same with soft.

**Proposition 2.7** Let $X$ be a paracompact space. Every fine sheaf is soft. Say

$$0 \to \mathcal{F}' \xrightarrow{\lambda} \mathcal{F} \xrightarrow{\mu} \mathcal{F}'' \to 0$$

is an exact sequence of sheaves and $\mathcal{F}'$ is soft. Then,

$$0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X) \to 0$$

is exact.

Again, if

$$0 \to \mathcal{F}' \xrightarrow{\lambda} \mathcal{F} \xrightarrow{\mu} \mathcal{F}'' \to 0$$

is an exact sequence of sheaves and if $\mathcal{F}'$ and $\mathcal{F}$ are soft, so is $\mathcal{F}''$. Every soft sheaf is cohomologically trivial ($H^p(X, \mathcal{F}) = (0)$ if $p > 0$).

**Proof.** Take $\mathcal{F}$ fine, $S$ closed and $\tau \in \mathcal{F}(S)$. There is an open cover of $S$ and sections, $\tau_\alpha \in \mathcal{F}(U_\alpha)$, so that $\tau_\alpha \upharpoonright U_\alpha \cap S = \tau \upharpoonright U_\alpha \cap S$. Let $U_0 = X - S$, an open, so that $U_0$ and the $U_\alpha$ cover $X$. By paracompactness, we may assume that the cover is locally finite. Take the $\eta_\alpha \in \text{Aut}(\mathcal{F})$ guaranteed as $\mathcal{F}$ is fine. Now, we have $\eta_\alpha(\tau_\alpha) = 0$ near the boundary of $U_\alpha$, so $\eta_\alpha(\tau_\alpha)$ extends to all of $X$ (as section) by zero, call it $\sigma_\alpha$. We have $\sigma_\alpha \in \mathcal{F}(X)$ and

$$\sigma = \sum_\alpha \sigma_\alpha$$

exists (by local finiteness).

As $\sigma_\alpha \upharpoonright U_\alpha \cap S = \tau_\alpha \upharpoonright U_\alpha \cap S$, we get

$$\sigma_\alpha = \eta_\alpha(\tau_\alpha) = \eta_\alpha(\tau) \quad \text{on } U_\alpha \cap S$$

and we deduce that

$$\sigma = \sum_\alpha \sigma_\alpha = \sum_\alpha \eta_\alpha(\tau_\alpha) = \sum_\alpha \eta_\alpha(\tau) = \left(\sum_\alpha \eta_\alpha\right)(\tau) = \tau; \quad \text{on } S.$$
Therefore, $\sigma$ is a lift of $\tau$ to $X$ from $S$.

Exactness of the sequence

$$0 \to \mathcal{F}' \xrightarrow{\lambda} \mathcal{F} \xrightarrow{\mu} \mathcal{F}'' \to 0$$

implies that for every $\sigma \in \mathcal{F}''(X)$, there is an open cover, $\{U_\alpha \to X\}$, and a family of sections, $\tau_\alpha \in \mathcal{F}(U_\alpha)$, so that $\mu(\tau_\alpha) = \sigma | U_\alpha$. By paracompactness, we may replace the $U_\alpha$'s by a locally finite family of closed sets, $S_\alpha$. Consider the set

$$S = \left\{ (\tau, S) \mid \begin{array}{l}
(1) \quad S = \bigcup S_\alpha, \text{ for some of our } S_\alpha \\
(2) \quad \tau \in \mathcal{F}(S), \quad \tau | S_\alpha = \tau_\alpha, \text{ for each } S_\alpha \text{ as in (1).}
\end{array} \right\}$$

The set $S$ is, as usual, partially ordered and it is inductive (DX). By Zorn's lemma, $S$ possesses a maximal element, $(\tau, S)$. I claim that $X = S$.

If $S \neq X$, then there is some $S_\beta$ with $S_\beta \subset S$. On $S \cap S_\beta$, we have

$$\mu(\tau - \tau_\beta) = \sigma - \sigma = 0,$$

where $\mu(\tau) = \sigma$, by (2), and $\mu(\tau_\beta) = \sigma$, by definition. By exactness, there is some $\zeta \in \mathcal{F}'(S \cap S_\beta)$ so that $\lambda(\zeta) = \tau - \tau_\beta$ on $S \cap S_\beta$. Now, as $\mathcal{F}'$ is soft, $\zeta$ extends to a global section of $\mathcal{F}'$, say, $z$. Define $\omega$ by

$$\omega = \begin{cases} 
\tau & \text{on } S \\
\tau_\beta + \lambda(z) & \text{on } S_\beta.
\end{cases}$$

On $S \cap S_\beta$, we have $\omega = \tau = \tau_\beta + \lambda(z) = \tau_\beta + \lambda(\zeta) = \tau$, so $\omega$ and $\tau$ agree. But then, $(\omega, S \cup S_\beta) \in S$ and $(\omega, S \cup S_\beta) > (\tau, S)$, a contradiction. Therefore, the sequence

$$0 \to \mathcal{F}' \xrightarrow{\lambda} \mathcal{F} \xrightarrow{\mu} \mathcal{F}'' \to 0$$

has globally exact sections.

Now, assume that $\mathcal{F}'$ and $\mathcal{F}$ are soft and take $\tau \in \mathcal{F}''$, with $S$ closed. Apply the above to $X = S$; as $\mathcal{F}'$ is soft, we deduce that $\mathcal{F}(S) \to \mathcal{F}''(S)$ is onto. As $\mathcal{F}$ and $\mathcal{F}'$ are soft, the commutative diagram

$$\begin{array}{ccc}
\mathcal{F}(X) & \to & \mathcal{F}''(X) \\
\downarrow & & \downarrow \\
\mathcal{F}(S) & \to & \mathcal{F}''(S) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

implies that $\mathcal{F}''(X) \to \mathcal{F}''(S)$ is surjective.

For the last part, we use induction. The induction hypothesis is: If $\mathcal{F}$ is soft, then $H^p(X, \mathcal{F}) = (0)$, for $0 < p \leq n$. When $n = 1$, we can embed $\mathcal{F}$ in a flasque sheaf, $Q$, and we have the exact sequence

$$0 \to \mathcal{F} \to Q \to \text{cok} \to 0. \quad (\dagger)$$

If we apply cohomology we get

$$0 \to H^1(X, \mathcal{F}) \to H^1(X, Q) = (0),$$

since $Q$ is flasque, so $H^1(X, \mathcal{F}) = (0)$. 

For the induction step, use (†) and note that cok is soft because \( F \) and \( Q \) are soft (\( Q \) is flasque and flasque sheaves are soft over a paracompact space, see Homework). When we apply cohomology, we get
\[
(0) = H^j(X, Q) \rightarrow H^j(X, \text{cok}) \rightarrow H^{j+1}(X, F) \rightarrow H^{j+1}(X, Q) = (0), \quad (j \geq 1)
\]
so \( H^j(X, \text{cok}) \cong H^{j+1}(X, F) \). As cok is soft, by the induction hypothesis, \( H^j(X, \text{cok}) = (0) \), so \( H^{j+1}(X, F) = (0) \).

**Corollary 2.8** Each of the resolutions \((p > 0)\)
\[
0 \longrightarrow \Omega^p_X \longrightarrow \bigwedge^p X \overset{\partial}{\longrightarrow} \bigwedge^{p+1} X \overset{\partial}{\longrightarrow} \cdots,
\]
(for \( p = 0 \), a resolution of \( \mathcal{O}_X \)),
\[
0 \longrightarrow \Omega^q_X \longrightarrow \bigwedge^q X \overset{\partial}{\longrightarrow} \bigwedge^{q+1} X \overset{\partial}{\longrightarrow} \cdots,
\]
and
\[
0 \longrightarrow \mathbb{R} \overset{c}{\longrightarrow} \bigwedge^0 X = C^\infty \overset{d}{\longrightarrow} \bigwedge^1 X \overset{d}{\longrightarrow} \cdots,
\]
is an acyclic resolution (i.e., the cohomology of \( \bigwedge^p X, \bigwedge^q X \) vanishes).

**Proof.** The sheaves \( \bigwedge^p X, \bigwedge^q X \) are fine, therefore soft, by Proposition 2.7. \( \square \)

Recall the spectral sequence of Čech cohomology (\( \check{S}S \)):
\[
E_2^{p,q} = \check{H}^p(X, \mathcal{H}^q(F)) \Rightarrow H^*(X, F),
\]
where
(1) \( F \) is a sheaf of abelian groups on \( X \)
(2) \( \mathcal{H}^q(F) \) is the presheaf defined by \( U \hookrightarrow H^q(U, F) \).

Now, we have the following vanishing theorem (see Godement [5]):

**Theorem 2.9** (Vanishing Theorem) Say \( X \) is paracompact and \( F \) is a presheaf on \( X \) so that \( F^\sharp (= \text{associated sheaf to } F) \) is zero. Then,
\[
\check{H}^p(X, F) = (0), \quad \text{all } p \geq 0.
\]

Putting the vanishing theorem together with the spectral sequence (\( \check{S}S \)), we get:

**Theorem 2.10** (Isomorphism Theorem) If \( X \) is a paracompact space, then for all sheaves, \( F \), the natural map
\[
\check{H}^p(X, F) \longrightarrow H^p(X, F)
\]
is an isomorphism for all \( p \geq 0 \).

**Proof.** The natural map \( \check{H}^p(X, F) \longrightarrow H^p(X, F) \) is just the edge homomorphism from (\( \check{S}S \)). By the handout on cohomology,
\[
\mathcal{H}^q(F)^\sharp = (0), \quad \text{all } q \geq 1.
\]
Thus, the vanishing says
\[
E_2^{p,q} = \check{H}^p(X, \mathcal{H}^q(F)) = (0), \quad \text{all } p \geq 0, q \geq 1,
\]
which implies that the spectral sequence (\( \check{S}S \)) degenerates and we get our isomorphism. \( \square \)
2.2. COHOMOLOGY, DE RHAM, DOLBEAULT

**Comments:** How to get around the spectral sequence \( (\hat{S}S) \).

(1) Look at the presheaf \( \mathcal{F} \) and the sheaf \( \mathcal{F}^\circ \). There is a map of presheaves, \( \mathcal{F} \rightarrow \mathcal{F}^\circ \), so we get a map, \( \hat{H}^p(X, \mathcal{F}) \rightarrow \hat{H}^p(X, \mathcal{F}^\circ) \). Let \( K = \text{Ker}(\mathcal{F} \rightarrow \mathcal{F}^\circ) \) and \( C = \text{Coker}(\mathcal{F} \rightarrow \mathcal{F}^\circ) \). We have the short exact sequences of presheaves

\[
0 \rightarrow K \rightarrow \mathcal{F} \rightarrow \text{Im} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \text{Im} \rightarrow \mathcal{F}^\circ \rightarrow C \rightarrow 0,
\]

where \( \text{Im} \) is the presheaf image \( \mathcal{F} \rightarrow \mathcal{F}^\circ \). The long exact sequence of \( \check{\text{C}} \)ech cohomology for presheaves gives

\[
\cdots \rightarrow \hat{H}^p(X, K) \rightarrow \hat{H}^p(X, \mathcal{F}) \rightarrow \hat{H}^p(X, \text{Im}) \rightarrow \hat{H}^{p+1}(X, K) \rightarrow \cdots
\]

and

\[
\cdots \rightarrow \hat{H}^{p-1}(X, C) \rightarrow \hat{H}^p(X, \text{Im}) \rightarrow \hat{H}^p(X, \mathcal{F}^\circ) \rightarrow \hat{H}^p(X, C) \rightarrow \cdots,
\]

and as \( K^\circ = C^\circ = (0) \), by the vanishing theorem, we get

\[
\hat{H}^p(X, \mathcal{F}) \cong \hat{H}^p(X, \text{Im}) \cong \hat{H}^p(X, \mathcal{F}^\circ).
\]

Therefore, on a paracompact space, \( \hat{H}^p(X, \mathcal{F}) \cong \hat{H}^p(X, \mathcal{F}^\circ) \).

