Chapter 2

Cohomology

2.1 Review of De Rham Cohomology

Let M is a smooth manifold (C^∞-manifold). We assume that the reader is familiar with differential forms. For a comprehensive treatment of differential forms and cohomology, see Warner [37] (notably Chapter 5) or Bott and Tu [2]. A more elementary but very detailed treatment of differential forms can be found in Guillemin and Pollack [14], and a comprehensive and clear treatment of differential forms and cohomology in Madsen and Tornehave [22]. The little book by Sato [32] conveys informally but very nicely some of the main concepts of algebraic topology, such as homology, cohomology, vector bundles. and spectral sequences. For those who read French, Serre's famous paper [33] known to the cognescenti as "FAC" is definitely worth reading, especially Chapter I, §1-29, on Čech cohomology.

The *de Rham complex* $\mathcal{A}^*(M)$ is the direct sum

$$\mathcal{A}^*(M) = \bigoplus_{p\geq 0} \mathcal{A}^p(M),$$

where $\mathcal{A}^p(M)$ is the the real vector space of C^∞ p-forms on M. The *differential*

$$d\colon \mathcal{A}(M) \to \mathcal{A}(M)$$

of the de Rham complex is the linear map defined by the family $d = (d_p)$ of exterior derivatives

$$d_p\colon \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M).$$

The differential d satisfies the properties

$$d^2 = 0$$

and

$$d(\tau\omega) = \omega d\tau + (-1)^p \tau d\omega,$$

where τ is a p-form and ω is a q-form. We say that d is a *skew-derivation*. For example, if $M = \mathbb{R}^3$,

$$d_0 \colon \mathcal{A}^0 \to \mathcal{A}^1$$

corresponds to *grad*,

$$d_1 \colon \mathcal{A}^1 \to \mathcal{A}^2$$

corresponds to *curl*, and

$$d_2 \colon \mathcal{A}^2 \to \mathcal{A}^3$$

corresponds to *div* (note that *grad*, *curl*, *div* are normally defined on vector fields, but there is a simple bijection between vector fields and differential forms on $\mathbb{R}^3$).

In fact, $\mathcal{A}^*(U)$ can be defined on any open subset U of the manifold M, and $\mathcal{A}^*$ is a sheaf of differential complexes. We let

$$Z^* = \bigoplus_{p \geq 0} Z^p$$

be the vector space of *closed forms*, where Z^p is the real vector space consisting of those p-forms ω such that

$$d\omega = 0,$$

and we let

$$B^* = \bigoplus_{p \geq 0} B^p$$

be the vector space of *exact forms*, where B^p is the real vector space consisting of those p-forms ω such that

$$\omega = d\tau$$

for some $(p-1)$-form τ, with $B^0 = 0$, the trivial vector space.

Since $d^2 = 0$, every exact form is closed, but the converse is generally false. The *de Rham cohomology* $H^*_{dR}(M)$ is the real vector space

$$H^*_{dR}(M) = \bigoplus_{p \geq 0} H^p_{dR}(M)$$

where the cohomology group (vector space) $H^p_{dR}(M)$ is the quotient vector space

$$H^p_{dR}(M) = Z^p(M)/B^p(M).$$

Thus, the cohomology group (vector space) $H^*_{dR}(M)$ gives some measure of the failure of closed forms to be exact.

Note that $H^*_{dR}(M)$ is a graded vector space. Recall that a *gradation of a vector space* V is a family (V_p) of subspaces $V_p \subseteq V$ such that

$$V = \bigoplus_p V_p.$$

A *filtration* on a vector space V is a nested family (V_p) of subspaces such that

$$V = V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots \supseteq V_n \supseteq \cdots$$

Actually, $H^*_{dR}(M)$ is a graded $\mathbb{R}$-algebra, where the multiplication on $H^*_{dR}(M)$ is induced by the exterior product of differential forms. Indeed, recall that

$$B^* \subseteq Z^* \subseteq \mathcal{A}^*.$$

The derivation property

$$d(\tau\omega) = \omega d\tau + (-1)^{deg \tau} \tau d\omega$$

implies that Z^* is a subring of $\mathcal{A}^*$, and that B^* is an ideal in Z^*. If $\omega = d\alpha$ and $d\tau = 0$, then

$$\omega\tau = d(\alpha\tau).$$

Thus, Z^*/B^* inherits a ring stucture from $\mathcal{A}^*$.

A variant of the de Rham cohomology is the *de Rham cohomology with compact support* $\mathcal{A}^*_c \subseteq \mathcal{A}^*$. This time, $\mathcal{A}^*_c$ consists of differential forms with compact support. As before, we have $B^*_c \subseteq Z^*_c$, and we let

$$\mathcal{A}^*_{dR,c} = Z^*_c/B^*_c.$$

The *Poincaré Lemma's* are the following results:

$$H^p(\mathbb{R}^n) = \begin{cases} 0 & \text{unless } p \neq 0; \\ \mathbb{R} & \text{if } p = 0, \end{cases}$$

and

$$H^p_c(\mathbb{R}^n) = \begin{cases} 0 & \text{unless } p \neq 0; \\ \mathbb{R} & \text{if } p = n. \end{cases}$$

2.2 Review of Complexes

More generally, a *(differential) complex* C is a $\mathbb{Z}$-graded vector space

$$C = \bigoplus_{p \in \mathbb{Z}} C^p$$

together with a linear map

$$d \colon C \to C$$

such that $dC^p \subseteq C^{p+1}$ and $d^2 = 0$.

Given a complex C, we define the $\mathbb{Z}$-graded vector spaces $B^* = \operatorname{Im} d$ and $Z^* = \operatorname{Ker} d$. We have

$$B^* \subseteq Z^* \subseteq C,$$

and we define the *cohomology spaces*

$$H^*(C) = \bigoplus_{p\in\mathbb{Z}} H^p(C),$$

where $H^p(C)$ is the quotient vector space

$$H^p(C) = \frac{\operatorname{Ker} d \cap C^p}{\operatorname{Im} d \cap C^p} = Z^p(C)/B^p(C).$$

Elements of $Z^p(C)$ are *p-cocyles* or *cocycles*, elements of $B^p(C)$ are *p-coboundaries* or *coboundaries*, elements of C^p are *p-cochains* or *cochains*. Given any cocycle $a \in Z^p(C)$, its *cohomology class* $a + \operatorname{Im} d^{p-1}$ is denoted by $[a]$. We will often drop the complex C when writing $Z^p(C)$, $B^p(C)$, or $H^p(C)$.

