2.5 Fiber Bundles and Vector Bundles

We begin by reviewing the definition of a fiber bundle. The following definition is not the most general definition of a fiber bundle, but it is sufficient for our needs:

Definition 2.5.1 A *fiber bundle with (typical) fiber F and structure group G* is a tuple $\xi = (E, B, \pi, F, G)$, where E, B, F are smooth manifolds, $\pi\colon E \to B$ is a smooth surjective map between E and B, G is a Lie group of diffeomorphisms of F, and there is some open cover $\mathcal{U} = (U_i)_{i\in I}$ of B and a family $\varphi = (\varphi_i)_{i\in I}$ of diffeomorphisms

$$\varphi_i\colon \pi^{-1}(U_i) \to U_i \times F$$

such that the following properties hold:

(a) The diagram

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\varphi_i"] \arrow[dr, "\pi"'] & & U_i \times F \arrow[dl, "p_1"] \\ & U_i & \end{tikzcd}$$

commutes for all $i \in I$, where $p_1\colon U_i \times F \to U_i$ is the first projection. Equivalently, for all $(b, y) \in U_i \times F$,

$$\pi \circ \varphi_i^{-1}(b, y) = b.$$

The space B is called the *base space*, E is called the *total space*, F is called the *(typical) fiber*, and each φ_i is called a *(local) trivialization*. For each $b \in B$, the space $\pi^{-1}(b)$ is called the *fiber above* b. It is also denoted by E_b, and $\pi^{-1}(U_i)$ is also denoted by $E|_{U_i}$.

For every (U_i, φ_i) and every $b \in U_i$, we have the diffeomorphism

$$(p_2 \circ \varphi_i)|_{E_b}\colon E_b \to F,$$

where $p_2\colon U_i \times F \to F$ is the second projection, which we denote by $\varphi_{i,b}$. Furthermore, for all U_i, U_j in $\mathcal{U}$ such that $U_i \cap U_j \neq \emptyset$, for every $b \in U_i \cap U_j$, there is a relationship between $\varphi_{i,b}$ and $\varphi_{j,b}$ which gives the twisting of the bundle:

(b) The diffeomorphism

$$\varphi_{i,b} \circ \varphi_{j,b}^{-1}\colon F \to F$$

is an element of the group G.

(c) The map $g_{ij}\colon U_i \cap U_j \to G$ defined by

$$g_{ij}(b) = \varphi_{i,b} \circ \varphi_{j,b}^{-1}$$

is smooth. The maps g_{ij} are called the *transition maps* of the fiber bundle.

A fiber bundle $\xi = (E, B, \pi, F, G)$ is also called a *fiber bundle over* B. Intuitively, a fiber bundle over B is a family $E = (E_b)_{b\in B}$ of spaces E_b (fibers) indexed by B and varying smoothly as b moves in B, such that every E_b is diffeomorphic to F. The bundle in which $E = B \times F$ and π is the first projection is called the *trivial bundle*. Note that if G is the trivial one-element group, then a fiber bundle is trivial.

A Möbius strip is an example of a nontrivial fiber bundle where the base space B is the circle S^1 and the fiber space F is the closed interval $[-1, 1]$. The total space E is the strip obtained by rotating the line segment $[-1, 1]$ around the circle, keeping its midpoint in contact with the circle, and gradually twisting the line segment so that after a full revolution, the segment has been tilted by π. A Klein bottle is also a fiber bundle for which both the base space and the fiber are the circle S^1.