(2) \( \check{\text{C}} \)ech cohomology is a \( \delta \)-functor on the category of sheaves for paracompact \( X \).

Say

\[
0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0
\]

is exact as sheaves. Then, if we write \( \text{Im} \) for \( \text{Im}(\mathcal{F} \rightarrow \mathcal{F}'') \) as presheaves, we have the short exact sequence of presheaves

\[
0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \text{Im} \rightarrow 0
\]

and \( \text{Im}^\circ = \mathcal{F}'' \). Then, for presheaves, we have

\[
\cdots \rightarrow \hat{H}^p(X, \mathcal{F}) \rightarrow \hat{H}^p(X, \text{Im}) \rightarrow \hat{H}^{p+1}(X, \mathcal{F}') \rightarrow \cdots
\]

and by (1), \( \hat{H}^p(X, \mathcal{F}) \cong \hat{H}^p(X, \mathcal{F}^\circ) \), so we get (2).

(3) One knows, for soft \( \mathcal{F} \) on a paracompact space, \( X \), we have \( \hat{H}^p(X, \mathcal{F}) = (0) \), for all \( p \geq 1 \). Each \( \mathcal{F} \) embeds in a flasque sheaf; flasque sheaves are soft, so \( \{ \hat{H}^p \} \) is an effaceable \( \delta \)-functor on the category of sheaves and it follows that \( \{ \hat{H}^p \} \) is universal. By homological algebra, we get the isomorphism theorem, again.

In fact, instead of (3), one can prove the following proposition:

**Proposition 2.11** Say \( X \) is paracompact and \( \mathcal{F} \) is a fine sheaf. Then, for a locally finite cover, \( \{ U_\alpha \rightarrow X \} \), we have

\[
\hat{H}^p(\{ U_\alpha \rightarrow X \}, \mathcal{F}) = (0), \quad \text{if} \ p \geq 1.
\]

**Proof.** Take \( \{ \eta_\alpha \} \), the sheaf partition of unity of \( \mathcal{F} \) subordinate to our cover, \( \{ U_\alpha \rightarrow X \} \). Pick \( \tau \in Z^p(\{ U_\alpha \rightarrow X \}, \mathcal{F}) \), with \( p \geq 1 \). So, we have \( \tau = \tau(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}) \). Write

\[
\omega = \sum_3 \eta_\beta(\tau(U_\beta \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})).
\]

Observe that \( \omega \) exists as section over \( U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \) as \( \eta_\beta \) is zero near the boundary of \( U_\beta \); so \( \omega \) can be extended from \( U_\beta \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \) to \( U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \) by zero. You check (usual computation): \( d\omega = \tau \).

**Corollary 2.12** If \( \mathcal{F} \) is fine (over a paracompact, \( X \)), then

\[
\hat{H}^p(X, \mathcal{F}) = (0), \quad \text{for all} \ p \geq 1.
\]
Theorem 2.13 (P. Dolbeault) If \( X \) is a complex manifold, then we have the isomorphisms
\[
H^q(X, \Omega^p_X) \cong H^{p,q}_\partial(X) \cong \check{H}^q(X, \Omega^p_X).
\]

Proof. The middle cohomology is computed from the resolution of sheaves
\[
0 \rightarrow \Omega^p_X \rightarrow \bigwedge^p X \rightarrow \bigwedge^{p,1} X \rightarrow \bigwedge^{p,2} X \rightarrow \ldots.
\]
Moreover, the \( \bigwedge^{p,q} X \) are acyclic for \( H^\bullet(X, -) \) and for \( \check{H}^\bullet(X, -) \). Yet, by homological algebra, we can compute \( H^q(X, \Omega^p_X) \) and \( \check{H}^q(X, \Omega^p_X) \) by any acyclic resolution (they are \( \delta \)-functors).

To prove de Rham’s theorem, we need to look at singular cohomology.

Proposition 2.14 If \( X \) is a real or complex manifold and \( \mathcal{F} \) is a constant sheaf (sheaf associated with a constant presheaf), then there is a natural isomorphism
\[
\check{H}^p(X, \mathcal{F}) \cong H^p_{\text{sing}}(X, \mathcal{F}),
\]
provided \( \mathcal{F} \) is torsion-free.

Proof. The space, \( X \), is triangulable, so we get a singular simplicial complex, \( \mathcal{K} \) (see Figure 2.1). Pick a vertex, \( v \), of \( \mathcal{K} \) and set
\[
\text{St}(v) = \bigcup \{ \sigma \in \mathcal{K} \mid v \in \sigma \},
\]
the open star of \( v \) (i.e., the union of the interiors of the simplices having \( v \) as a vertex). If \( v_0, \ldots, v_p \) are vertices, consider
\[
\text{St}(v_0) \cap \cdots \cap \text{St}(v_p) = U_{v_0, \ldots, v_p}.
\]
We have
\[
U_{v_0, \ldots, v_p} = \begin{cases} 
\emptyset & \text{if } v_0, \ldots, v_p \text{ are not the vertices of a } p\text{-simplex} \\
\text{a connected set} & \text{if } v_0, \ldots, v_p \text{ are the vertices of a } p\text{-simplex}.
\end{cases}
\]
Observe that \( \{ U_v \rightarrow X \}_{v \in \text{vert}(\mathcal{K})} \) is an open cover of \( X \) and as \( \mathcal{F} \) is a constant sheaf, we get
\[
\mathcal{F}(U_{v_0, \ldots, v_p}) = \begin{cases} 
\emptyset & \text{if } (v_0, \ldots, v_p) \notin \mathcal{K} \\
\mathcal{F} & \text{if } (v_0, \ldots, v_p) \in \mathcal{K}.
\end{cases}
\]
Let \( \tau \) be a Čech \( p \)-cochain, then \( \tau(U_{v_0, \ldots, v_p}) \in \mathcal{F} \) and let
\[
\Theta(\tau)((v_0, \ldots, v_p)) = \tau(U_{v_0, \ldots, v_p}),
\]
where \( (v_0, \ldots, v_p) \in \mathcal{K} \). Note that \( \Theta(\tau) \) is a \( p \)-simplicial cochain and the map \( \tau \mapsto \Theta(\tau) \) is an isomorphism
\[
C^p(\{ U_v \rightarrow X \}, \mathcal{F}) \cong C^p_{\text{sing}}(X, \mathcal{F})
\]
that commutes with the coboundary operators on both sides. So, we get the isomorphism
\[
\check{H}^p(\{ U_v \rightarrow X \}, \mathcal{F}) \cong H^p_{\text{sing}}(X, \mathcal{F}).
\]
We can subdivide \( \mathcal{K} \) simplicially and we get refinements of our cover and those are arbitrarily fine. Subdivision does not change the right hand side and if we take right limits we get
\[
\check{H}^p(X, \mathcal{F}) \cong H^p_{\text{sing}}(X, \mathcal{F}).
\]
As a consequence, we obtain
Theorem 2.15 (de Rham) On a real or complex manifold, we have the isomorphisms

\[ H^p(X, \mathbb{R}) \cong \hat{H}^p(X, \mathbb{C}) \cong H^p_{\text{sing}}(X, \mathbb{C}) \cong H^p_{\text{DR}}(X, \mathbb{C}) \]

Proof. The isomorphism of singular cohomology with Čech cohomology follows from Proposition 2.14. The isomorphism of derived functor cohomology with Čech cohomology follows since \( X \) is paracompact. Also de Rham cohomology is the cohomology of the resolution

\[ 0 \to \mathbb{R} \to \mathcal{C}_d^\infty \to \bigwedge^1 X \to \bigwedge^2 X \to \cdots, \]

and the latter is an acyclic resolution, so it computes \( H^p \) or \( \hat{H}^p \).

Explicit Connection: de Rham \( \rightsquigarrow \) Singular.

Take a singular \( p \)-chain, \( \sum_j a_j \Delta_j \), where \( \Delta_j = f_j(\Delta) \); \( f_j \in \mathcal{C}(\Delta) \); \( \Delta = \) the usual \( p \)-simplex (\( a_j \in \mathbb{Z} \), or \( a_j \in \mathbb{R} \), or \( a_j \in \mathbb{C} \), ...). We say that this \( p \)-chain is piecewise smooth, for short, ps, if the \( f_j \)’s actually are \( \mathcal{C}^\infty \)-functions on a small neighborhood around \( \Delta \). By the usual \( \mathcal{C}^\infty \)-approximation (using convolution), each singular \( p \)-chain is approximated by a ps \( p \)-chain in such a way that cocycles are approximated by ps cocycles and coboundaries, too. In fact, the inclusion

\[ C^\text{ps}_p(X, \mathbb{R}) \hookrightarrow C^\text{sing}_p(X, \mathbb{R}) \]

is a chain map and induces an isomorphism

\[ H^p_{\text{ps}}(X, \mathbb{R}) \cong H^p_{\text{sing}}(X, \mathbb{R}). \]

Say \( \omega \in \bigwedge^p X \), a de Rham \( p \)-cochain, i.e., a \( p \)-form. If \( \sigma \in C^\text{ps}_p(X, \mathbb{R}) \), say \( \sigma = \sum_j a_j f_j(\Delta) \) (with \( a_j \in \mathbb{R} \)), then define \( \Phi(\omega) \) via:

\[ \Phi(\omega)(\sigma) = \int_\sigma \omega \overset{\text{def}}{=} \sum_j a_j \int_{f_j(\Delta)} \omega \overset{\text{def}}{=} \sum_j a_j \int_\Delta f_j^* \omega \in \mathbb{R}. \]

The map \( \Phi(\omega) \) is clearly a linear map on \( C^\text{ps}_p(X, \mathbb{R}) \), so we have \( \Phi(\omega) \in C^\text{ps}_p(X, \mathbb{R}) \). Also, observe that

\[ \Phi(d\omega)(\tau) = \int_\tau \omega = \int_{\partial \tau} \omega \ (\text{by Stokes}) = \Phi(\omega)(\partial \tau), \]

from which we conclude that \( \Phi(d\omega)(\tau) = (\partial \Phi)(\omega)(\tau) \), and thus, \( \Phi(d\omega) = \partial \Phi(\omega) \). This means that

\[ \int : \bigwedge^p (X, \mathbb{R}) \to C^\text{ps}_p(X, \mathbb{R}) \]

is a cochain map and so, we get our map

\[ H^p_{\text{DR}}(X, \mathbb{R}) \to H^p_{\text{sing}}(X, \mathbb{R}). \]

2.3 Hodge I, Analytic Preliminaries

Let \( X \) be a complex analytic manifold. An Hermitian metric on \( X \) is a \( C^\infty \)-section of the vector bundle \((T^1_{X,0} \otimes \overline{T^1_{X,0}})^D \), which is Hermitian symmetric and positive definite. This means that for each \( z \in X \), we have a map \((-,-)_z : T^1_{X,z} \otimes T^1_{X,z} \to \mathbb{C} \) which is linear in its first argument, Hermitian symmetric and positive definite, that is:
(1) \((v, u)_z = (u, v)_z\) (Hermitian symmetric)

(2) \((u_1 + u_2, v)_z = (u_1, v)_z + (u_2, v)_z\) and \((u, v_1 + v_2)_z = (u, v_1)_z + (u, v_2)_z\).