A *chain map* $f\colon B \to C$ between two complexes B and C is a family $f = (f^p)$ of maps $f^p\colon B^p \to C^p$, such that all the squares in the following diagram commute:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & B^p & \xrightarrow{d_B} & B^{p+1} & \xrightarrow{d_B} & B^{p+2} & \longrightarrow & \cdots \\
\vdots & & \big\downarrow{\scriptstyle f^p} & & \big\downarrow{\scriptstyle f^{p+1}} & & \big\downarrow{\scriptstyle f^{p+2}} & & \vdots \\
\cdots & \longrightarrow & C^p & \xrightarrow[d_C]{} & C^{p+1} & \xrightarrow[d_C]{} & C^{p+2} & \longrightarrow & \cdots
\end{array}$$

or equivalently, $d_C \circ f^p = f^{p+1} \circ d_B$.

A map of complexes $f\colon B \to C$ induces a map between cohomology spaces $f^*\colon H^*(B) \to H^*(C)$, which means that each map $f^p\colon B^p \to C^p$ induces a homomorphism $f^{p*}\colon H^p(B) \to H^p(C)$. The map f^{p*} is defined as follows: For every cohomology class $[a] \in H^p(B)$, we let

$$f^{p*}([a]) = [f^p(a)].$$

The following lemma shows that this definition makes sense:

Lemma 2.2.1 *The map $f^{p*}\colon H^p(B) \to H^p(C)$ is well defined.*

Proof. If $[a]$ is a cohomology class in $H^p(B)$, where $a \in B^p$ and $d_B(a) = 0$ (a is a cocycle), we need to check that $[f^p(a)]$ does not depend on the choice of $a \in [a]$ and that $f^p(a)$ is a cocycle in C^p. If $[a] = [b]$ where $a, b \in B^p$ and $d_B(a) = d_B(b) = 0$, then $a - b \in \operatorname{Im} d_B^{p-1}$, so that $a - b = d_B^{p-1}(x)$ for some $x \in B^{p-1}$, and since

$$d_C \circ f^{p-1} = f^p \circ d_B,$$

we get

$$f^p(a - b) = f^p \circ d_B^{p-1}(x) = d_C \circ f^{p-1}(x),$$

so that $f^p(a) - f^p(b) \in \operatorname{Im} d_C$, which means that $[f^p(a)] = [f^p(b)]$. Thus, $f^{p*}([a])$ is uniquely defined. Also, since $d_B(a) = 0$ and

$$d_C \circ f^p = f^{p+1} \circ d_B,$$

we get

$$d_C \circ f^p(a) = f^{p+1} \circ d_B(a) = 0,$$

which shows that $f^p(a) \in C^p$ is a cocycle. $\square$

There are situations, for instance when defining Čech cohomology groups, where we have different maps $f\colon B \to C$ and $g\colon B \to C$ between two complexes B and C, and yet we would like the induced maps $f^*\colon H^*(B) \to H^*(C)$ and $f^*\colon H^*(B) \to H^*(C)$ to be indentical, i.e., $f^* = g^*$. A sufficient condition is the existence of a chain homotopy between f and g.

Given two chain maps $f\colon B \to C$ and $g\colon B \to C$ between two complexes B and C, a *chain homotopy* between f and g is a family $s = (s^i)_{i\in\mathbb{Z}}$ of linear maps $s^p\colon B^p \to C^{p-1}$ such that

$$d_C^{p-1} \circ s^p + s^{p+1} \circ d_B^p = f^p - g^p$$

for all $p \in \mathbb{Z}$. As a diagram, a chain homotopy is given by the slanted arrows, where it is convenient to let $h = f - g$:

$$\begin{tikzcd}
\cdots \arrow[r] & B^{p-1} \arrow[r, "d_B"] \arrow[d, "h^{p-1}"] & B^p \arrow[r, "d_B"] \arrow[d, "h^p"] \arrow[dl, "s^p"] & B^{p+1} \arrow[r] \arrow[d, "h^{p+1}"] \arrow[dl, "s^{p+1}"] & \cdots \arrow[dl, "s^{p+2}"] \\
\cdots \arrow[r] & C^{p-1} \arrow[r, "d_C"'] & C^p \arrow[r, "d_C"'] & C^{p+1} \arrow[r] & \cdots
\end{tikzcd}$$

The following lemma clarifies this somewhat mysterious definition:

Lemma 2.2.2 *Given two chain maps $f\colon B \to C$ and $g\colon B \to C$ between two complexes B and C, if s is a chain homotopy between f and g, then $f^* = g^*$.*

Proof. If $[a]$ is a cohomology class in $H^p(B)$, where $a \in B^p$ is a cocycle ($d_B^p a = 0$), we have

$$(f^* - g^*)[a] = [f^p(a) - g^p(a)] = [d_C^{p-1} \circ s^p(a) + s^{p+1} \circ d_B^p(a)],$$

and since a is a cocycle, $d_B^p a = 0$, so that

$$(f^* - g^*)[a] = [d_C^{p-1} \circ s^p(a)] = 0,$$

since $d_C^{p-1} \circ s^p(a)$ is a coboundary in C^{p-1}. $\square$

Recall that a sequence

$$\cdots \longrightarrow A_p \xrightarrow{f_p} A_{p+1} \xrightarrow{f_{p+1}} A_{p+2} \longrightarrow \cdots$$

is *exact* if $\mathrm{Im}\, f_p = \mathrm{Ker}\, f_{p+1}$ for all p (where the A_p are groups, vector spaces, algebras, and the f_p are homomorphisms of the right type). A short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an sequence which is exact at A, B, C, which means that g is injective, g is surjective, and $\mathrm{Im}\, f = \mathrm{Ker}\, g$. Given a short exact sequence of complexes