Remark:

(1) The above definition is slightly different (but equivalent) to the definition given in Bott and Tu [2], page 47-48. Definition 2.5.1 is closer to the one given in Hirzebruch [17]. Bott and Tu and Hirzebruch assume that G acts effectively on the left on the fiber F. This means that there is a smooth map $\cdot\colon G \times F \to F$ such that

$$(gh) \cdot x = g \cdot (h \cdot x), \quad \text{and} \quad e \cdot x = x,$$

for all $g, h \in G$ and all $x \in F$, where $e \in G$ is the identity element of G. The group G acts effectively on F if for every $g \in G$,

$$g \cdot x = x$$

for all $x \in F$ implies that $g = e$. Every $g \in G$ defines a diffeomorphism $\varphi_g\colon F \to F$ defined by

$$\varphi_g(x) = g \cdot x$$

for all $x \in F$. Indeed, the inverse of φ_g is $\varphi_{g^{-1}}$, since

$$\varphi_{g^{-1}} \circ \varphi_g(x) = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = e \cdot x = x,$$

so that $\varphi_{g^{-1}} \circ \varphi_g = \mathrm{id}$, and similarly, $\varphi_g \circ \varphi_{g^{-1}} = \mathrm{id}$. The fact that G acts effectively on F means that the map $g \mapsto \varphi_g$ is injective. This justifies viewing G as a group of diffeomorphims of F, and from now on, we will denote $\varphi_g(x)$ by $g(x)$. Then, Hirzebruch defines the sheaf G_∞ such that $\Gamma(U, G_\infty)$ is the group of smooth functions $g\colon U \to G$, where U is some open subset of B and G is a Lie group acting effectively (on the left) on the fiber F. The group operation on $\Gamma(U, G_\infty)$ is induced by multiplication in G, that is, given two (smooth) functions $g\colon U \to G$ and $h\colon U \to G$,

$$gh(b) = g(b)h(b)$$

for all $b \in U$.

Beware that gh is **not** function composition, unless G itself is a group of functions, which is the case for vector bundles.

Our conditions (b) and (c) are then replaced by the following equivalent condition: For all U_i, U_j in $\mathcal{U}$ such that $U_i \cap U_j \neq \emptyset$, there is some $g_{ij} \in \Gamma(U_i \cap U_j, G_\infty)$ such that

$$\varphi_i \circ \varphi_j^{-1}(b, x) = (b, g_{ij}(b)(x))$$

for all $b \in U_i \cap U_j$ and all $x \in F$.

The most comprehensive treatment of fiber bundles and vector bundles is probably given in Husemoller [18]. However, we can hardly recommend this book. We find the presentation overly formal, and intuitions are absent.

(2) The definition of a fiber bundle ξ involves some open cover $\mathcal{U} = (U_i)_{i \in I}$ of B and a family $\varphi = (\varphi_i)_{i \in I}$ of trivializations. If we want to indicate this fact explicitly, we denote the fiber bundle by $(E, B, \pi, F, G, \mathcal{U}, \varphi)$. This dependency on a specific open cover is somehwat of a nuisance, but the following remark shows that the situation is not as bad as it looks: If $\mathcal{V} = (V_j)_{j \in J}$ is another open cover of B and $\mathcal{V}$ is a refinement of $\mathcal{U}$, which means that there is a map $\tau \colon J \to I$ such that $V_j \subseteq U_{\tau(j)}$ for all $j \in J$, any family $\varphi = (\varphi_i)_{i \in I}$ of trivializations of ξ relative to $\mathcal{U}$ yields a family $\varphi' = (\varphi'_j)_{j \in J}$ of trivializations of ξ relative to $\mathcal{V}$. Indeed, for every $V_j \in \mathcal{V}$, since $V_j \subseteq U_{\tau(j)}$, observe that the restriction of $\varphi_{\tau(j)}$ to V_j is a trivialization

$$\varphi'_j \colon \pi^{-1}(V_j) \to V_j \times F,$$

and that conditions (b) and (c) are still satisfied. This fact allows us to assume that if $\xi_1 = (E_1, B, \pi_1, F, G)$ and $\xi_2 = (E_2, B, \pi_2, F, G)$ are two fiber bundles over the same base space B, then we can assume that we can pick trivializing families φ for ξ_1 and φ' for ξ_2 relative to the same open cover $\mathcal{U} = (U_i)_{i \in I}$. Indeed, if ξ_1 and ξ_2 are defined using a trivializing family φ relative to some open cover $\mathcal{U}$ for ξ_1 and a trivializing family φ' relative to some open cover $\mathcal{V}$ for ξ_2, we can use the common refinement $\mathcal{W} = \{U_i \cap V_j \mid U_i \in \mathcal{U},\, V_j \in \mathcal{V}\}$, of $\mathcal{U}$ and $\mathcal{V}$ and the corresponding trivializations. From now on, we make this assumption when dealing with fiber bundles over the same base.