(3) \((\lambda u, v)_z = \lambda(u, v)_z\) and \((u, \mu v)_z = \overline{\mu}(u, v)_z\).

(4) \((u, u)_z \geq 0\), for all \(u\), and \((u, u)_z = 0\) iff \(u = 0\) (positive definite).

(5) \(z \mapsto h(z) = (\alpha, \beta)_z\) is a \(C^\infty\)-function.

**Remark:** Note that (2) and (3) is equivalent to saying that we have a \(\mathbb{C}\)-linear map, \(T^{1,0}_{X,z} \otimes T^{0,1}_{X,z} \to \mathbb{C}\).

In local coordinates, since \((T^{1,0}_X)^D = \Lambda^{1,0}T^D_X\) and \(\overline{T^{1,0}_X} = T^{0,1}_X\) and since \(\{dz_j\}, \{d\overline{z}_j\}\) are bases for \(\Lambda^{1,0}T^D_{X,z}\) and \(\Lambda^{0,1}T^D_{X,z}\), we get

\[
h(z) = \sum_{k,l} h_{kl}(z)dz_k \otimes d\overline{z}_l,
\]

for some matrix \((h_{kl}) \in M_n(\mathbb{C})\). Now, \((-\cdot)_z\) is an Hermitian inner product, so locally on a trivializing cover for \(T^{1,0}_X\), \(T^{0,1}_X\), by Gram-Schmidt, we can find \((1,0)\)-forms, \(\varphi_1, \ldots, \varphi_n\), so that

\[
(-\cdot)_z = \sum_{j=1}^n \varphi_j(z) \otimes \overline{\varphi_j(z)}.
\]

The collection \(\varphi_1, \ldots, \varphi_n\) is called a coframe for \((-\cdot)_z\) (on the respective open of the trivializing cover). Using a partition of unity subordinate to a trivializing cover, we find all these data exist on any complex manifold.

Consider \(\Re(-\cdot)_z\) and \(\Im(-\cdot)_z\). For \(\lambda \in \mathbb{R}\), (1), (2), (3), (4) imply that \(\Re(-\cdot)_z\) is a positive definite bilinear form, \(C^\infty\) as a function of \(z\), i.e., as \(T^*_{X,z}\) real tangent space \(\cong T^{1,0}_{X,z}\); we see that \(\Re(-\cdot)_z\) is a \(C^\infty\)-Riemannian metric on \(X\). Hence, we have concepts such as length, area, volume, curvature, etc., associated to an Hermitian metric, namely, those concepts for the real part of \((-\cdot)_z\), i.e., the associated Riemannian metric.

If we look at \(\Im(-\cdot)_z\), then (1), (2), (3) and (5) imply that for \(\lambda \in \mathbb{R}\), we have an alternating real bilinear nondegenerate form on \(T^{1,0}_{X,z}\), \(C^\infty\) in \(z\). That is, we get an element of \((T^{1,0}_{X,z} \wedge T^{1,0}_{X,z})^D \subseteq \Lambda^2(T^D_{X,z} \otimes \mathbb{C})\). In fact, this is a \((1,1)\)-form. Look at \(\Im(-\cdot)_z\) in a local coframe. Say \(\varphi_k = \alpha_k + i\beta_k\), where \(\alpha_k, \beta_k \in T^D_{X,z}\). We have

\[
\sum_k \varphi_k(z) \otimes \overline{\varphi_k(z)} = \sum_k (\alpha_k(z) + i\beta_k(z)) \otimes (\alpha_k(z) - i\beta_k(z)) = \sum_k (\alpha_k(z) \otimes \alpha_k(z) + \beta_k(z) \otimes \beta_k(z) + i(\beta_k(z) \otimes \alpha_k(z) - \alpha_k(z) \otimes \beta_k(z))).
\]

Now, a symmetric bilinear form yields a linear form on \(S^2T_{X,z} = S^2T^{1,0}_{X,z}\); consequently, the real part of the Hermitian inner product is \(\Re(-\cdot)_z = \sum_k (\alpha_k(z))^2 + (\beta_k(z))^2\). We usually write \(ds^2\) for \(\sum_k \varphi_k \otimes \overline{\varphi_k}\) and \(\Re(ds^2)\) is the associated Riemannian metric. For \(\Im(ds^2)\), we have a form in \(\Lambda^2(T^{1,0}_X)^D\):

\[
\Im(ds^2) = -2 \sum_{k=1}^n \alpha_k \wedge \overline{\alpha_k}.
\]

We let

\[
\omega_{ds^2} = \omega = -\frac{1}{2} \Im(ds^2)
\]
2.3. Hodge I, Analytic Preliminaries

and call it the associated \((1,1)\)-form to the Hermitian \(ds^2\). If we write \(\varphi_k = \alpha_k + i\beta_k\), we have

\[
\sum_{k=1}^{n} \varphi_k \wedge \overline{\varphi_k} = \sum_{k=1}^{n} (\alpha_k + i\beta_k) \wedge (\alpha_k - i\beta_k) = -2i \sum_{k=1}^{n} \alpha_k \wedge \beta_k.
\]

Therefore,

\[
\omega = \sum_{k=1}^{n} \alpha_k \wedge \beta_k = \frac{i}{2} \sum_{k=1}^{n} \varphi_k \wedge \overline{\varphi_k},
\]

which shows that \(\omega\) is a \((1,1)\)-form.

Remark: The expression for \(\omega\) in terms of \(\Im(ds^2)\) given above depends on the definition of \(\wedge\). In these notes,

\[
\alpha \wedge \beta = \frac{1}{2}(\alpha \otimes \beta - \beta \otimes \alpha),
\]

but in some books, one finds

\[
\omega = -\Im(ds^2).
\]

Conversely, suppose we are given a real \((1,1)\)-form. This means, \(\omega\) is a \((1,1)\)-form and for all \(\xi\),

\[
\omega(\xi) = \overline{\omega(\xi)} \quad \text{(reality condition)}.
\]

Define an “inner product” via

\[
H(v, w) = \omega(v \wedge iw).
\]

We have

\[
H(w, v) = \omega(w \wedge iw) \\
= -\omega(iw \wedge w) \\
= \omega(iw \wedge \overline{w}) \\
= \overline{\omega(iw \wedge \overline{w})} \\
= \omega(v \wedge i\overline{w}) \\
= \overline{H(v, w)}.
\]

(Note we could also set \(H(v, w) = -\omega(v \wedge i\overline{w})\).) Consequently, \(H(v, w)\) will be an inner product provided \(H(v, v) > 0\) iff \(v \neq 0\). So, we need \(\omega(v \wedge iw) = -i\omega(v \wedge \overline{v}) > 0\), for all \(v \neq 0\). Therefore, we say \(\omega\) is positive definite iff

\[
-i\omega(v \wedge \overline{v}) > 0, \quad \text{for all } v \neq 0.
\]

Thus, \(\omega = -(1/2)\Im(ds^2)\) recaptures all of \(ds^2\). You check (DX) that \(\omega\) is positive definite iff in local coordinates

\[
\omega = \frac{i}{2} \sum_{k,l} h_{kl}(z) dz_k \wedge d\overline{z}_l,
\]

where \((h_{kl})\) is a Hermitian positive definite matrix.

Example 1. Let \(X = \mathbb{C}^n\), with \(ds^2 = \sum_{k=1}^{n} dz_k \otimes d\overline{z}_k\). As usual, if \(z_k = x_k + iy_k\), we have

(a) \(\Re(ds^2) = \sum_{k=1}^{n} (dx_k^2 + dy_k^2)\), the ordinary Euclidean metric.

(b) \(\omega = -(1/2)\Im(ds^2) = (i/2) \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k\), a positive definite \((1,1)\)-form.
Remark: Assume that \( f: Y \to X \) is a complex analytic map and that we have an Hermitian metric on \( X \). Then, \( Df: T_Y \to T_X \) maps \( T_{Y,y}^{1,0} \) to \( T_{X,f(y)}^{1,0} \), for all \( y \in Y \). We define an “inner product” on \( Y \) via
\[
\left( \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} \right)_y = \left(Df \frac{\partial}{\partial y_k}, Df \frac{\partial}{\partial y_l}\right)_{f(y)}.
\]
We get a Hermitian symmetric form on \( Y \). If we assume that \( Df \) is everywhere an injection, then our Hermitian metric, \( ds^2 \), on \( X \) induces one on \( Y \); in particular, this holds if \( Y \hookrightarrow X \).

Assume \( Df \) is injective everywhere. We have the dual map, \( f^*: T_Y^0 \to T_X^0 \), i.e., \( f^*: \wedge^{1,0} X \to \wedge^{1,0} Y \). Pick \( U \) small enough in \( Y \) so that
\begin{enumerate}
\item \( T_Y \mid U \) is trivial
\item \( T_X \mid f(U) \) is trivial.
\item We have a local coframe, \( \varphi_1, \ldots, \varphi_n \), on \( T_X \mid f(U) \) and \( f^*(\varphi_{m+1}) = \cdots = f^*(\varphi_n) = 0 \), where \( m = \dim(Y) \) and \( n = \dim(X) \).
\end{enumerate}

Then,
\[
f^* \omega_X = f^* \left( \frac{i}{2} \sum_{k=1}^{n} \varphi_k \wedge \overline{\varphi_k} \right) = \frac{i}{2} \sum_{k=1}^{m} f^*(\varphi_k) \wedge f^*(\overline{\varphi_k}) = \omega_Y.
\]
Hence, the \((1,1)\)-form of the induced metric on \( Y \) (from \( X \)) is the pullback of the \((1,1)\)-form of the metric on \( X \).

Consequently (Example 1), on an affine variety, we get an induced metric and an induced form computable from the embedding in some \( \mathbb{C}^N \).