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

we obtain a long exact sequence of cohomology groups of the following form:

$$\begin{array}{ccccccccc}
\downarrow & & & & & & & & \\
 & \longrightarrow & H^p(A) & \xrightarrow{f^*} & H^p(B) & \xrightarrow{g^*} & H^p(C) & \longrightarrow & \\
 & & & & & & & & \downarrow \\
 & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \\
\downarrow & & & & & & & & \\
 & \longrightarrow & H^{p+1}(A) & \xrightarrow{f^*} & H^{p+1}(B) & \xrightarrow{g^*} & H^{p+1}(C) & \longrightarrow & \\
 & & & & & & & & \downarrow
\end{array}$$

The map

$$\partial^p \colon H^p(C) \to H^{p+1}(A)$$

is defined as follows:

$$\partial^p([c]) = \left[(f^{p+1})^{-1}\left(d_B^p\left((g^p)^{-1}(c)\right)\right)\right],$$

where $[c]$ is any cocycle in C^p. It remains to verify that ∂^p is well defined and that it makes the long sequence exact, which is left as an exercise to the reader (or, see Madsen and Tornehave [22], Chapter 4, Definition 4.5).

A good example of the above is the *Mayer-Vietoris sequence*. Given a smooth manifold M, assume that $M = U_1 \cup U_2$, where U_1, U_2 are two open subsets of M. We have the short exact sequence

$$0 \longrightarrow \mathcal{A}^*(M) \xrightarrow{f} \mathcal{A}^*(U_1) \oplus \mathcal{A}^*(U_2) \xrightarrow{g} \mathcal{A}^*(U_2 \cap U_2) \longrightarrow 0,$$

with $f = \langle i_1^*, i_2^* \rangle$ and $g = j_1^* - j_2^*$, where $i_k \colon U_k \to M$ and $j_k \colon U_1 \cap U_2 \to U_k$ are the obvious inclusion maps. We get the Mayer-Vietoris long exact sequence

$$\begin{array}{ccccccccc}
\downarrow & & & & & & & & \\
 & \longrightarrow & H^p(M) & \xrightarrow{f^*} & H^p(U_1) \oplus H^p(U_2) & \xrightarrow{g^*} & H^p(U_1 \cap U_2) & \longrightarrow & \\
 & & & & & & & & \downarrow \\
 & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \\
\downarrow & & & & & & & & \\
 & \longrightarrow & H^{p+1}(M) & \xrightarrow{f^*} & H^{p+1}(U_1) \oplus H^{p+1}(U_2) & \xrightarrow{g^*} & H^{p+1}(U_1 \cap U_2) & \longrightarrow & \\
 & & & & & & & & \downarrow
\end{array}$$

which yields a way of computing $H^{p+1}(M)$ from $H^p(U_1 \cap U_2)$, $H^{p+1}(U_1)$, and $H^{p+1}(U_2)$.

2.3 Review of Čech Cohomology

For a comprehensive presentation of Čech cohomology of sheaves, see Warner [37] (notably Chapter 5), Bott and Tu [2] (Chapter 2), Hirzebruch [17] (Chapter 2), Chern [3] (Chapter 4), and for those who read French, Serre's "FAC" [33].

Let M be a smooth manifold, and let $\mathcal{F}$ be a sheaf on M. The final goal is to define the cohomology groups $\check{H}^p(X, \mathcal{F})$. However, in general, this requires a limit process involving the computation of cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ associated with open covers $\mathcal{U} = (U_j)_{j \in J}$ of the manifold M. Given two open covers $\mathcal{U}$ and $\mathcal{V}$, we can define when $\mathcal{V}$ is a refinement of $\mathcal{U}$, and then, we define the cohomology group $\check{H}^p(X, \mathcal{F})$ as the direct limit of the directed system of groups $\check{H}^p(\mathcal{U}, \mathcal{F})$. When the sheaf $\mathcal{F}$ has some special properties and when particularly nice covers exist, the limit process can be bypassed.

We begin by defining the cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ associated with an an open cover $\mathcal{U} = (U_j)_{j \in J}$ of M. We assume that J is a countable totally ordered set. Given any finite ordered subsequence $I = (i_1, \ldots, i_p)$ of elements in J, we let

$$U_{i_1 \ldots i_p} = U_{i_1} \cap \ldots \cap U_{i_p}.$$

Since J is totally ordered, for every ordered subsequence (i_0, i_1), we have inclusion maps

$$\partial_0 \colon U_{i_0} \cap U_{i_1} \to U_{i_1}, \quad \partial_1 \colon U_{i_0} \cap U_{i_1} \to U_{i_0},$$

for every ordered sequence (i_0, i_1, i_2), we have inclusion maps

$$\partial_0 \colon U_{i_0} \cap U_{i_1} \cap U_{i_2} \to U_{i_1} \cap U_{i_2}, \quad \partial_1 \colon U_{i_0} \cap U_{i_1} \cap U_{i_2} \to U_{i_0} \cap U_{i_2}, \quad \partial_2 \colon U_{i_0} \cap U_{i_1} \cap U_{i_2} \to U_{i_0} \cap U_{i_1},$$

and more generally for every ordered sequence $(i_0, \ldots i_p)$ of length $p+1$ $(p \geq 0)$, we have the $p+1$ inclusion maps

$$\partial_k \colon U_{i_0 \cdots i_p} \to U_{i_0 \cdots \widehat{i_k} \cdots i_p}$$

(∂_k "ignores" the kth open set, $0 \leq k \leq p$), where $U_{i_0 \cdots \widehat{i_k} \cdots i_p}$ denotes the intersection of the p open subsets obtained by omitting U_{i_k} from $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$ (the intersection of $p+1$ open subsets).