(3) The family of transition functions (g_{ij}) satisfies the *cocycle condition*

$$g_{ij}(b) g_{jk}(b) = g_{ik}(b)$$

for all i, j, k such that $U_i \cap U_j \cap U_k \neq \emptyset$ and all $b \in U_i \cap U_j \cap U_k$. Setting $i = j = k$, we get

$$g_{ii} = \mathrm{id},$$

and setting $k = i$, we get

$$g_{ji} = g_{ij}^{-1}.$$

Again, beware that this means that $g_{ji}(b) = g_{ij}^{-1}(b)$, where $g_{ij}^{-1}(b)$ is the inverse of $g_{ji}(b)$ in G. In general, g_{ij}^{-1} is **not** the functional inverse of g_{ji}.

The cocycle condition makes it possible to view a fiber bundle over B as a member of a certain (Čech) cohomology set $\check{H}^1(B, \mathcal{G})$, where $\mathcal{G}$ denotes a certain sheaf of functions from the manifold B into the Lie group G, as explained in Hirzebruch [17], Section 3.2. However, this requires defining a noncommutative version of Čech cohomology (at least, for $\check{H}^1$), and clarifying when two open covers and two trivializations define the same fiber bundle over B, or equivalently, defining when two fiber bundles over B are equivalent. If the bundles under considerations are line bundles (see Definition 2.5.8), then $\check{H}^1(B, \mathcal{G})$ is actually a group. In this case, $G = \mathrm{GL}(1, \mathbb{R}) \approx \mathbb{R}^*$ in the real case and $G = \mathrm{GL}(1, \mathbb{C}) \approx \mathbb{C}^*$ in the complex case (where $\mathbb{R}^* = \mathbb{R} - \{0\}$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$), and the sheaf $\mathcal{G}$ is the sheaf of smooth (real-valued or complex-valued) functions vanishing nowhere. The group $\check{H}^1(B, \mathcal{G})$ plays an important role, especially when the bundle is a holomorphic line bundle over a complex manifold. In the latter case, it is called the *Picard group* of B.

A map of fiber bundles is defined as follows:

Definition 2.5.2 Given two fiber bundles $\xi_1 = (E_1, B_1, \pi_1, F, G)$ and $\xi_2 = (E_2, B_2, \pi_2, F, G)$ with the same typical fiber F and the same structure group G, a *bundle map (or bundle morphism)* $f \colon \xi_1 \to \xi_2$ is a pair $f = (f_E, f_B)$ of smooth maps $f_E \colon E_1 \to E_2$ and $f_B \colon B_1 \to B_2$ such that the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "f_E"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "f_B"'] & B_2 \end{tikzcd}$$

A bundle map $f \colon \xi_1 \to \xi_2$ is an *isomorphism* if there is some bundle map $g \colon \xi_2 \to \xi_1$ called the *inverse of* f such that

$$g_E \circ f_E = \mathrm{id}, \qquad f_E \circ g_E = \mathrm{id}.$$

The bundles ξ_1 and ξ_2 are called *isomorphic*. Given two fiber bundles $\xi_1 = (E_1, B, \pi_1, F, G)$ and $\xi_2 = (E_2, B, \pi_2, F, G)$ over the same base space B, a *bundle map (or bundle morphism)* $f \colon \xi_1 \to \xi_2$ is a pair $f = (f_E, f_B)$ where $f_B = \mathrm{id}$ (the identity map). Such a bundle map is an *isomorphism* if it has an inverse as defined above. In this case, we say that the bundles ξ_1 and ξ_2 over B are *isomorphic*.