Example 2: Fubini-Study Metric on \( \mathbb{P}^n \). Let \( \pi \) be the canonical projection, \( \pi: \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n \), let \( z_0, \ldots, z_n \) be coordinates on \( \mathbb{C}^{n+1} \) and let \( (Z_0: \cdots : Z_n) \) be homogeneous coordinates on \( \mathbb{P}^n \). For a small open \( U \), pick some holomorphic section, \( F: U \to \mathbb{C}^{n+1} - \{0\} \), of \( \pi \) (so that \( \pi \circ F = \text{id}_U \)). For any \( p \in U \), consider
\[
\|F(p)\|^2 = \sum_{j=0}^{n} F_j(p) \overline{F_j(p)} \neq 0.
\]
Pick \( U \) small enough so that \( \log \|F\|^2 \) is defined. Now, set
\[
\omega_F = \frac{i}{2\pi} \partial \overline{\partial} \log \|F\|^2.
\]
We need to show that this definition does not depend on the choice of the holomorphic section, \( F \). So, let \( S \) be another holomorphic section of \( \pi \) over \( U \). As \( \pi \circ S = \pi \circ F = \text{id}_U \), we have
\[
(S_0(p): \cdots : S_n(p)) = (F_0(p): \cdots : F_n(p)), \quad \text{for all } p \in U,
\]
so, there is a holomorphic function, \( \lambda \), on \( U \), so that
\[
\lambda(p)S(p) = F(p), \quad \text{for all } p \in U.
\]
We have
\[
\|F\|^2 = \overline{F} F = \lambda \overline{S} S = \lambda \overline{\lambda} \|S\|^2,
\]
so we get
\[
\log \|F\|^2 = \log \lambda + \log \overline{\lambda} + \log \|F\|^2.
\]
Consequently,

\[ \omega_F = \frac{i}{2\pi} \partial \overline{\partial} (\log \lambda + \log \overline{\lambda}) + \omega_S = \omega_S, \]

since \( \lambda \) is holomorphic, \( \overline{\lambda} \) is anti-holomorphic, \( \partial (\text{holo}) = 0, \overline{\partial} (\text{holo}) = 0 \), \( \partial \overline{\partial} = -\overline{\partial} \partial \) and \( \partial (\text{anti-holo}) = 0 \). Clearly, our \( \omega_F \) are \((1,1)\)-forms. Now, cover \( \mathbb{P}^n \) by opens, as above; pick any section on each such open, use a partition of unity and get a \textit{global} \((1,1)\)-form on \( \mathbb{P}^n \) which is \( C^\infty \). We still need to check positivity, but since the unitary group, \( U(n + 1) \), acts transitively on \( \mathbb{C}^{n+1} \), we see that \( \mathbb{P}U(n) \) acts transitively on \( \mathbb{P}^n \) and our form is invariant. Therefore, it is enough to check positivity at one point, say \((1:0: \cdots :0)\). This point lies in the open \( Z_0 \neq 0 \). Lift \( Z_0 \) to \( \mathbb{C}^{n+1} - \{0\} \) via

\[ F: (Z_0: \cdots : Z_n) \mapsto (1, z_1, \ldots, z_n), \quad \text{where} \quad z_j = \frac{Z_j}{Z_0}. \]

Thus, \( \|F\|^2 = 1 + \sum_{k=1}^n z_k \bar{z}_k \), and we get

\[
\partial \overline{\partial} \log \left( 1 + \sum_{k=1}^n z_k \bar{z}_k \right) = \partial \left( \frac{\sum_{k=1}^n z_k d\bar{z}_k}{1 + \sum_{k=1}^n z_k \bar{z}_k} \right)
= \left( \sum_{k=1}^n dz_k \wedge d\bar{z}_k \right) \left( 1 + \sum_{k=1}^n z_k \bar{z}_k \right) - \left( \sum_{k=1}^n \bar{z}_k dz_k \right) \wedge \left( \sum_{l=1}^n z_l d\bar{z}_l \right)
\]
\[
\left( 1 + \sum_{k=1}^n z_k \bar{z}_k \right) \]

When we evaluate the above at \((1:0: \cdots :0)\), we get \( \sum_{k=1}^n dz_k \wedge d\bar{z}_k \) and so

\[ \omega_F(1:0: \cdots :0) = \frac{i}{2\pi} \sum_{k=1}^n dz_k \wedge d\bar{z}_k, \]

which is positive. Therefore, we get a Hermitian metric on \( \mathbb{P}^n \), this is the \textit{Fubini-Study metric}. As a consequence, every projective manifold inherits an Hermitian metric from the Fubini-Study metric.

From now on, assume that \( X \) is \textit{compact} manifold (or each object has compact support). Look at the bundles \( \bigwedge^{p,q} \) and choose once and for all an Hermitian metric on \( X \) and let \( \omega \) be the associated positive \((1,1)\)-form. So, locally in a coframe,

\[ \omega = \frac{i}{2} \sum_{k=1}^n \varphi_k \wedge \overline{\varphi}_k. \]

At each \( z \), a basis for \( \bigwedge^z^{p,q} \) is just \( \{ \varphi_I \wedge \overline{\varphi}_J \} \), where \( I = \{i_1 < \cdots < i_p\} \), \( J = \{j_1 < \cdots < j_q\} \) and

\[ \varphi_I \wedge \overline{\varphi}_J = \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p} \wedge \overline{\varphi}_{j_1} \wedge \cdots \wedge \overline{\varphi}_{j_q}. \]

We can define an orthonormal basis of \( \bigwedge^z^{p,q} \) if we decree that the \( \varphi_I \wedge \overline{\varphi}_J \) are pairwise orthogonal, and we set

\[ \|\varphi_I \wedge \overline{\varphi}_J\|^2 = (\varphi_I \wedge \overline{\varphi}_J, \varphi_I \wedge \overline{\varphi}_J) = 2^{p+q}. \]

This gives \( \bigwedge^z^{p,q} \) a \( C^\infty \)-varying Hermitian inner product. To understand where \( 2^{p+q} \) comes from, look at \( \mathbb{C} \). Then, near \( z \), we have \( \varphi = dz, \overline{\varphi} = d\overline{z} \), so

\[ dz \wedge d\overline{z} = (dx + idy) \wedge (dx - i dy) = -i(dx \wedge dy + dx \wedge dy) = -2i dx \wedge dy. \]

Therefore, \( \|dz \wedge d\overline{z}\| = 2 \) and \( \|dz \wedge d\overline{z}\|^2 = 4 = 2^{1+1} \) (here, \( p = 1 \) and \( q = 1 \)).

Let us write \( \bigwedge^z^{p,q}(X) \) for the set of global \( C^\infty \)-sections, \( \Gamma_{C^\infty}(X, \bigwedge^z^{p,q}) \). Locally, on an open, \( U \), we have

\[ \omega = \frac{i}{2} \sum_{k=1}^n \varphi_k \wedge \overline{\varphi}_k \in \bigwedge^1(U) \]
and so, we deduce that
\[ \omega^n = \left( \frac{i}{2} \right)^n n! (-1)^{\frac{n}{2}} \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \bar{\varphi}_1 \wedge \cdots \wedge \bar{\varphi}_n. \]

We call \( \Phi(z) = \omega^n(z)/n! = C_n \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \bar{\varphi}_1 \wedge \cdots \wedge \bar{\varphi}_n \) the volume form and \( C_n = (\frac{i}{2})^n (-1)^{\frac{n}{2}} \) the twisting constant. We can check that \( \Phi \) is a real, positive form, so we can integrate w.r.t. to it. For \( \xi, \eta \in \bigwedge^{p,q}(X) \), set
\[ (\xi, \eta) = \int_X (\xi, \eta) \Phi(z) \in \mathbb{C}. \]

This makes \( \bigwedge^{p,q}(X) \) a complex (infinite-dimensional) inner-product space. We have
\[ \partial : \bigwedge^{p,q-1}(X) \to \bigwedge^{p,q}(X) \]
and say (as in the finite dimensional case) \( \overline{\partial} \) is a closed operator (i.e., \( B^{p,q}_\partial \) is closed in \( \bigwedge^{p,q}(X) \)). Pick some \( \xi \in Z^{p,q}_\partial \), i.e., \( \partial \xi = 0 \). All the cocycles representing the class of \( \xi \) (an element of \( H^{p,q}_\partial \)) form the translates \( \xi + B^{p,q}_\partial \subseteq \bigwedge^{p,q}(X) \). This translate is a closed and convex subset of \( \bigwedge^{p,q}(X) \).

Does there exist a smallest (in the norm we've just defined) cocycle in this cohomology class—if so, how to find it?

Now, we can ask if \( \overline{\partial} \) has an adjoint. If so, call it \( \overline{\partial}^* \) and then, \( \overline{\partial}^* : \bigwedge^{p,q}(X) \to \bigwedge^{p,q-1}(X) \) and
\[ (\overline{\partial}^* (\xi), \eta) = (\xi, \overline{\partial}(\eta)), \text{ for all } \xi, \eta. \]

Then, Hodge observed the

**Proposition 2.16** The cocycle, \( \xi \), is of smallest norm in its cohomology class iff \( \overline{\partial}^* (\xi) = 0 \).

**Proof.**

(\( \Leftarrow \)). Compute
\[ \| \xi + \overline{\partial} \eta \|^2 = (\xi + \overline{\partial} \eta, \xi + \overline{\partial} \eta) = \| \xi \|^2 + \| \eta \|^2 + 2 \Re(\xi, \overline{\partial} \eta). \]
But, \( (\xi, \overline{\partial} \eta) = (\overline{\partial}^* (\xi), \eta) = 0 \), by hypothesis, so
\[ \| \xi + \overline{\partial} \eta \|^2 = \| \xi \|^2 + \| \eta \|^2, \]
which shows the minimality of \( \| \xi \| \) in \( \xi + B^{p,q}_\partial \) and the uniqueness of such a \( \xi \).

(\( \Rightarrow \)). We know that \( \| \xi + \overline{\partial} \eta \|^2 \geq \| \xi \|^2 \), for all our \( \eta \)'s. Make
\[ f(t) = (\xi + t \overline{\partial} \eta, \xi + t \overline{\partial} \eta). \]
The function \( f(t) \) has a global minimum at \( t = 0 \) and by calculus, \( f'(t) \big|_{t=0} = 0 \). We get
\[ \left( (\overline{\partial} \eta, \xi + t \overline{\partial} \eta) + (\xi + t \overline{\partial} \eta, \overline{\partial} \eta) \right)_{t=0} = 0, \]
that is, \( \Re(\xi, \overline{\partial} \eta) = 0 \). But, \( i \eta \) is another element of \( \bigwedge^{p,q-1} X \). So, let
\[ g(t) = (\xi + it \overline{\partial} \eta, \xi + it \overline{\partial} \eta). \]
Repeating the above argument, we get \( \Im(\xi, \overline{\partial} \eta) = 0 \). Consequently, we have \( (\xi, \overline{\partial} \eta) = 0 \), for all \( \eta \). Since \( (\overline{\partial}^* (\xi), \eta) = (\xi, \overline{\partial}(\eta)) \), we conclude that \( (\overline{\partial}^* (\xi), \eta) = 0 \), for all \( \eta \), so \( \overline{\partial}^* (\xi) = 0 \), as required. \( \square \)

If the reasoning can be justified, then
(1) In each cohomology class of $H^{p,q}_{\partial}$, there is a unique (minimal) representative.

(2) $H^{p,q}_{\partial}(X) \cong \{ \xi \in p,q \bigwedge X \mid (a) \partial \xi = 0 \}$. We know from previous work that $H^{p,q}_{\partial}(X) \cong H^{q}(X, \Omega^p_X)$.