Given a presheaf (or sheaf) $\mathcal{F}$ of abelian groups on M, we define the *Čech Complex $C^*(\mathcal{U}, \mathcal{F})$ of the cover $\mathcal{U}$ with values in $\mathcal{F}$* as follows: We let

$$C^0(\mathcal{U}, \mathcal{F}) = \prod_{i \in J} \mathcal{F}(U_i),$$
$$C^1(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < i_1} \mathcal{F}(U_{i_0 i_1}),$$

and generally,

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0 \cdots i_p}).$$

The elements of $C^p(\mathcal{U}, \mathcal{F})$ are called *cochains on $\mathcal{U}$ with values in $\mathcal{F}$*. The coboundary maps

$$\delta_p\colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$$

are defined as follows:

$$\begin{aligned}\delta_0 &= \mathcal{F}(\partial_0) - \mathcal{F}(\partial_1),\\ \delta_1 &= \mathcal{F}(\partial_0) - \mathcal{F}(\partial_1) + \mathcal{F}(\partial_2),\end{aligned}$$

and generally

$$\delta_p = \mathcal{F}(\partial_0) - \mathcal{F}(\partial_1) + \cdots + (-1)^{p+1}\mathcal{F}(\partial_{p+1}).$$

A bit more explicitly,

$$(\delta_p\omega)_{i_0\cdots i_{p+1}} = \sum_{k=0}^{p+1}(-1)^k \omega_{i_0\cdots\widehat{i_k}\cdots i_{p+1}}|U_{i_0\ldots i_{p+1}},$$

for any $\omega \in C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0<\cdots<i_p} \mathcal{F}(U_{i_0\cdots i_p})$, where $\omega_{i_0\cdots\widehat{i_k}\cdots i_{p+1}}|U_{i_0\ldots i_{p+1}}$ denotes the restriction of $\omega_{i_0\cdots\widehat{i_k}\cdots i_{p+1}}$ from $U_{i_0\cdots\widehat{i_k}\cdots i_{p+1}}$ to $U_{i_0\ldots i_{p+1}}$. We often omit the index p in δ_p.

For example, given a family $(\omega_i)_{i\in I} \in C^0(\mathcal{U}, \mathcal{F})$, the family $(\delta\omega)_{ij} \in C^1(\mathcal{U}, \mathcal{F})$ is defined by

$$(\delta\omega)_{ij} = \omega_j - \omega_i,$$

for all i, j such that $U_{ij} = U_i \cap U_j \neq \emptyset$. More precisely, restriction to U_{ij} should be applied on the right-hand side:

$$(\delta\omega)_{ij} = \omega_j|_{U_{ij}} - \omega_i|_{U_{ij}}.$$

Such families $(\delta\omega)_{ij}$ form the group of coboundaries $B^1(\mathcal{U}, \mathcal{F})$. The group of cocycles $Z^0(\mathcal{U}, \mathcal{F})$ consists of the families $(\omega_i)_{i\in I} \in C^0(\mathcal{U}, \mathcal{F})$ such that

$$\omega_j = \omega_i$$

for all i, j such that $U_i \cap U_j \neq \emptyset$ (supressing the restriction to U_{ij} to simplify the notation).

Given a family $(\omega_{ij}) \in C^1(\mathcal{U}, \mathcal{F})$, the family $(\delta\omega)_{ijk} \in C^2(\mathcal{U}, \mathcal{F})$ is defined by

$$(\delta\omega)_{ijk} = \omega_{jk} - \omega_{ik} + \omega_{ij},$$

for all i, j, k such that $U_{ijk} = U_i \cap U_j \cap U_k \neq \emptyset$. More precisely, restriction to U_{ijk} should be applied on the right-hand side:

$$(\delta\omega)_{ijk} = \omega_{jk}|_{U_{ijk}} - \omega_{ik}|_{U_{ijk}} + \omega_{ij}|_{U_{ijk}}.$$

Such families $(\delta\omega)_{ijk}$ form the group of coboundaries $B^2(\mathcal{U}, \mathcal{F})$. The group of cocycles $Z^1(\mathcal{U}, \mathcal{F})$ consists of the families $(\omega_{ij}) \in C^1(\mathcal{U}, \mathcal{F})$ such that

$$\omega_{ij} + \omega_{jk} = \omega_{ik}$$

for all i, j, k such that $U_i \cap U_j \cap U_k \neq \emptyset$ (supressing the restriction to U_{ijk} to simplify the notation).

In general,

$$Z^p(\mathcal{U}, \mathcal{F}) = \operatorname{Ker} \delta_p \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$$

is the group of cocycles and

$$B^p(\mathcal{U}, \mathcal{F}) = \operatorname{Im} \delta_{p-1} \colon C^{p-1}(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{U}, \mathcal{F})$$

is the group of coboundaries, with B^0 the trivial group. It is easy to check that

$$\delta^2 = 0,$$

and we obtain the *Čech cohomology $H^*(\mathcal{U}, \mathcal{F})$ of the cover $\mathcal{U}$ with values in $\mathcal{F}$* from the complex $(C^*(\mathcal{U}, \mathcal{F}), \delta)$ in the usual way, i.e.

$$H^p(\mathcal{U}, \mathcal{F}) = \frac{\operatorname{Ker} \delta \cap C^p(\mathcal{U}, \mathcal{F})}{\operatorname{Im} \delta \cap C^p(\mathcal{U}, \mathcal{F})} = Z^p(\mathcal{U}, \mathcal{F})/B^p(\mathcal{U}, \mathcal{F}).$$

The cohomology groups $H^p(\mathcal{U}, \mathcal{F})$ are also denoted by $\check{H}^p(\mathcal{U}, \mathcal{F})$.

If $\mathcal{F}$ is a sheaf, it is fairly obvious that $\check{H}^0(\mathcal{U}, \mathcal{F})$ is the group $\Gamma(M, \mathcal{F})$ of global sections. However, for $p \geq 1$, the dependency of the cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ on the cover $\mathcal{U}$ is somewhat problematic, since we would like to define the cohomology $\check{H}^*(M, \mathcal{F})$ of the manifold M (with values in $\mathcal{F}$) independently of the cover $\mathcal{U}$.

If the sheaf $\mathcal{F}$ is the sheaf of locally constant functions on M, then $\check{H}^*(\mathcal{U}, \mathcal{F})$ is independent of the cover $\mathcal{U}$ as long as $\mathcal{U}$ is a "good cover" (see see Bott and Tu [2], Corollary 8.9.1, page 99). A good cover $\mathcal{U}$ is a cover such that every intersection $U_{i_0 \ldots i_p}$ is diffeomorphic to $\mathbb{R}^n$ (or, is contractible). The resulting Čech cohomology is denoted by $\check{H}^*(M, \mathcal{U})$.