When f is an isomorphism, the surjectivity of π_1 and π_2 implies that

$$g_B \circ f_B = \mathrm{id}, \qquad f_B \circ g_B = \mathrm{id}.$$

Thus, when $f = (f_E, f_B)$ is an isomorphism, both f_E and f_B are diffeomorphisms. Some authors require the "preservation" of fibers, that is, for every $b \in B_1$, the map of fibers

$$f_E|_{\pi_1^{-1}(b)} \colon \pi_1^{-1}(b) \to \pi_2^{-1}(f_B(b))$$

must be a diffeomorphism. This is automatic for isomorphisms.

We can also define the notion of equivalence for fiber bundles over the same base space B (see Hirzebruch [17], Section 3.2, Chern [3], Section 5, and Husemoller [18], Chapter 5). Equivalence of bundles implies that they are isomorphic.

Definition 2.5.3 Given two fiber bundles $\xi_1 = (E_1, B, \pi_1, F, G)$ and $\xi_2 = (E_2, B, \pi_2, F, G)$ over the same base space B, we say that ξ_1 and ξ_2 are *equivalent* if there is an open cover $\mathcal{U} = (U_i)_{i\in I}$ for B, a family $\varphi = (\varphi_i)_{i\in I}$ of trivializations

$$\varphi_i \colon \pi_1^{-1}(U_i) \to U_i \times F$$

for ξ_1, a family $\varphi' = (\varphi'_i)_{i\in I}$ of trivializations

$$\varphi'_i \colon \pi_2^{-1}(U_i) \to U_i \times F$$

for ξ_2, and a family $(\rho_i)_{i\in I}$ of smooth maps $\rho_i \colon U_i \to G$ such that

$$\varphi_i \circ {\varphi'_i}^{-1}(b, x) = (b, \rho_i(b)(x))$$

for all $b \in U_i$ and all $x \in F$.

Since the trivializations are bijections, the family $(\rho_i)_{i\in I}$ is unique. The following lemma shows that equivalent fiber bundles are isomorphic:

Lemma 2.5.4 *If two fiber bundles $\xi_1 = (E_1, B, \pi_1, F, G)$ and $\xi_2 = (E_2, B, \pi_2, F, G)$ over the same base space B are equivalent, then*

$$g'_{ij}(b) = \rho_i(b)^{-1} g_{ij}(b) \rho_j(b)$$

for all $b \in U_i \cap U_j$, where the g_{ij}s are the transition functions associated with the φ_is and the g'_{ij}s are the transition functions associated with the φ'_is. Furthermore, ξ_1 and ξ_2 are equivalent.

Proof. We only check the first part, leaving the second as an exercise (or consult Husemoller [18], Chapter 5). Recall that

$$\varphi_i \circ \varphi_j^{-1}(b, x) = (b, g_{ij}(b)(x))$$

for all $b \in U_i \cap U_j$ and all $x \in F$. This is equivalent to

$$\varphi_j^{-1}(b, x) = \varphi_i^{-1}(b, g_{ij}(b)(x)),$$

and it is desirable to introduce ψ_i such that $\psi_i = \varphi_i^{-1}$. Then, we have

$$\psi_j(b, x) = \psi_i(b, g_{ij}(b)(x)),$$

and

$$\varphi_i \circ {\varphi'_i}^{-1}(b, x) = (b, \rho_i(b)(x))$$

becomes

$$\psi'_i(b, x) = \psi_i(b, \rho_i(b)(x)).$$

We have

$$\psi'_j(b, x) = \psi_j(b, \rho_j(b)(x)) = \psi_i(b, g_{ij}(b)(\rho_j(b)(x)))$$

and

$$\psi'_i(b, g'_{ij}(b)(x)) = \psi_i(b, \rho_i(b)(g'_{ij}(b)(x))).$$

Since we also have

$$\psi'_j(b, x) = \psi'_i(b, g'_{ij}(b)(x)),$$

we get

$$\psi_i(b, g_{ij}(b)(\rho_j(b)(x))) = \psi_i(b, \rho_i(b)(g'_{ij}(b)(x))),$$

which implies that

$$g_{ij}(b)(\rho_j(b)(x)) = \rho_i(b)(g'_{ij}(b)(x))),$$

i.e.