Making $\partial^*$. First, we make the Hodge $\ast$ operator:

$$\ast : \bigwedge X \to \bigwedge^{n-p,n-q} X$$

by pure algebra. We want $(\xi(z), \eta(z)) \mapsto \Phi(z) = \xi(z) \wedge \ast \eta(z)$ for all $\xi$.

We need to define $\ast$ on basis elements, $\xi = \phi_I \wedge \phi_J$. We want $(\phi_I \wedge \overline{\phi}_J, \sum_{K,L} \eta_{K,L} \phi_K \wedge \overline{\phi}_L) C_n \phi_1 \wedge \cdots \wedge \phi_n \wedge \overline{\phi}_1 \wedge \cdots \wedge \overline{\phi}_n = \phi_I \wedge \overline{\phi}_J \wedge \sum_{|M|=n-p \atop |N|=n-q} a_{M,N} \phi_M \wedge \overline{\phi}_N$,

where $|I| = |K| = p$ and $|J| = |L| = q$. The left hand side is equal to $2^{p+q} \eta_{I,J} C_n \phi_1 \wedge \cdots \wedge \phi_n \wedge \overline{\phi}_1 \wedge \cdots \wedge \overline{\phi}_n$ and the right hand side is equal to

$$\sum_{|M|=n-p \atop |N|=n-q} a_{M,N} \phi_I \wedge \overline{\phi}_J \wedge \phi_M \wedge \overline{\phi}_N = a_{I^0,J^0} \phi_I \wedge \overline{\phi}_J \wedge \phi_{I^0} \wedge \overline{\phi}_{J^0},$$

where $I^0 = \{1, \ldots, n\} - I$ and $J^0 = \{1, \ldots, n\} - J$. The right hand side has $\phi_1 \wedge \cdots \wedge \phi_n \wedge \overline{\phi}_1 \wedge \cdots \wedge \overline{\phi}_n$ in scrambled order. Consider the permutation

$$(1, 2, \ldots, n; \tilde{1}, \tilde{2}, \ldots, \tilde{n}) \mapsto (i_1, \ldots, i_p, \tilde{j}_1, \ldots, \tilde{j}_q, i_{p+1}, \ldots, i_{n-p}, j_1, \ldots, j_{n-q}).$$

If we write $\text{sgn}_{I,J}$ for the sign of this permutation, we get

$$a_{I^0,J^0} = 2^{p+q-n} i^n(-1)^{n/2} \eta_{I,J} \text{sgn}_{I,J}.$$

Therefore,$$\ast \eta = \ast \sum_{K,L} \eta_{K,L} \phi_K \wedge \overline{\phi}_L = 2^{p+q-n} i^n(-1)^{n/2} \sum_{|K^0|=n-p \atop |L^0|=n-q} \text{sgn}_{K,L} \phi_{K^0} \wedge \overline{\phi}_{L^0}.$$

Now, set

$$\overline{\partial}^* = -\ast \circ \overline{\partial} \circ \ast,$$

where $\overline{\partial}^* : \bigwedge^{p,q} X \to \bigwedge^{n-p,n-q} X \to \bigwedge^{n-p,n-q+1} X \to \bigwedge^{p,q-1} X$.

I claim that $-\ast \circ \overline{\partial} \circ \ast$ is the formal adjoint, $\overline{\partial}^*$, we seek. Consider

$$\langle \overline{\partial} \xi, \eta \rangle = \int_X \langle \overline{\partial} \xi, \eta \rangle \Phi(z) = \int_X \overline{\partial} \xi \wedge \ast \eta,$$
where \( \xi \in \bigwedge^{p,q-1}(X) \) and \( \eta \in \bigwedge^{p,q}(X) \). Now, \( \partial(\xi \wedge \ast \eta) = \partial \xi \wedge \ast \eta + (\ast)(\partial \xi \wedge \ast \eta) \), so we get
\[
\int_X (\partial(\xi \wedge \ast \eta) = (\partial \xi, \ast \eta) + (\ast)(\partial \xi \wedge \ast \eta).
\]

Also, \( \xi \wedge \ast \eta \in \bigwedge^{p,q-1}(X) \wedge \bigwedge^{n-p,n-q}(X) \), i.e., \( \xi \wedge \ast \eta \in \bigwedge^{n,n-1}(X) \). But, \( d = \partial + \bar{\partial} \), so
\[
d(\xi \wedge \ast \eta) = \partial(\xi \wedge \ast \eta) + \bar{\partial}(\xi \wedge \ast \eta) = \bar{\partial}(\xi \wedge \ast \eta),
\]
and we deduce that
\[
\int_X \bar{\partial}(\xi \wedge \ast \eta) = \int_X d(\xi \wedge \ast \eta) = \int_{\partial X} \xi \wedge \ast \eta = 0,
\]
if either \( X \) is compact (in which case \( \partial X = \emptyset \)), or the forms have compact support (and hence, vanish on \( \partial X \)). So, we have
\[
(\bar{\partial} \xi, \eta) = (\ast)(\partial \xi \wedge \ast \eta).
\]

Check (DX): For \( \eta \in \bigwedge^{p,q}(X) \), we have
\[
\ast \ast \eta = (\ast)(\partial)(\ast \eta).
\]
As \( \ast \eta \in \bigwedge^{n-p,n-q}(X) \), we have \( \bar{\partial}(\ast \eta) \in \bigwedge^{n-p,n-q+1}(X) \), and so,
\[
\ast \ast \bar{\partial}(\ast \eta) = (\ast)(\partial)(\ast)(\ast \eta) = (\ast)(\partial) = \ast \bar{\partial} \ast \eta.
\]
We conclude that
\[
(\bar{\partial} \xi, \eta) = -\int_X \xi \wedge \ast \partial(\ast \eta)
\]
\[
= \int_X \xi \wedge \ast(\ast \partial \ast \eta)
\]
\[
= (\xi, \ast \partial \ast \eta).
\]

Therefore, \( \bar{\partial} = -\ast \partial \ast \), as contended.

Now, we define the Hodge Laplacian, or Laplace-Beltrami operator, \( \Box \), by:
\[
\Box = \bar{\partial} \partial + \partial \bar{\partial} : \bigwedge^{p,q}(X) \longrightarrow \bigwedge^{p,q}(X).
\]

You check (DX) that \( \Box \) is formally self-adjoint.

Claim: \( \Box(\varphi) = 0 \) iff both \( \bar{\partial} \varphi = 0 \) and \( \partial^\ast \varphi = 0 \).

First, assume \( \Box(\varphi) = 0 \) and compute \( (\varphi, \Box(\varphi)) \). We get
\[
(\varphi, \Box(\varphi)) = (\varphi, \partial \bar{\partial} \varphi) + (\varphi, \bar{\partial} \partial \varphi)
\]
\[
= (\partial \varphi, \bar{\partial} \varphi) + (\bar{\partial} \varphi, \partial \varphi)
\]
\[
= (\bar{\partial} \varphi, \bar{\partial} \varphi) + \| \partial \varphi \|^2
\]
\[
= \| \varphi \|^2 + \| \partial \varphi \|^2.
\]

Therefore, if \( \Box(\varphi) = 0 \), then \( \bar{\partial} \varphi = 0 \) and \( \partial^\ast \varphi = 0 \). The converse is obvious by definition of \( \Box(\varphi) \).

Consequently, our minimality is equivalent to \( \Box(\varphi) = 0 \), where \( \Box \) is a second-order differential operator.

To understand better what the operator \( \Box \) does, consider the special case where \( X = \mathbb{C}^n \) (use compactly supported “gadgets”), with the standard inner product, and \( \bigwedge^{0,0}(X) = C_0^\infty \). Pick \( f \in C_0^\infty \), then again, \( \Box(f) \in C_0^\infty \) and on those \( f \), we have \( \bar{\partial}^\ast \partial^\ast f = 0 \). Consequently,
\[
\Box(f) = \partial \bar{\partial} f = \partial \bar{\partial} \left( \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \right).
\]
We also have\[ * \left( \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} \, dz_j \right) = 2^{1-n} i^n (-1)^{\frac{n}{2}} \sum_{j=1}^{n} \left( \frac{\partial f}{\partial z_j} \right) dz_1 \wedge \cdots \wedge dz_n \wedge d\xi_j, \]
and it is uniquely determined by\[ G = 2^{1-n} i^n (-1)^{\frac{n}{2}} \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} \, sgn_{\emptyset, \{j\}} \, dz_1 \wedge \cdots \wedge dz_n \wedge d\xi_j. \]
Taking $\overline{\partial}$ of the above expression, we get\[ 2^{1-n} i^n (-1)^{\frac{n}{2}} \sum_{k,j=1}^{n} \frac{\partial^2 f}{\partial z_k \partial z_j} \, sgn_{\emptyset, \{j\}} \, d\xi_k \wedge d\xi_j \wedge dz_1 \wedge \cdots \wedge dz_n \wedge d\xi_j = 2^{1-n} i^n (-1)^{\frac{n}{2}} (-1)^n \sum_{j=1}^{n} \frac{\partial^2 f}{\partial z_j \partial z_j} \, sgn_{\emptyset, \{j\}} \, dz_1 \wedge \cdots \wedge dz_n \wedge d\xi_j \wedge d\xi_j. \]
Taking $-\ast$ of the above, we get\[ -2i^{2n} (-1)^{\frac{n}{2}} (-1)^n \sum_{j=1}^{n} \frac{\partial^2 f}{\partial z_j \partial z_j} = -2 \sum_{j=1}^{n} \frac{\partial^2 f}{\partial z_j \partial z_j}. \]
But,\[ \frac{4\partial^2 f}{\partial z_j \partial z_j} = \frac{\partial^2 f}{\partial x_j^2} + \frac{\partial^2 f}{\partial y_j^2}, \]
and this implies that on $\bigwedge^{0,0}(X)$, $\Box(f)$ up to a constant $(-1/2)$ is just the usual Laplacian.

Write $\mathcal{H}^{p,q}(X)$ for the kernel of $\Box$ on $\bigwedge^{p,q}(X)$, the space of harmonic forms. Here is Hodge’s theorem.