Given two covers $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_j)_{j \in J}$, we say that *$\mathcal{V}$ is a refinement of $\mathcal{U}$*, denoted by $\mathcal{U} < \mathcal{V}$, if there is a map $\varphi \colon J \to I$ such that

$$V_j \subseteq U_{\varphi(j)}$$

for all $j \in J$. It can be shown that every cover can be refined to a good cover (see Bott and Tu [2], Section §5, page 42-43). It can also be shown that if $\mathcal{U}$ is a good cover of M then

$$H^*_{dR}(M) \simeq \check{H}^*(M, \mathcal{U}),$$

where the isomorphism is canonical (see Bott and Tu [2], Theorem 8.9, page 98).

The proof of the isomorphism $H^*_{dR}(M) \simeq \check{H}^*(M, \mathcal{U})$ (where $\mathcal{U}$ is a good cover) involves the double complex $\mathcal{AC}^{*,*}$ induced by the sheaf of differential forms $\mathcal{A}^*$, defined such that

$$\mathcal{AC}^{p,q} = \bigoplus_{I, |I| = p+1} \Gamma(U_I, \mathcal{A}^q).$$

There are two differentials

$$d\colon \mathcal{AC}^{p,q} \to \mathcal{AC}^{p,q+1} \quad \text{and} \quad \delta\colon \mathcal{AC}^{p,q} \to \mathcal{AC}^{p+1,q},$$

and we have $d^2 = 0$ and $\delta^2 = 0$. We can associate to the double complex $\mathcal{AC}^{*,*}$ the single complex $\mathcal{AC}^*$ defined such that

$$\mathcal{AC}^n = \bigoplus_{p+q=n} \mathcal{AC}^{p,q},$$

with the differential

$$D_n\colon \mathcal{AC}^n \to \mathcal{AC}^{n+1}$$

given by

$$D_n = \delta + (-1)^n d.$$

It is easily verifed that

$$D_{n+1} \circ D_n = 0.$$

Then, the cohomology of the complex $(\mathcal{AC}^*, D)$ is denoted by $H_D\{\mathcal{AC}^*(\mathcal{U}, \mathcal{A}^*)\}$, and it can be shown that there is a natural isomorphism

$$H^*_{dR}(M) \simeq H_D\{\mathcal{AC}^*(\mathcal{U}, \mathcal{A}^*)\}.$$

If $\mathcal{U}$ is a good cover, it can be shown that there is a natural isomorphism

$$\check{H}^*(M, \mathcal{U}) \simeq H_D\{\mathcal{AC}^*(\mathcal{U}, \mathcal{A}^*)\}.$$

These two isomorphisms imply that

$$H^*_{dR}(M) \simeq \check{H}^*(M, \mathcal{U})$$

for any good cover. For details, see Bott and Tu [2], Section §8.

When $\mathcal{F}$ is a more general sheaf on M, the Čech cohomology $\check{H}^*(M, \mathcal{F})$ of the manifold M can still be defined by forming a direct limit. This direct limit arises in the following way:

First, it can be shown that for any two covers $\mathcal{U} = (U_i)_{i\in I}$ and $\mathcal{V} = (V_j)_{j\in J}$, if $\mathcal{V}$ is a refinement of $\mathcal{U}$, then there is a well-defined map

$$f^{\mathcal{U}}_{\mathcal{V}}\colon \check{H}^*(\mathcal{U}, \mathcal{F}) \to \check{H}^*(\mathcal{V}, \mathcal{F}).$$

This is shown as follows: If $\varphi\colon J \to I$ is a function such that $V_j \subseteq U_{\varphi(j)}$ for all $j \in J$, we get a map

$$\varphi^{\#}\colon C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{V}, \mathcal{F})$$

defined such that

$$(\varphi^{\#}\omega)(V_{j_0\cdots j_p}) = \omega(U_{\varphi(j_0)\cdots\varphi(j_p)}).$$

Then, it is easily shown that $\varphi^{\#}$ is a chain map. Furthermore, if $\psi: J \to I$ is another function such that $V_j \subseteq U_{\psi(j)}$, we show that the maps

$$\kappa_p: C^p(\mathcal{U}, \mathcal{F}) \to C^{p-1}(\mathcal{V}, \mathcal{F})$$

defined such that

$$(\kappa_p\omega)(V_{j_0 \cdots j_{p-1}}) = \sum (-1)^i \omega(U_{\varphi(j_0)\cdots\varphi(j_i)\psi(j_i)\cdots\psi(j_{p-1})})$$

define a chain homotopy between $\varphi^{\#}$ and $\psi^{\#}$. By Lemma 2.2.2, $\varphi^{\#}$ and $\psi^{\#}$ induce the same map

$$f^{\mathcal{U}}_{\mathcal{V}}: \check{H}^*(\mathcal{U}, \mathcal{F}) \to \check{H}^*(\mathcal{V}, \mathcal{F}).$$

Next, a *direct system of groups* is a family $(G_i)_{i\in I}$ of groups indexed by a preordered directed set I (which means that there is a symmetric and transitive relation $\leq$ on I, and that for all $i, j \in I$, there is some $k \in I$ such that $i, j \leq k$), and for all $i, j \in I$ with $i \leq j$, there is a group homomorphism $f^i_j: G_i \to G_j$ such that:

$$\begin{aligned} f^i_i &= \mathrm{id}, \\ f^i_k &= f^j_k \circ f^i_j \end{aligned}$$

whenever $i \leq j \leq k$. The *direct limit* of the system $(G_i)_{i\in I}$, denoted by $\lim_{i\in I} G_i$, is the group defined as follows: Form the disjoint union $\coprod_{i\in I} G_i$, and define the equivalence relation $\sim$ on $\coprod_{i\in I} G_i$ as follows: for any $a_i \in G_i$ and $a_j \in G_j$,

$$a_i \sim a_j \quad \text{iff} \quad f^i_k(a_i) = f^j_k(a_j) \quad \text{for some upper bound } k \text{ of } i, j.$$

Then,

$$\lim_{i\in I} G_i = (\coprod_{i\in I} G_i)/\sim,$$

with the group operation defined such that

$$[a_i] + [a_j] = [f^i_k(a_i) + f^j_k(a_j)].$$

Since the maps

$$f^{\mathcal{U}}_{\mathcal{V}}: \check{H}^*(\mathcal{U}, \mathcal{F}) \to \check{H}^*(\mathcal{V}, \mathcal{F})$$

make $(\check{H}^*(\mathcal{U}, \mathcal{F}))_{\mathcal{U}}$ into a direct system of groups, we define the *Čech cohomology of M with values in $\mathcal{F}$* as the direct limit

$$\check{H}^*(M, \mathcal{F}) = \lim_{\mathcal{U}} \check{H}^*(\mathcal{U}, \mathcal{F}).$$

Since good covers are cofinal in the set of covers of M, the above limit can be restricted to good covers. We now turn to spectral sequences, a crucial tool for computing cohomology groups.