$$g'_{ij}(b) = \rho_i(b)^{-1} g_{ij}(b) \rho_j(b).$$

□

In general, isomorphic fiber bundles over the same base B may not be equivalent, because a smooth map $h: U \times F \to F$ may not arise from a smooth map $\rho: U \to G$, in the sense that

$$h(b, x) = \rho(b)(x)$$

for all $b \in U$ and all $x \in F$. However, this will be the case when $G = \mathrm{GL}(n, \mathbb{R}))$ (or $G = \mathrm{GL}(n, \mathbb{C})))$ and when h is a smooth map linear in x for every fixed b. This is the case for vector bundles, to be considered shortly. Following Hirzebruch [17], it is possible to modify the notion of a map of fiber bundles over B so that isomorphism of bundles implies bundle equivalence.

Definition 2.5.5 Given two fiber bundles $\xi_1 = (E_1, B, \pi_1, F, G)$ and $\xi_2 = (E_2, B, \pi_2, F, G)$ over the same base space B, a *strong bundle map* $f: \xi_1 \to \xi_2$ is a bundle map as in Definition 2.5.2, and the following additional conditions hold:

(a) There is an open cover $\mathcal{U} = (U_i)_{i \in I}$ for B, a family $\varphi = (\varphi_i)_{i \in I}$ of trivializations

$$\varphi_i: \pi_1^{-1}(U_i) \to U_i \times F$$

for ξ_1, a family $\varphi' = (\varphi'_i)_{i \in I}$ of trivializations

$$\varphi'_i: \pi_2^{-1}(U_i) \to U_i \times F$$

for ξ_2.

(b) For every $b \in B$, there are some trivializations $\varphi_i\colon \pi_1^{-1}(U_i) \to U_i \times F$ and $\varphi_i'\colon \pi_2^{-1}(U_i) \to U_i \times F$, with $b \in U_i$, and some smooth map $\rho_i\colon U_i \to G$ such that

$$\varphi_i' \circ f \circ \varphi_i^{-1}(u, x) = (u, \rho_i(b)(x))$$

for all $u \in U_i$ and all $x \in F$.

A strong bundle map is an isomorphism if it has an inverse as in Definition 2.5.2. In this case, we say that the bundles ξ_1 and ξ_2 over B are *strongly isomorphic*.

The following lemma is not hard to prove (see Husemoller [18], Chapter 5):

Lemma 2.5.6 *Two fiber bundles $\xi_1 = (E_1, B, \pi_1, F, G)$ and $\xi_2 = (E_2, B, \pi_2, F, G)$ over the same base space B are equivalent iff they are strongly isomorphic.*

Given a fiber bundle $\xi = (E, B, \pi, F, G)$, we observed that the family $g = (g_{ij})$ of transition maps $g_{ij}\colon U_i \cap U_j \to G$ induced by a trivializing family $\varphi = (\varphi_i)_{i\in I}$ relative to the open cover $\mathcal{U} = (U_i)_{i\in I}$ for B satisfies the *cocycle condition*

$$g_{ij}(b)g_{jk}(b) = g_{ik}(b)$$

for all i, j, k such that $U_i \cap U_j \cap U_k \neq \emptyset$ and all $b \in U_i \cap U_j \cap U_k$. Without altering anything, we may assume that g_{ij} is the (unique) function from $\emptyset$ to G when $U_i \cap U_j = \emptyset$. Then, we call a family $g = (g_{ij})_{(i,j)\in I\times I}$ as above a *$\mathcal{U}$-cocycle*, or simply, a *cocycle*. Remarkably, given such a cocycle g relative to $\mathcal{U}$, a fiber bundle ξ_g over B with fiber F and structure group G having g as family of transition functions can be constructed. In view of Lemma 2.5.4, we say that two cocycles $g = (g_{ij})_{(i,j)\in I\times I}$ and $g' = (g_{ij})_{(i,j)\in I\times I}$ are *equivalent* if there is a family $(\rho_i)_{i\in I}$ of smooth maps $\rho_i\colon U_i \to G$ such that