**Theorem 2.17** (*Hodge, (1941)*) Let $X$ be a complex manifold and assume that $X$ is compact. Then,

1. The space $\mathcal{H}^{p,q}(X)$ is finite-dimensional.
2. There exist a projection, $\mathcal{H}^{p,q}(X) \to \mathcal{H}^{p,q}(X)$, so that we have the orthogonal decomposition (Hodge decomposition)
   \[ \bigwedge^p(X) = \mathcal{H}^{p,q}(X) \bigoplus \mathcal{H}^{p,q-1}(X) \bigoplus \mathcal{H}^{p,q+1}(X). \]
3. There exists a parametrix (= pseudo-inverse), $G$, (Green’s operator) for $\Box$, and it is uniquely determined by
   \[ \text{(a) id} = \mathcal{H}^{p,q}(X) \bigoplus \mathcal{H}^{p,q-1}(X) \bigoplus \mathcal{H}^{p,q+1}(X). \]
   \[ \text{(b)} G\overline{\partial} = \overline{\partial} G, G\overline{\partial}^* = \overline{\partial}^* G \text{ and } G \mid \mathcal{H}^{p,q}(X) = 0. \]

**Remarks:** (1) If a decomposition “à la Hodge” exists, it must be an orthogonal decomposition. Say $\xi \in \overline{\partial} \bigwedge^{p,q-1}(X)$ and $\eta \in \overline{\partial}^* \bigwedge^{p,q+1}(X)$, then
   \[ (\xi, \eta) = (\overline{\partial} \xi_0, \overline{\partial}^* \eta_0) = (\overline{\partial} \overline{\partial} \xi_0, \eta_0) = 0, \]
and so, $\overline{\partial} \bigwedge^{p,q-1}(X) \perp \overline{\partial}^* \bigwedge^{p,q+1}(X)$. Observe that we can write the Hodge decomposition as
   \[ \bigwedge^p(X) = \mathcal{H}^{p,q}(X) \bigoplus \Box \bigwedge^{p,q}(X). \]
For, if \( \xi \in \Box \wedge^{p,q}(X) \), then \( \xi = \overline{\partial}(\overline{\partial}^* \xi_0) + \overline{\partial}^* (\overline{\partial} \xi_0) \), and this implies
\[
\Box \wedge^{p,q}(X) \subseteq \overline{\partial} \wedge^{p,q-1}(X) + \overline{\partial}^* \wedge^{p,q+1}(X).
\]
However, the right hand side is an orthogonal decomposition and it follows that
\[
\mathcal{H}^{p,q}(X) + \Box \wedge^{p,q}(X) = \mathcal{H}^{p,q}(X) + \overline{\partial} \wedge^{p,q-1}(X) \perp \overline{\partial}^* \wedge^{p,q+1}(X) = \wedge^{p,q}(X).
\]
For perpendicularity, as \( \Box \) is self-adjoint, for \( \xi \in \mathcal{H}^{p,q}(X) \), we have
\[
(\xi, \Box(\eta)) = (\Box(\xi), \eta) = 0,
\]
since \( \Box(\xi) = 0 \).

(2) We can give a n.a.s.c. that \( \Box(\xi) = \eta \) has a solution, given \( \eta \). Namely, by (3a),
\[
\eta = \kappa(\eta) + \Box(G(\eta)).
\]
If \( \kappa(\eta) = 0 \), then \( \eta = \Box(G(\eta)) \) and we can take \( \xi = G(\eta) \). Conversely, orthogonality implies that if \( \eta = \Box(\xi) \), then \( \kappa(\eta) = 0 \). Therefore, \( \kappa(\eta) \) is the obstruction to solving \( \Box(\xi) = \eta \).

How many solutions does \( \Box(\xi) = \eta \) have?

The solutions of \( \Box(\xi) = \eta \) are in one-to-one correspondence with \( \xi_0 + \mathcal{H}^{p,q}(X) \), where \( \xi_0 \) is a solution and if we take \( \xi_0 \in \text{Ker} \kappa \), then \( \xi_0 \) is unique, given by \( G(\eta) \).

(3) Previous arguments, once made correct, give us the isomorphisms
\[
\mathcal{H}^{p,q}(X) \cong H^{p,q}_\partial \cong H^q(X, \Omega^p_X).
\]
Therefore, \( H^q(X, \Omega^p_X) \) is a finite-dimensional vector space, for \( X \) a compact, complex manifold.

For the proof of Hodge’s theorem, we need some of the theory of distributions. At first, restrict to \( C^\infty_0(U) \) (smooth functions of compact support) on some open, \( U \subseteq \mathbb{C}^n \). One wants to understand the dual space, \( (C^\infty_0(U))^D \). Consider \( g \in L^2(U) \), then for any \( \varphi \in C^\infty_0(U) \), we set
\[
\lambda_g(\varphi) = \int_U \varphi \overline{g} d\mu.
\]
(Here, \( \mu \) is the Lebesgue measure on \( \mathbb{C}^n \).) So, we have \( \lambda_g \in C^\infty_0(U)^D \). Say \( \lambda_g(\varphi) = 0 \), for all \( \varphi \). Take \( E \), a measurable subset of \( U \) of finite measure with \( \overline{E} \) compact. Then, as \( \chi_E \) is \( L^2 \), the function \( \chi_E \) is \( L^2 \)-approximable by \( C^\infty_0(U) \)-functions. So, there is some \( \varphi \in C^\infty_0(U) \) so that
\[
\| \varphi - \chi_E \|_2 < \epsilon.
\]
As \( \chi_E = \chi_E - \varphi + \varphi \), we get
\[
\int_E \overline{g} d\mu = \int_U \chi_E \overline{g} d\mu = \int_U (\chi_E - \varphi) \overline{g} d\mu + \int_U \varphi \overline{g} d\mu = \int_U (\chi_E - \varphi) \overline{g} d\mu
\]
(by hypothesis, \( \lambda_g(\varphi) = 0 \)). Therefore,
\[
\left| \int_E \overline{g} d\mu \right| \leq \| \chi_E - \varphi \|_2 \| \overline{g} \|_2 < \| g \|_2 \epsilon,
\]
which implies that \( g \equiv 0 \) almost everywhere. It follows that \( L^2(U) \to (C_0^\infty(U))^D \). The same argument applies for \( g \in C(U) \) and uniform approximations by \( C_0^\infty \)-functions, showing that \( C(U) \to (C_0^\infty(U))^D \).

**Notation.** Set
\[
D_j = \frac{1}{i} \frac{\partial}{\partial X_j} = -i \frac{\partial}{\partial X_j},
\]
where \( X_1, \ldots, X_n \) are real coordinates in \( \mathbb{C}^n \), and if \( \alpha = (\alpha_1, \ldots, \alpha_n) \), with \( \alpha_j \in \mathbb{Z} \) and \( \alpha_j \geq 0 \), set
\[
D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}
\]
and \( |\alpha| = \sum_{j=1}^n \alpha_j \). Also, for any \( n \)-tuple \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n \), we let \( \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \) and \( |\xi|^\alpha = |\xi_1|^{\alpha_1} \cdots |\xi_n|^{\alpha_n} \). The reason for the factor \( 1/i \) is this: Say \( v \) is a function and look at
\[
D_j(v) = -i \frac{\partial v}{\partial X_j}.
\]
Therefore,
\[
D_j(uv) = (D_ju)v + uD_jv = (D_ju)v - uD_jv.
\]
Consider \( u, v \in C_0^\infty(U) \); then,
\[
(D_ju, v) = \int_U (D_ju)v = \int_U D_j(uv) + \int_U u D_j(v).
\]
The first term on the right hand side is zero as \( u \) and \( v \) have compact support, so we get
\[
(D_ju, v) = \int_U u \overline{D_j(v)} = (u, D_jv),
\]
which says that the \( D_j \)'s are formally self-adjoint. Repeated application of the above gives
\[
(D^\alpha u, v) = (u, D^\alpha v)
\]
and also
\[
\int_U (D^\alpha u)v = \int_U u(D^\alpha v).
\]

**Definition 2.4** Let \( \tilde{D}(U) = C_0^\infty(U)^{alg, D} \) be the set of (complex-valued) linear functionals on \( C_0^\infty(U) \). Now define, \( D(U) \), the *space of distributions* on \( U \), so that \( \lambda \in D(U) \) iff \( \lambda \in C_0^\infty(U)^{alg, D} \) and \( \lambda \) is “continuous”, i.e., there is some \( k \geq 0 \) and some \( C_\lambda \), so that for all \( \varphi \in C_0^\infty(U) \),
\[
|\lambda(\varphi)| \leq C_\lambda \max_{|\alpha| \leq k} ||D^\alpha \varphi||_\infty.
\]

As an example of a distribution, if \( g \in C_0(U) \), so \( g \) is bounded (all we need is boundedness and integrability), then
\[
\lambda_g(\varphi) = \int_U \varphi g d\mu.
\]
Then, we have
\[
|\lambda_g(\varphi)| \leq ||\varphi||_\infty ||g||_1,
\]
so we can take \( C_{\lambda_g} = ||g||_1 \) and we get a distribution. The intuition in (⋆) is that the bigger \( k \) is, the “worse” \( \lambda \) is as a distribution (\( k \) indicates how many derivatives we need to control).

We can differentiate distributions: Take \( g \in C^1 \), we have
\[
\lambda_{D_j g}(\varphi) = \int_U \varphi \overline{D_j g} d\mu = \int_U (D_j \varphi) \overline{g} d\mu = \lambda_g(D_j \varphi).
\]
This gives the reason behind the
Definition 2.5 If $\lambda \in \mathcal{D}(U)$, let $D^\alpha \lambda \in \tilde{\mathcal{D}}(U)$, defined by

$$(D^\alpha \lambda)(\varphi) = \lambda(D^\alpha \varphi).$$

Claim: If $\lambda \in \mathcal{D}(U)$, then $D^\alpha \lambda \in \mathcal{D}(U)$.

Indeed, we have

$$|((D^\alpha \lambda)(\varphi))| = |\lambda(D^\alpha \varphi)| \leq C_\lambda \max_{|\beta| \leq k} \|D^{\alpha+\beta}(\varphi)\|_\infty \leq C_\lambda \max_{|\gamma| \leq k+|\alpha|} \|D^{\gamma}(\varphi)\|_\infty.$$

Therefore, $D^\alpha \lambda$ is again a distribution. Given a multi-index, $\alpha$, write

$$\sigma(\alpha) = |\alpha| + \left\lceil \frac{n}{2} \right\rceil + 1.$$

This is the Sobolev number of $\alpha$ ($n =$ dimension of the underlying space). Now, we can define the Sobolev norm and the Sobolev spaces, $H_s$ ($s \in \mathbb{Z}, s \geq 0$). If $\varphi \in C_0^\infty(U)$, set

$$\|\varphi\|^2_s = \sum_{|\alpha| \leq s} \|D^\alpha \varphi\|^2_{L^2}.$$

This is the Sobolev $s$-norm. It comes from an inner product

$$(\varphi, \psi)_s = \sum_{|\alpha| \leq s} (D^\alpha \varphi, D^\alpha \psi).$$

If we complete $C_0^\infty(U)$ in this norm, we get a Hilbert space, the Sobolev space, $H_s$.

Say $s > r$, then for all $\varphi \in C_0^\infty(U)$, we have

$$\|\varphi\|^2_s \leq \|\varphi\|^2_r.$$

Hence, if $\{\varphi_i\}$ is a Cauchy sequence in the $s$-norm, it is also a Cauchy sequence in the $r$-norm and we get a continuous embedding

$$H_s \subseteq H_r \quad \text{if} \quad s > r.$$

Let $H_\infty = \bigcap_{s \geq 0} H_s$.

**Theorem 2.18 (Sobolev Inequality and Embedding Theorem)** For all $\varphi \in C_0^\infty(U)$, for all $\alpha$, we have

$$\|D^\alpha \varphi\|_{\infty} \leq K_\alpha \|\varphi\|_{\sigma(\alpha)} \quad \text{and} \quad H_s(U) \subseteq C^m(\overline{U}),$$

provided $U$ has finite measure, $m \geq 0$ and $\sigma(m) \leq s$. Furthermore, $H_s(U) \subseteq L^{\frac{2n}{n-2s}}(U)$ if $n > 2s$.