2.4 Spectral Sequences

We begin with exact couples.

Definition 2.4.1 An *exact couple* is a tuple (A, B, i, j, k) where $i\colon A \to A$, $j\colon A \to B$, and $k\colon B \to A$, are homomorphisms of abelian groups A, B such that the triangle

$$\begin{array}{ccc} A & \xrightarrow{i} & A \\ & {\scriptstyle k}\searrow \quad \swarrow{\scriptstyle j} & \\ & B & \end{array}$$

is exact at every vertex, which means that

$$\begin{aligned} \operatorname{Ker} k &= \operatorname{Im} j, \\ \operatorname{Ker} i &= \operatorname{Im} k, \\ \operatorname{Ker} j &= \operatorname{Im} i. \end{aligned}$$

Given an exact couple (A, B, i, j, k), we get a *derived couple*

$$\begin{array}{ccc} A' & \xrightarrow{i'} & A' \\ & {\scriptstyle k'}\searrow \quad \swarrow{\scriptstyle j'} & \\ & B' & \end{array}$$

as follows: Let $d = j \circ k$, and observe that

$$d^2 = 0$$

since

$$d^2 = d \circ d = j \circ k \circ j \circ k = j \circ (k \circ j) \circ k = 0$$

and $\operatorname{Ker} k = \operatorname{Im} j$. We let

$$\begin{aligned} A' &= i(A) = \operatorname{Ker} j, \\ B' &= \operatorname{Ker} d/\operatorname{Im} d = H_d(B), \end{aligned}$$

and roughly stated, we let

$$\begin{aligned} i' &= i|A', \\ j' &= j \circ i^{-1}, \\ k' &= k. \end{aligned}$$

Let us clarify the meaning of the above equations. The map i' is the restriction of i to A'. The map j' is defined as follows: For every $i(x) \in A'$, since

$$d(j(x)) = j \circ k \circ j(x) = 0$$

because $\operatorname{Ker} k = \operatorname{Im} j$, we can define j' by

$$j'(i(x)) = [j(x)],$$

the cohomology class of $j(x)$. The map j' is well defined because if $i(x) = i(y) \in A'$, from $\operatorname{Ker} i = \operatorname{Im} k$, we have $x - y = k(z)$ for some $z \in B$, and

$$j(x) - j(y) = j \circ k(z) = d(z),$$

which shows that $[j(x)] = [j(y)]$ is the same cohomology class. Finally, k' is defined such that

$$k'([y]) = k(y)$$

for every $[y] \in B'$, where $y \in \operatorname{Ker} d$. Since

$$j \circ k(y) = d(y) = 0$$

and $\operatorname{Ker} j = \operatorname{Im} i$, we have $k(y) \in i(A) = A'$. Also, if $[y] = [y']$, we have $y' - y = d(z)$, and from this, we get

$$k(y') - k(y) = k \circ d(z) = k \circ j \circ k(z) = 0$$

since $\operatorname{Ker} k = \operatorname{Im} j$, which shows that k' is well defined. We can also prove that the new triangle is exact at every vertex.

Definition 2.4.2 Given an exact couple

$$\begin{array}{ccccc} A & & \xrightarrow{i} & & A \\ & {\scriptstyle k} \searrow & & \swarrow {\scriptstyle j} & \\ & & E & & \end{array}$$

we can define inductively a sequence of derived couples

$$\begin{array}{ccccc} A_n & & \xrightarrow{i_n} & & A_n \\ & {\scriptstyle k_n} \searrow & & \swarrow {\scriptstyle j_n} & \\ & & E_n & & \end{array}$$

using the above definition, i.e.

$$\begin{aligned} A_{n+1} &= i_n(A_n) = \operatorname{Ker} j_n, \\ E_{n+1} &= \operatorname{Ker} d_n / \operatorname{Im} d_n = H_{d_n}(E_n), \\ i_{n+1} &= i_n | A_{n+1}, \\ j_{n+1} &= j_n \circ i_n^{-1}, \\ k_{n+1} &= k_n, \end{aligned}$$

and with

$$d_n = j_n \circ k_n.$$

The sequence of differential groups (E_n, d_n) is called the *spectral sequence* of (A, E, i, j, k). If there is some integer $m > 0$ such that $d_n = 0$ for all $n \geq m$, then $\mathrm{Ker}\, d_m = E_m$ and $\mathrm{Im}\, d_m = 0$, so that

$$E_m = E_{m+1} = E_{m+2} = \cdots$$

In such a situation, we say that the spectral sequence *converges* and we denote by E_∞ the common abelian group E_m.

In particular, consider a filtered differential group (K, D) where

$$K = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq \cdots,$$

and we also let $K_{-n} = K$ for $n < 0$. The direct sum

$$GK = \bigoplus_{p \in \mathbb{N}} K_p / K_{p+1}$$

is called the *associated graded complex* of K. Let

$$A = \bigoplus_{p \in \mathbb{Z}} K_p,$$

and let $i \colon A \to A$ be the inclusion map induced by the inclusion maps $K_{p+1} \subseteq K_p$. Let $B = A/i(A)$, so that we have a short exact sequence

$$0 \longrightarrow A \stackrel{i}{\longrightarrow} A \stackrel{j}{\longrightarrow} B \longrightarrow 0,$$

which can also represented by the exact couple

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "0"] & \end{tikzcd}$$

From this short exact sequence, we get a long exact sequence of cohomologies

$$\begin{tikzcd} H(A) \arrow[rr, "i_*"] & & H(A) \arrow[dl, "j_*"] \\ & H(B) \arrow[ul, "\partial"] & \end{tikzcd}$$

and we let

$$\begin{aligned} A_1 &= H(A), \\ E_1 &= H(B), \\ i_1 &= i_*, \\ j_1 &= j_*, \\ k_1 &= \partial. \end{aligned}$$