$$g'_{ij}(b) = \rho_i(b)^{-1} g_{ij}(b) \rho_j(b)$$

for all $b \in U_i \cap U_j$.

Lemma 2.5.7 *Given two smooth manifolds B and F, a Lie group G acting effectively on F, an open cover $\mathcal{U} = (U_i)_{i\in I}$ of B, and a cocycle $g = (g_{ij})_{(i,j)\in I\times I}$, there is a fiber bundle $\xi_g = (E, B, \pi, F, G)$ whose transition maps are the maps in the cocycle g. Furthermore, if g and g' are equivalent cocycles, then ξ_g and $\xi_{g'}$ are equivalent.*

Proof sketch. First, we define the space Z as the disjoint sum

$$Z = \coprod_{i\in I} U_i \times F.$$

We define the relation $\simeq$ on $Z \times Z$ as follows: For all $(b, x) \in U_j \times F$ and $(b, y) \in U_i \times F$, if $U_i \cap U_j \neq \emptyset$,

$$(b, x) \simeq (b, y) \quad \text{iff} \quad y = g_{ij}(b)(x).$$

We let $E = Z/\simeq$, and we give E the largest topology such that the the injections $\eta_i\colon U_i \times F \to Z$ are smooth. The cocycle condition insures that $\simeq$ is indeed an equivalence relation. We define $\pi\colon E \to B$ by $\pi([b, x]) = b$. If $p\colon Z \to E$ is the the quotient map, observe that the maps $p \circ \eta_i\colon U_i \times F \to E$ are injective, and that

$$\pi \circ p \circ \eta_i(b, x) = b.$$

Thus,

$$p \circ \eta_i\colon U_i \times F \to \pi^{-1}(U_i)$$

is a bijection, and we define the trivializating maps by setting

$$\varphi_i = (p \circ \eta_i)^{-1}.$$

It is easily verified that the corresponding transition functions are the original g_{ij}. There are some details to check, for instance, see Husemoller [18], Chapter 5, or Wells [38]. The fact that ξ_g and $\xi_{g'}$ are equivalent when g and g' are equivalent follows from Lemma 2.5.4 (see Husemoller [18], Chapter 5). $\square$

Obviously, if we start with a fiber bundle $\xi = (E, B, \pi, F, G)$ whose transition maps are the cocycle $g = (g_{ij})$ and form the fiber bundle ξ_g, the bundles ξ and ξ_g are equivalent. This leads to a characterization of the set of equivalence classes of fiber bundles over a base space B as the cohomology set $\check{H}^1(B, \mathcal{G})$. In the present case, the sheaf $\mathcal{G}$ is defined such that $\Gamma(U, \mathcal{G})$ is the group of smooth maps from the open subset U of B to the Lie group G. Since G is not abelian, the coboundary maps have to be interpreted multiplicatively. It is natural to define

$$\delta_0\colon C^0(\mathcal{U}, \mathcal{G}) \to C^1(\mathcal{U}, \mathcal{G})$$

by

$$(\delta_0 g)_{ij} = g_i^{-1} g_j,$$

for any $g = (g_i)$, with $g_i \in \Gamma(U_i, \mathcal{G})$. As to

$$\delta_1\colon C^1(\mathcal{U}, \mathcal{G}) \to C^2(\mathcal{U}, \mathcal{G}),$$