(We have $\sigma(m) \leq s$ iff $m < s - \left\lceil \frac{n}{2} \right\rceil$.)

**Theorem 2.19 (Rellich Lemma)** The continuous embedding, $\rho_s^*: H_s \hookrightarrow H_r$, (for $s > r$) is a compact operator. That is, for any bounded set, $B$, the image $\rho_s^*(B)$ has a compact closure. Alternatively, if $\{\varphi_j\}$ is a bounded sequence in $H_s$, then $\{\rho_s^*(\varphi_j)\}$ possesses a converging subsequence in $H_r$.

To connect with distributions, we use the Fourier Transform. If $\varphi \in C_0(U)$, we set

$$\hat{\varphi}(\theta) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{C}^n} \varphi(x)e^{-i(x,\theta)} \, dx,$$
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where \((x, \theta) = \sum_{j=1}^{n} x_j \bar{\theta}_j\). (Recall that over \(\mathbb{R}\), we are in \(\mathbb{R}^{2n}\).) The purpose of the fudge factor in front of the integral is to insure that Fourier transform of the Gaussian

\[
\varphi(x) = e^{-\frac{\|x\|^2}{2}}
\]

is itself. As

\[
\int_{\mathbb{R}^n} e^{-\frac{\|x\|^2}{2}} \, dx = \left(\sqrt{2\pi}\right)^n,
\]
we determine that the “fudge factor” is \((2\pi)^{-n}\). It is also interesting to see what \(\hat{D}_j \varphi(\theta)\) is. We have

\[
\begin{align*}
\hat{D}_j \varphi(\theta) &= \int_{\mathbb{C}^n} (D_j \varphi)(x) e^{-i(x, \theta)} \, dx = \int_{\mathbb{C}^n} \varphi(x) D_j e^{i(x, \theta)} \, dx.
\end{align*}
\]

Now,

\[
\frac{\partial}{\partial x_j} e^{i \sum x_k \bar{\theta}_k} = i \bar{\theta}_k e^{i \sum x_k \bar{\theta}_k}
\]

and

\[
D_j e^{i \sum x_k \bar{\theta}_k} = -i \frac{\partial}{\partial x_j} e^{i \sum x_k \bar{\theta}_k} = \bar{\theta}_k e^{i \sum x_k \bar{\theta}_k}.
\]

It follows that

\[
\hat{D}_j \varphi(\theta) = \theta_j \hat{\varphi}(\theta),
\]

that is, \(D_j\) turns into multiplication by \(\theta_j\) by the Fourier transform. We also get

**Theorem 2.20 (Plancherel)** If \(\varphi \in C_0^\infty\), then

\[
\|\varphi\|_{L^2} = \|\hat{\varphi}\|_{L^2}.
\]

As a consequence, we can compute the Sobolev norm using the Fourier transform:

\[
\|\varphi\|_s^2 = \sum_{|\alpha| \leq s} \|D^\alpha \varphi\|_{L^2}^2
\]

and

\[
\sum_{|\alpha| \leq s} \|D^\alpha \varphi\|_{L^2}^2 = \sum_{|\alpha| \leq s} \int_{\mathbb{C}^n} \theta^\alpha \hat{\varphi}(\theta) \bar{\theta}^\alpha \hat{\varphi}(\theta) \, d\theta
\]

\[
= \int_{\mathbb{C}^n} \sum_{|\alpha| \leq s} |\theta|^{2\alpha} |\hat{\varphi}(\theta)|^2 \, d\theta
\]

\[
\leq \int_{\mathbb{C}^n} (1 + |\theta|^2)^s |\hat{\varphi}(\theta)|^2 \, d\theta
\]

\[
\leq \text{Const} \int_{\mathbb{C}^n} |\theta|^{2\alpha} |\hat{\varphi}(\theta)|^2 \, d\theta = \text{Const} \|\varphi\|_s^2.
\]

(Using Plancherel in the last step.) Therefore, the norm

\[
\|\varphi\|_s^2 = \int_{\mathbb{C}^n} (1 + |\theta|^2)^s |\hat{\varphi}(\theta)|^2 \, d\theta
\]

satisfies

\[
\|\varphi\|_s^2 \leq \|\hat{\varphi}\|_s^2 \leq \text{Const} \|\varphi\|_s^2,
\]
that is, these norms are equivalent and we can measure $\varphi$ by the Sobolev norm on the Fourier transform.

Observe that we can define $H_{-s}$ ($s > 0$) via the completion of $C_0^\infty$ in the norm $\int_{C^n} (1 + |\theta|^2)^{-s} |\hat{\varphi}(\theta)|^2 d\theta$. Clearly, we get the chain of inclusions

$$\cdots \supseteq H_{-n} \supseteq H_{-n+1} \supseteq \cdots \supseteq H_{1} \supseteq L^2 \supseteq H_{0} \cdots \supseteq H_{n} \supseteq \cdots \supseteq H_{\infty}.$$ 

This suggests defining $H_{-\infty}$ by

$$H_{-\infty} = \bigcup_{n \in \mathbb{Z}} H_n.$$ 

The Sobolev embedding lemma implies $H_{\infty} \subseteq C^\infty(U)$ and $C_0^\infty(U) \subseteq H_{\infty}$. Now, $H_{-s}$ defines linear functionals on $H_s$; say $\psi \in H_{-s}$ and $\varphi \in H_s$. Consider

$$\psi(\varphi) := \int (\varphi \overline{\psi})(\theta) d\theta = \int \sqrt{1 + |\theta|^2}^s \varphi \frac{1}{\sqrt{1 + |\theta|^2}^s} \overline{\psi} d\theta.$$ 

By Cauchy-Schwarz,

$$|\psi(\varphi)| = |\langle \varphi, \psi \rangle| = \int (\varphi \overline{\psi})(\theta) d\theta \leq \|\varphi\| \|\psi\|_{-s}.$$ 

Therefore, we have a map $H_{-s} \mapsto H_s^D$ and it follows that $H_{-s} \cong H_s^D$, up to conjugation.

**Remark:** If $\varphi \in C_0^\infty(U)$ and $\lambda \in D(U)$, then

$$|\lambda(\varphi)| \leq C_\lambda \max_{|\alpha| \leq k} \|D^\alpha \varphi\|_{\infty}, \quad \text{for some } k.$$ 

By Sobolev’s inequality,

$$|\lambda(\varphi)| \leq C_\lambda K_n \|\varphi\|_{\sigma(\alpha)},$$

for some suitable $\alpha$ so that $|\alpha| \leq k$. Thus, if $\lambda \in D(U)$, then there exist some $\alpha$ such that $\lambda$ is a continuous functional on $C_0^\infty(U)$ in the $\sigma(\alpha)$-norm. But then, $\lambda$ extends to an element of $H_s^D_{\sigma(\alpha)}$ (by completion) and we conclude that $D(U) = H_{-\infty}$.

**Proof of Theorem 2.19 (Rellich Lemma).** Given a bounded sequence, $\{\varphi_k\}_{k=1}^\infty$, there is some $C > 0$ so that, for every $k$,

$$\int_{\mathbb{R}^n} (1 + |\theta|^2)^s |\hat{\varphi}_k(\theta)|^2 d\theta \leq C.$$ 

Thus, for every $\theta$, the sequence of $(1 + |\theta|^2)^s |\hat{\varphi}_k(\theta)|^2$ is a bounded sequence of complex numbers. Therefore, for every $\theta$, we have a Cauchy subsequence in $\mathbb{C}$. As there exists a countable dense subset of $\theta$’s in $\mathbb{R}^n$, the $K_0$-diagonalization procedure yields a subsequence of the $\varphi_k$’s so that this subsequence is Cauchy at every $\theta$ (i.e., $(1 + |\theta|^2)^s |\hat{\varphi}_k(\theta)|^2$ is Cauchy at every $\theta$) and, of course, we replace the $\varphi_k$’s by this subsequence. Now, pick $\epsilon > 0$, and write $U_0$ for the set of all $\theta$’s such that

$$\frac{1}{(1 + |\theta|^2)^{s-r}} \geq \epsilon.$$ 

Look at

$$\|\varphi_k - \varphi_l\|_r^2 = \int_{\mathbb{R}^n} (1 + |\theta|^2)^r |(\hat{\varphi}_k - \hat{\varphi}_l)(\theta)|^2 d\theta$$

$$= \int_{U_0} (1 + |\theta|^2)^r |(\hat{\varphi}_k - \hat{\varphi}_l)(\theta)|^2 d\theta + \int_{\mathbb{R}^n - U_0} (1 + |\theta|^2)^r |(\hat{\varphi}_k - \hat{\varphi}_l)(\theta)|^2 d\theta.$$
But, as \( \{(1 + |\theta|^2)^r |\hat{\varphi}_k(\theta)|^2\} \) is Cauchy, there is some large \( N \) so that for all \( k, l \geq N \),

\[
(1 + |\theta|^2)^r (|\hat{\varphi}_k - \hat{\varphi}_l|) = (1 + |\theta|^2)^r (|\hat{\varphi}_k - \hat{\varphi}_l|) < \epsilon/\mu(U_0)
\]

for all \( \theta \). Then, the first integral is at most \( \epsilon \). In the second integral,

\[
(1 + |\theta|^2)^r (|\hat{\varphi}_k - \hat{\varphi}_l|) = \frac{(1 + |\theta|^2)^s}{(1 + |\theta|^2)^{s-r}} (|\hat{\varphi}_k - \hat{\varphi}_l|) < \epsilon \text{ numerator.}
\]

But then,

\[
\int_{\mathbb{R}^n - U_0} (1 + |\theta|^2)^r (|\hat{\varphi}_k - \hat{\varphi}_l|) d\theta < \epsilon \int_{\mathbb{R}^n} \text{ numerator < } C\epsilon.
\]

Therefore, \( \{\varphi_k\} \) is Cauchy in \( H_r \), and since \( H_r \) is complete, the sequence \( \{\varphi_k\} \) converges in \( H_r \). \( \square \)

**Proof of Theorem 2.18 (Sobolev’s Theorem).** Pick \( \varphi \in C_0^\infty(U) \) and take \( s = 1 \). Then, for every \( j \), as

\[
|\varphi(x)| \leq \int_{-\infty}^{\infty} |D_j \varphi(x)| dx_j,
\]

we get

\[
|\varphi(x)|^n \leq \prod_{j=1}^{n} \left( \int_{-\infty}^{\infty} |D_j \varphi(x)| dx_j \right).
\]

Thus, we have

\[
|\varphi(x)|^{n/(n-1)} \leq \prod_{j=1}^{n} \left( \int_{-\infty}^{\infty} |D_j \varphi(x)| dx_j \right)^{1/(n-1)}. \tag{*}
\]

We will use the generalized Hölder inequality: If

\[
\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1,
\]

and if \( \varphi_j \in L^{p_j} \), for \( j = 1, \ldots, m \), then \( \varphi_1 \cdots \varphi_m \in L^1 \) and

\[
\|\varphi_1 \cdots \varphi_m\|_{L^1} \leq \|\varphi_1\|_{L^{p_1}} \cdots \|\varphi_m\|_{L^{p_m}}.
\]