We can define the spectral sequence (E_p, d_p) starting from the exact couple $(A_1, E_1, i_1, j_1, k_1)$. If the filtration

$$K = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_l \supseteq \{0\}$$

is finite, then

$$A_1 = \bigoplus_{p \in \mathbb{Z}} H(K_p)$$

where

$$H(K) = H(K_0) \hookleftarrow H(K_1) \hookleftarrow H(K_2) \hookleftarrow \cdots \hookleftarrow H(K_l) \hookleftarrow \{0\},$$

which is not an exact sequence. Then,

$$A_2 = i(A_1) = \bigoplus_{p \in \mathbb{Z}} iH(K_p),$$

where

$$H(K) = H(K_0) \supseteq iH(K_1) \hookleftarrow iH(K_2) \hookleftarrow \cdots \hookleftarrow iH(K_l) \hookleftarrow \{0\},$$

and $iH(K_1) \subseteq H(K_0)$ is an inclusion. Similarly

$$A_3 = i(A_2) = \bigoplus_{p \in \mathbb{Z}} i^2 H(K_p)$$

is the direct sum of the groups in the sequence

$$H(K) = H(K_0) \supseteq iH(K_1) \supseteq iiH(K_2) \hookleftarrow iiH(K_3) \hookleftarrow \cdots \hookleftarrow iiH(K_l) \hookleftarrow \{0\},$$

and

$$A_4 = i(A_3) = \bigoplus_{p \in \mathbb{Z}} i^3 H(K_p)$$

where

$$H(K) = H(K_0) \supseteq iH(K_1) \supseteq iiH(K_2) \supseteq iiiH(K_3) \hookleftarrow \cdots \hookleftarrow iiiH(K_l) \hookleftarrow \{0\},$$

and so on.

Generally, we have

$$A_q = i(A_{q-1}) = \bigoplus_{p \in \mathbb{Z}} i^{q-1} H(K_p)$$

where

$$H(K) = H(K_0) \supseteq iH(K_1) \supseteq iiH(K_2) \supseteq iiiH(K_3) \hookleftarrow \cdots \hookleftarrow i^{q-1}H(K_l) \hookleftarrow \{0\}.$$

If we keep going until A_{l+1}, the map i_{l+1} is an inclusion, $k_{l+1} = 0$, and then $d_{l+1} = 0$, and the sequence stabilizes with $A_\infty = A_{l+1}$. Letting $F_p = i^p H(K_p)$, we have an induced filtration

$$H(K) = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_p \supseteq \cdots$$

making $H(K)$ into a filtered complex, and the stationary value $E_\infty = E_{l+1}$ of E_p $(p \geq l+1)$ is the associated graded complex

$$E_\infty = \bigoplus_p F_p/F_{p+1} = G(H(K)).$$

Note that

$$E_1 = H(B) = \bigoplus_p K_p/K_{p+1} = H(GK),$$

where GK is the graded complex associated with K. The map $i_\infty = i_{l+1}$ associated with i is injective and

$$E_\infty = A_\infty/i_\infty A_\infty = G(H(K)).$$

If $H(K)$ is a vector space, then $G(H(K))$ is isomorphic to $H(K)$, although not canonically.

We can also define spectral sequences associated with a graded filtered differential complex complex (K, D), where

$$K = \bigoplus_{n\in\mathbb{Z}} K^n,$$

and where K has a filtration

$$K = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_p \supseteq \cdots$$

To distinguish p from n, we call n the *dimension* of K^n. The filtration (K_p) on K induces a filtration in each dimension n: If $K_p^n = K^n \cap K_p$, then (K_p^n) is a filtration on K^n. Let us assume that the filtration (K_p^n) has finite length $l(n)$ for every n, and let $K_{-n} = K_0$ for all $n \in \mathbb{N}$.

Lemma 2.4.3 *Given a graded filtered differential complex (K, D) as above (where the filtration (K_p^n) has finite length $l(n)$ for every n), there is a spectral sequence (E_p, d_p) induced by the short exact sequence*

$$0 \longrightarrow \bigoplus_p K_{p+1} \xrightarrow{i} \bigoplus_p K_p \xrightarrow{j} \bigoplus_p K_p/K_{p+1} \longrightarrow 0$$

which converges to $H_D^*(K)$.

Proof. As in the discussion before Lemma 2.4.3, let

$$A_q = i(A_{q-1}) = \bigoplus_{p\in\mathbb{Z}} i^{q-1}H(K_p)$$

and $F_q = i^q H(K_q)$. For $r \geq p+1$, we have $i^r H(K_p) = F_p$, $i^r H(K_{p+1}) = F_{p+1}$ and the map $i\colon F_{p+1} \to F_p$ is an inclusion. Given n, let $r \geq l(n+1)+1$, where $l(n)$ is the length of the sequence (K_p^n). Then, for every integer p,

$$i^r H^{n+1}(K_{p+1}) = F_{p+1}^{n+1}$$

and

$$i\colon i^r H^{n+1}(K_{p+1}) \to i^r H^{n+1}(K_p)$$

is an inclusion. It follows that

$$i_r\colon A_r^{n+1} \to A_r^{n+1}$$

is an inclusion and that

$$k_r\colon E_r^n \to A_r^{n+1}$$

is the zero map. Consequently, as r increases, the E_r^n become stationary. We get