since the cocycle condition in the usual case is

$$g_{ij} + g_{jk} = g_{ik},$$

we set

$$(\delta_1 g)_{ijk} = g_{ij} g_{jk} g_{ik}^{-1},$$

for any $g = (g_{ij})$, with $g_{ij} \in \Gamma(U_i \cap U_j, \mathcal{G})$. Note that a cocycle $g = (g_{ij})$ is indeed an element of $Z^1(\mathcal{U}, \mathcal{G})$, and the condition for being in the kernel of

$$\delta_1\colon C^1(\mathcal{U}, \mathcal{G}) \to C^2(\mathcal{U}, \mathcal{G})$$

is the cocycle condition

$$g_{ij}(b)g_{jk}(b) = g_{ik}(b)$$

for all $b \in U_i \cap U_j \cap U_k$. In the commutative case, two cocycles g and g' are equivalent if their difference is a boundary, which can be stated as

$$g_{ij} + \rho_j = g'_{ij} + \rho_i = \rho_i + g'_{ij},$$

where $\rho_i \in \Gamma(U_i, \mathcal{G})$ for all $i \in I$. In the present case, two cocycles g and g' are equivalent iff there is a family $(\rho_i)_{i\in I}$, with $\rho_i \in \Gamma(U_i, \mathcal{G})$, such that

$$g'_{ij}(b) = \rho_i(b)^{-1} g_{ij}(b) \rho_j(b)$$

for all $b \in U_i \cap U_j$. This is the same condition of equivalence defined earlier. Thus, it is easily seen that if $g, h \in Z^1(\mathcal{U}, \mathcal{G})$, then ξ_g and ξ_h are equivalent iff g and h correspond to the same element of the cohomology set $\check{H}^1(B, \mathcal{G})$. As usual, $\check{H}^1(B, \mathcal{G})$ is defined as the direct limit of the directed system of sets $\check{H}^1(\mathcal{U}, \mathcal{G})$, over the preordered directed family of open covers. For details, see Hirzebruch [17], Section 3.1. In summary, there is a bijection between the equivalence classes of fiber bundles over B (with fiber F and structure group G) and the cohomology set $\check{H}^1(B, \mathcal{G})$. In the case of line bundles, it turns out that $\check{H}^1(B, \mathcal{G})$ is in fact a group.

Vector bundles are fibre bundles for which the typical fiber is a finite-dimensional vector space V, and the structure group is a subgroup of the group of linear isomorphisms ($\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$).

Definition 2.5.8 A *rank n real smooth vector bundle with fiber V* is a tuple $\xi = (E, B, \pi, V)$ such that $(E, B, \pi, V, \mathrm{GL}(n, \mathbb{R}))$ is a smooth fiber bundle and the fiber V is a real vector space of dimension n.

A *rank n complex smooth vector bundle with fiber V* is a tuple $\xi = (E, B, \pi, V)$ such that $(E, B, \pi, V, \mathrm{GL}(n, \mathbb{C}))$ is a smooth fiber bundle and the fiber V is an n-dimensional complex vector space (viewed as a real smooth manifold). When $n = 1$, a vector bundle is called a *line bundle*.

Maps of vector bundles are maps of fiber bundles such that the isomorphisms between fibers are linear.

Definition 2.5.9 Given two vector bundles $\xi_1 = (E_1, B_1, \pi_1, V)$ and $\xi_2 = (E_2, B_2, \pi_2, V)$ with the same typical fiber V a *bundle map (or bundle morphism)* $f\colon \xi_1 \to \xi_2$ is a pair $f = (f_E, f_B)$ of smooth maps $f_E\colon E_1 \to E_2$ and $f_B\colon B_1 \to B_2$ such that the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "f_E"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "f_B"'] & B_2 \end{tikzcd}$$

and such that for every $b \in B_1$, the map of fibers

$$f_E|_{\pi_1^{-1}(b)}\colon \pi_1^{-1}(b) \to \pi_2^{-1}(f_B(b))$$

is a linear map. A bundle map *isomorphism* $f\colon \xi_1 \to \xi_2$ is defined as in Definition 2.5.2. Given two vector bundles $\xi_1 = (E_1, B, \pi_1, V)$ and $\xi_2 = (E_2, B, \pi_2, V)$ over the same base space B, a *bundle map (or bundle morphism)* $f\colon \xi_1 \to \xi_2$ and *isomorphism of vector bundles over* B are also defined as in Definition 2.5.2.