Assume that \( n \geq 2 \) and set \( p_j = n - 1 \), for \( 1 \leq j \leq n - 1 \). Integrate \((*)\) w.r.t. \( x_1, x_2, \ldots, x_n \), but in between integration, use the Hölder inequality:

\[
\int_{-\infty}^{\infty} |\varphi(x)|^{n/(n-1)} dx_1 \leq \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |D_1 \varphi(x)| dx_1 \right)^{1/(n-1)} \prod_{j=2}^{n} \left( \int_{-\infty}^{\infty} |D_j \varphi(x)| dx_j \right)^{1/(n-1)} dx_1
\]

\[
\leq \left[ \int_{-\infty}^{\infty} |D_1 \varphi(x)| dx_1 \right]^{1/(n-1)} \prod_{j=2}^{n} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D_j \varphi(x)| dx_j dx_1 \right]^{1/(n-1)}.
\]

If we repeat this procedure, we get

\[
\int_{U} |\varphi(x)|^{n/(n-1)} dx \leq \left[ \prod_{j=1}^{n} \int_{U} |D_j \varphi(x)| dx \right]^{1/(n-1)}.
\]
Raising the above to the power \((n - 1)/n\), we get
\[
\|\varphi\|_{L^{n/(n-1)}} \leq \left( \prod_{j=1}^{n} \int_{U} |D_j(\varphi)| \, dx \right)^{1/n} \leq \frac{1}{n} \left( \sum_{j=1}^{n} \int_{U} |D_j(\varphi)| \, dx \right),
\]
by the arithmetic-geometric mean inequality. Apply this to \(\varphi^\gamma\), for some appropriate choice of \(\gamma\). For the rest of this argument, we need \(n > 2\) and we choose \(\gamma\) to satisfy
\[
\gamma \left( \frac{n}{n-1} \right) = 2(\gamma - 1).
\]
We deduce that
\[
\gamma = \frac{2(n-1)}{n-2} > 0,
\]
as \(n > 2\). We plug \(\varphi^\gamma\) in the above and we get
\[
\|\varphi^\gamma\|_{L^{n/(n-1)}} \leq \frac{1}{n} \left( \sum_{j=1}^{n} \int_{U} |\varphi^\gamma| \, dx \right)^{1/n} \leq \frac{2}{n} \left( \sum_{j=1}^{n} \int_{U} |\varphi^{\gamma-1}| \, dx \right)^{1/n} \leq \frac{2}{n} \left( \sum_{j=1}^{n} \|\varphi^{\gamma-1}\|_{L^2} \|D_j(\varphi)\|_{L^2} \right),
\]
by Cauchy-Schwarz. The left hand side is equal to
\[
\left( \int_{U} |\varphi^{\frac{n\gamma}{n-2}}| \, dx \right)^{\frac{n-1}{n}} = \left( \int_{U} |\varphi^{2(\gamma-1)}| \, dx \right)^{\frac{n-1}{n}}.
\]
On the right hand side, the term \(\|\varphi^{\gamma-1}\|_{L^2}\) is common to the summands, so pull it out. This factor is
\[
\left( \int_{U} |\varphi^{2(\gamma-1)}| \, dx \right)^{\frac{1}{2}}.
\]
When we divide both sides by this factor, we get
\[
\left( \int_{U} |\varphi^{2(\gamma-1)}| \, dx \right)^{\frac{n-1}{n}-\frac{1}{2}} \leq \frac{\gamma}{n} \sum_{j=1}^{n} \|D_j(\varphi)\|_{L^2}.
\]
But,
\[
2(\gamma - 1) = \frac{\gamma n}{n-1} = \frac{2n}{n-2}
\]
and
\[
\frac{n-1}{n} - \frac{1}{2} = \frac{n-2}{2n}.
\]
We obtain
\[
\left( \int_{U} |\varphi^{\frac{2n}{n-2}}| \, dx \right)^{\frac{n-2}{2n}} \leq \frac{2(n-1)}{n(n-2)} \sum_{j=1}^{n} \|D_j(\varphi)\|_{L^2}.
\]
Therefore, we get the Sobolev inequality for the case \(s = 1\) and \(n > 2\): For every \(\varphi \in C_0^\infty(U)\), we have
\[
\|\varphi\|_{L^{\frac{2n}{n-2}}} \leq K(n) \|\varphi\|_{1},
\]
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Now, say \( \psi \in H_1 \), then there is a sequence, \( \{ \varphi_q \} \), converging to \( \psi \) in the \( \| \|_1 \)-norm, with \( \varphi_q \in C^\infty_0(U) \). Consequently, this is a Cauchy sequence in the \( \| \|_1 \)-norm and so,

\[ \| \varphi_q - \varphi_r \|_1 < \epsilon \quad \text{for all } q, r \text{ sufficiently large} \]

which implies that

\[ \| \varphi_q - \varphi_r \|_{L^{2n/2}} < \epsilon \quad \text{for all } q, r \text{ sufficiently large}. \]

Therefore, the \( \varphi_q \) converge to a limit, \( \psi_0 \in L^{2n/2} \).

(a) The map \( \psi \mapsto \psi_0 \) does not depend on the choice of the Cauchy sequence.

(b) This map is an injection.

As a consequence, we get the Sobolev embedding when \( s = 1 \):

\[ H_1 \hookrightarrow L^{2n/2}, \quad \text{if } n > 2. \]

If we pass to the limit in \((*)\), we get: For every \( \psi \in H_1 \),

\[ \| \psi \|_{L^{2n/2}} \leq K(n) \| \psi \|_1. \]

Now, we want the Sobolev inequality on \( \| D^\alpha \varphi \|_\infty \) when \( s = 1 \). In this case, \( \sigma(\alpha) \leq s \) implies \( |\alpha| + 1 + \left\lceil \frac{n}{2} \right\rceil \leq 1 \). Thus, \( n = 1 \) and \( \alpha = 0 \). Therefore, we have to prove

\[ \| \varphi \|_\infty \leq K \| \varphi \|_1. \]

In the present case, \( U \subseteq \mathbb{R} \) and \( \varphi \in C^\infty_0(U) \). Then, we have

\[ \varphi(x) = \int_{-\infty}^x \varphi'(t)dt, \]

so

\[ |\varphi(x)| \leq \int_{-\infty}^x |\varphi'(t)| dt \leq \|1\|_{L^2} \|D\varphi\|_{L^2} \leq \sqrt{\mu(U)} \| \varphi \|_1, \]

where we used Cauchy Schwarz in the first inequality. If we take sup’s, we get the following Sobolev inequality for the case \( s = n = 1 \):

\[ \| \varphi \|_\infty \leq K \| \varphi \|_1. \]

(\( ** \))

Next, consider the embedding property. Here, we have \( 0 \leq m \leq s - \left\lceil \frac{n}{2} \right\rceil \), so \( m = 0 \). Take \( \psi \in H_1 \) and, as before, approximate \( \psi \) by some sequence, \( \{ \varphi_q \} \), where \( \varphi_q \in C^\infty_0(U) \). Then, \((**\)) implies that

\[ \| \varphi_q - \varphi_r \|_\infty \leq K \| \varphi_q - \varphi_r \|_1. \]

As the right hand side is smaller than \( \epsilon \) for all \( q, r \geq N \) (for some large \( N \)), we deduce that the \( \varphi_q \) converge uniformly to some \( \psi_0 \in C^0(U) \). Then, again, the map \( \psi \mapsto \psi_0 \) is well-defined and an embedding. Therefore,

\[ H_1(U) \subseteq C^0(U), \]

which is the Sobolev embedding in the case \( s = n = 1 \).

To prove the general case, we use induction on \( s \) and iterate the argument. The induction hypothesis is
(a) If \( n > 2s \), then for all \( \varphi \in \mathcal{C}^\infty(U) \),

\[
\|\varphi\|_{L^{\frac{2n}{n-2}}} \leq K(n) \|\varphi\|_s.
\]

\((*)\)

(a') There is an embedding, \( H_s(U) \hookrightarrow L^{\frac{2n}{n-2}} \), so \((*)\) holds for all \( \psi \in H_2 \).

(b) If \( 0 \leq m \leq s - \left\lceil \frac{n}{2} \right\rceil \) \( (\sigma(m) \leq s) \), then

\[
\| D^\alpha \varphi \|_\infty \leq K \| \varphi \|_{\sigma(\alpha)} \leq K \| \varphi \|_s.
\]

\(((**)\)

(Here, \( \sigma(\alpha) \leq s \).)

(b') There is an embedding, \( H_s(U) \hookrightarrow C^m(U) \), i.e., \((**\)) holds for all \( \psi \in H_s \).

(a) Actually, this part does not require induction. As the case \( s = 1 \) has been settled, we may assume \( s > 1 \) (and \( n > 2s \)). We need to show that for any \( \varphi \in \mathcal{C}^\infty_0(U) \),

\[
\|\varphi\|_{L^{\frac{2n}{n-2}}} \leq \|\varphi\|_s.
\]

We have

\[
\|\varphi\|_1 \leq \|\varphi\|_s
\]

and as \( n > 2s > 2 \), by the \( s = 1 \) case,

\[
\|\varphi\|_{L^{\frac{2n}{n-2}}} \leq \|\varphi\|_1.
\]

We conclude immediately that

\[
\|\varphi\|_{L^{\frac{2n}{n-2}}} \leq \|\varphi\|_s.
\]

Note that \((a')\) is a consequence of \((a)\) in the same way as before.

(b) Assume \( 0 \leq m \leq s + 1 - \left\lceil \frac{n}{2} \right\rceil \), i.e., \( m - 1 < s - \left\lceil \frac{n}{2} \right\rceil \). Pick \( \varphi \in \mathcal{C}^\infty_0(U) \) and look at \( D_j \varphi \) and \( \sigma(\beta) \leq s \). Observe that \( m - 1 \) is such a \( |\beta| \). By \((**)\),

\[
\| D^\beta D_j \varphi \|_\infty \leq K \| D_j \varphi \|_{\sigma(\beta)}, \text{ for all } j.
\]

But all \( D^\alpha \varphi \) are of this form, for some \( \beta \) with \( \sigma(\beta) \leq s \). Therefore,

\[
\| D^\alpha \varphi \|_\infty \leq K \| D_j \varphi \|_{\sigma(\beta)} \leq K \| \varphi \|_{s+1}, \text{ by } (\dagger)
\]

which is exactly \((**)\). By the induction hypothesis, each \( D_j \varphi \in C^{m-1}(\overline{U}) \) and we conclude that \( \varphi \in C^m(\overline{U}) \).

\(\square\)

**Notion of a Weak Solution to**, say \( \square \varphi = \psi \).

**Definition 2.6** Given \( \psi \in D(U) \) (but, usually, \( \psi \in C^\infty(U) \)), we call \( \varphi \in D(U) \) a **weak solution** of \( \square \varphi = \psi \) iff for every \( \eta \in \mathcal{C}^\infty_0(U) \), we have

\[
\varphi(\square \eta) = \psi(\eta).
\]

Motivation: We know that \( \square \varphi \) is defined by

\[
(\square \varphi)(\eta) = \varphi(\square \eta).
\]

Therefore, \( \square \varphi = \psi \) in \( D(U) \) when and only when \( \varphi \) is a weak solution.