$$A_\infty^n = \bigoplus_{p\in\mathbb{Z}} F_p^n,$$

and since i_∞ send every F_{p+1}^n to F_p^n for every n,

$$i_\infty\colon \bigoplus F_{p+1} \to \bigoplus F_p$$

is an inclusion, and

$$E_\infty = \bigoplus F_p/F_{p+1}$$

is the associated graded complex of $H_D^*(K)$. $\square$

We now consider a double complex

$$K = \bigoplus K^{p,q}$$

with horizontal differential

$$\delta\colon K^{p,q} \to K^{p+1,q}$$

and vertical differential

$$d\colon K^{p,q} \to K^{p,q+1}.$$

We define the grading (K^n) by letting

$$K^n = \bigoplus_{p+q=n} K^{p,q}$$

and the differential $D\colon K^n \to K^{n+1}$ such that

$$D = \delta + (-1)^p d$$

on $K^{p,q}$ (with $p+q=n$). Then, we have the filtration (K_p) on K defined by

$$K_p = \bigoplus_{i\geq p} \bigoplus_{q} K^{i,q}.$$

We let

$$A = \bigoplus_{p} K^p = \bigoplus_{p} \bigoplus_{i\geq p} \bigoplus_{q} K^{i,q},$$

which is also a double complex, viewing A as

$$A = \bigoplus_{p} \bigoplus_{q} \bigoplus_{i\geq p} K^{i,q}.$$

We can form a single complex $A = \bigoplus_k A^k$ out of this double complex by letting

$$A^k = \bigoplus_{p+q=k} \bigoplus_{i\geq p} K^{i,q}.$$

There is an inclusion map given by

$$i\colon A^k \cap K_{p+1} \to A^k \cap K_p,$$

and the single complex A inherits the differential $D = \delta + (-1)^p d$ from K. Similarly,

$$B = \bigoplus K_p/K_{p+1}$$

can be made into a single complex with differential D. Actually, on B, we have $D = (-1)^p d$, and thus

$$E_1 = H_D(B) = H_d(K).$$

We have the short exact sequence

$$0 \longrightarrow A \stackrel{i}{\longrightarrow} A \longrightarrow B \longrightarrow 0,$$

from which we can form a spectral sequence (E_r, d_r). The following lemma can be shown:

Lemma 2.4.4 *Given a double complex $K = \bigoplus K^{p,q}$, there is a spectral sequence (E_r, d_r) converging to the cohomology $H_D(K)$ (of the single complex K), such that each E_r has a bigrading with*

$$d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1},$$

and

$$E_1^{p,q} = H_d^{p,q}(K), \qquad E_2^{p,q} = H_\delta^{p,q} H_d(K).$$

Furthermore, the associated graded complex of the cohomology $H_D(K)$ is given by

$$GH_D^n(K) = \bigoplus_{p+q=n} E_\infty^{p,q}(K).$$

The proof can be foound in Bott and Tu [2], Theorem 14.14, page 161-165.

Instead of using the filtration (K_p) on the double complex $K = (K^{p,q})$ defined by

$$K_p = \bigoplus_{i \geq p} \bigoplus_{q} K^{i,q},$$

where we sum over the rows, we could have considered the filtration (K'_p) defined by

$$K'_p = \bigoplus_{p} \bigoplus_{j \geq q} K^{p,j},$$

where we sum over the columns. This yields a second spectral sequence (E'_r, d'_r) which also converges to the cohomology $H_D(K)$ of the single complex K, but with

$$d'_r \colon {E'}_r^{p,q} \to {E'}_r^{p-r+1,q+r},$$

and

$${E'}_1^{p,q} = H_\delta^{p,q}(K), \qquad {E'}_2^{p,q} = H_d^{p,q} H_\delta(K).$$

Using these two spectral sequences associated with a double complex, we can also prove the equivalence of the Čech cohomology and the de Rham cohomology (see Bott and Tu [2], Example 14.16, page 166-167). We consider the double complex

$$K^{p,q} = C^p(\mathcal{U}, \mathcal{A}^q)$$

where $\mathcal{U}$ is a good cover on the smooth manifold M. We consider the two spectral sequences formed from $(K^{p,q})$.

Considering the first spectral sequence, we get

$$E_1^{p,q} = H_d^{p,q}(K),$$

and by the Poincaré Lemma, we have

$$H_d^{p,q}(K) = \begin{cases} C^p(\mathcal{U}, \mathbb{R}) & \text{if } q = 0, \\ 0 & \text{if } q \neq 0, \end{cases}$$

so that

$$E_2^{p,q} = \begin{cases} \check{H}^p(M, \mathbb{R}) & \text{if } q = 0, \\ 0 & \text{if } q \neq 0, \end{cases}$$

and

$$E_2 = E_3 = \cdots = E_\infty = H_D(K).$$

Considering the second spectral sequence, we get

$${E'}_1^{p,q} = H_\delta^{p,q}(K),$$

and by the Mayer-Vietoris principle, we have

$$H_\delta^{p,q}(K) = \begin{cases} \Gamma(\mathcal{A}^q) & \text{if } p = 0, \\ 0 & \text{if } p \neq 0, \end{cases}$$

so that

$$E'^{p,q}_2 = \begin{cases} H^q_{dR}(M, \mathbb{R}) & \text{if } p = 0, \\ 0 & \text{if } p \neq 0, \end{cases}$$

and

$$E'_2 = E'_3 = \cdots = E'_\infty = H_D(K).$$

As a consequence,

$$H_{dR}(M) = \check{H}(M, \mathbb{R}).$$

The proof of the Mayer-Vietoris principle uses a partition of unity which is used to define a homotopy operator for (C, δ). For details, see Bott and Tu [2], Section §8.

Proving that

$$H_d^{p,q}(K) = \begin{cases} C^p(\mathcal{U}, \mathbb{R}) & \text{if } q = 0, \\ 0 & \text{if } q \neq 0, \end{cases}$$

immediately reduces to proving that

$$H^q_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

The proof of the Poincaré Lemma proceeds by induction. Using the projection

$$\pi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

and its section

$$s\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$$

defined such that $s(x) = (x, 0)$, we can construct a homotopy operator K on $\mathcal{A}^*(\mathbb{R}^n \times \mathbb{R})$ such that

$$1 - \pi^* \circ s^* = \pm(dK \pm Kd).$$

From this, we get an isomorphism

$$H^*_{dR}(\mathbb{R}^n \times \mathbb{R}) \approx H^*_{dR}(\mathbb{R}^n),$$

and finally, we get

$$H^q_{dR}(\mathbb{R}^n) = H^q_{dR}(point) = \begin{cases} \mathbb{R} & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

For details, see Bott and Tu [2], Section §4.

Another application of spectral sequences is the computation of the de Rham cohomology of a fiber bundle using Leray spectral sequences.