Some authors require the preservation of fibers, that is, the map

$$f_E|_{\pi_1^{-1}(b)}\colon \pi_1^{-1}(b) \to \pi_2^{-1}(f_B(b))$$

is a bijective linear map.

A *holomorphic vector bundle* is a fiber bundle where E, B are complex manifolds, V is a complex vector space of dimension n, π is holomorphic, the φ_i are biholomorphic, and the transistion functions g_{ij} are holomorphic. When $n = 1$, a holomorphic vector bundle is called a *holomorphic line bundle*.

Definition 2.5.3 also applies to vector bundles and defines the notion of equivalence of vector bundles over B. Lemma 2.5.4 also holds for equivalent vector bundles. This time, because the fiber is a finite-dimensional vector space, two vector bundles over the same base space B are isomorphic iff they are equivalent.

Lemma 2.5.10 *Two vector bundles $\xi_1 = (E_1, B, \pi_1, F, G)$ and $\xi_2 = (E_2, B, \pi_2, F, G)$ over the same base space B are equivalent iff they are isomorphic.*

For more on fiber bundles and vector bundles, see Bott and Tu [2], Madsen and Tornehave [22], Sato [32], Griffith and Harris [13], Wells [38], Hirzebruch [17], Chern [3], Milnor and Stasheff [25], Narasimham [29], and also Husemoller [18], which covers much more, including characteristic classes.

We now explain how the Leray spectral sequence can be used to compute the cohomology of a fiber bundle. Let $(E, M, \pi, F, \mathcal{U}, \varphi)$ be a fiber bundle with fiber F, and assume that $\mathcal{U}$ is a good cover of M. Then, $\pi^{-1}\mathcal{U}$ is also a cover on E, and we can form the double complex

$$K^{p,q} = C^p(\pi^{-1}\mathcal{U}, \mathcal{A}^q).$$

Then, if we let $\mathcal{H}^q$ be the sheaf of fiber cohomologies defined such that for every $U \in \mathcal{U}$,

$$\mathcal{H}^q(U) = H^q(\pi^{-1}(U)),$$

the sheaf $\mathcal{H}^q$ is a locally constant sheaf with group $H^q(F)$ (because $\mathcal{U}$ is a good cover). Since $d_1 = \delta$ on E_1, we get

$$E_2^{p,q} = H_\delta^p(\mathcal{U}, \mathcal{H}^q),$$

and the spectral sequence associated with $K = (K^{p,q})$ converges to $H^*_D(K)$, which, by the generalized Mayer-Vietoris principle, is equal to $H^*(E)$, since $\pi^{-1}\mathcal{U}$ is a good cover of E.

Another application of spectral sequences is the *Künneth formula*, which says that if M and F are manifolds and the cohomology of F is finite-dimensional, then

$$H^*(M \times F) = H^*(M) \otimes H^*(F),$$

which means that

$$H^n(M \times F) = \bigoplus_{p+q=n} H^p(M) \otimes H^q(F).$$

We consider the trivial bundle $\pi\colon M \times F \to M$. In this case, we can check that $\mathcal{H}^q$ is the constant sheaf $H^q(F)$, and

$$E_2^{p,q} = H^p(\mathcal{U}, \mathbb{R}) \otimes H^q(F) = H^p(M) \otimes H^q(F).$$

We also have

$$d_2 = d_3 = \cdots = d_\infty,$$

and thus $E_2 = E_\infty = H^*(M \times F)$. From this, we get

$$H^n(M \times F) = \bigoplus_{p+q=n} H^p(M) \otimes H^q(F).$$

2.6 Product Structures and the Gysin Sequence

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