A projective Shortcut:

If X is a hypersurface in $\mathbb{A}^n$ defined by the equation

$$f(x_1, \ldots, x_n) = 0,$$

observe that showing that X is singular at some point $x \in X$ entails finding some $(x_1, \ldots, x_n)$ solution of the set of equations

$$\begin{aligned} f(x_1, \ldots, x_n) &= 0, \\ \frac{\partial f}{\partial x_1}(x_1, \ldots, x_n) &= 0, \\ &\cdots\cdots \\ \frac{\partial f}{\partial x_n}(x_1, \ldots, x_n) &= 0. \end{aligned}$$

When X is a projective hypersurface, it is possible to take a shortcut. In this case, we have a homogeneous polynomial $F(x_0, \ldots, x_n)$, and if F is of degree m, by Euler's identity, we have

$$mF = x_0 \frac{\partial F}{\partial x_1} + \cdots + x_n \frac{\partial F}{\partial x_n}.$$

Thus,

$$\frac{\partial F}{\partial x_j}(x_0, \ldots, x_n) = 0$$

for all j, $0 \leq j \leq n$, implies that

$$F(x_0, \ldots, x_n) = 0.$$

As a consequence, a point $x = (x_0, \ldots, x_n)$ is singular iff it is a solution of the set of equations

$$\begin{aligned} \frac{\partial F}{\partial x_0}(x_0, \ldots, x_n) &= 0, \\ &\cdots\cdots \\ \frac{\partial F}{\partial x_n}(x_0, \ldots, x_n) &= 0. \end{aligned}$$

For example, the affine curve of equation

$$xy - 1 = 0$$

becomes the projective curve of equation

$$xy - z^2 = 0,$$

and the affine curve of equation

$$xy = 0$$

remains the same. In the first case, the partials are

$$(y, x, -2z),$$

and since x, y, z are homogeneous coordinates, we can't have $x = y = z = 0$. However, in the second case, the partials are

$$(y, x, 0),$$

and they are null for $x = y = 0$, $z = 1$.

1.10 Constructions Of Varieties

In this Section, we discuss products, cones, and projections.

Products:

The question is, given two varieties X, Y, how do we define their product $X \times Y$ so that it is a variety. This is done in several steps listed below:

First, we deal with the affine case.

Step (1 a) We let

$$\mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}.$$

Step (1 b) Define $X \times Y$, where $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ are affine varieties (see below).

Next, we deal with the projective case.

Step (2 a) Define $\mathbb{P}^m \times \mathbb{P}^n$ (see below).

Step (2 b) Define $X \times Y$, where $X \subseteq \mathbb{P}^m$ and $Y \subseteq \mathbb{P}^n$ are projective varieties. This is similar to (2 a), but it uses homogeneous ideals.

Step (2 c) For abstract varieties X, Y, use (2 b) and glueing.

The details for each step are as follows;

(1 b) We can assume that X is defined by an ideal I whose polynomials use the variables $x_1, \ldots, x_m$, and that Y is is defined by an ideal J whose polynomials use the variables $x_{m+1}, \ldots, x_{m+n}$. Then, $X \times Y$ is defined by the ideal in $\mathbb{C}[x_1, \ldots, x_{m+n}]$ generated by $I \cup J$, or equivalently, if I' and J' are the ideals in $\mathbb{C}[x_1, \ldots, x_{m+n}]$ generated by I and J, $X \times Y$ is defined by the ideal

$$I' + J' = \{f + g \mid f \in I',\, g \in J'\}.$$

For example, if $X \subseteq \mathbb{A}^2$ is the circle of equation

$$x^2 + y^2 - 1 = 0$$

and $Y \subseteq \mathbb{A}^1$ is the point of equation $z = 0$, the variety $X \times Y$ is the circle in $\mathbb{A}^3$ defined by the two equations

$$x^2 + y^2 - 1 = 0, \quad z = 0.$$

On the other hand, if $Y = \mathbb{A}^1$, then $X \times Y$ is the cylinder of equation

$$x^2 + y^2 - 1 = 0.$$

(2 a) We use the *Segre embedding* of the set $\mathbb{P}^m \times \mathbb{P}^n$ into $\mathbb{P}^{mn+m+n}$. Note that $mn + m + n = (m+1)(n+1) - 1$. The *Segre embedding* $\sigma_{m,n}$ is defined a follows. Given any point $x = (x_0, \ldots, x_m) \in \mathbb{P}^m$ and any point $y = (y_0, \ldots, y_n) \in \mathbb{P}^n$,

$$\sigma_{m,n}(x, y) = (x_0y_0, \ldots, x_0y_n, \ldots, x_my_0, \ldots, x_my_n).$$

We can also give a coordinate-free definition. If $\mathbf{P}(V)$ is the projective space associated with the vector space V of dimension $m+1$ and $\mathbf{P}(W)$ is the projective space associated with the vector space W of dimension $n+1$, the Segre embedding is the biprojective map induced by the natural bilinear map

$$\otimes: V \times W \to V \otimes W.$$

The image of the set $\mathbb{P}^m \times \mathbb{P}^n$ in $\mathbb{P}^{mn+m+n}$ under the Segre embedding is an algebraic variety defined by equations of degree 2. Letting $Z = (z_{i,j})$ be the tuple of $(m+1)(n+1)$ variables corresponding to the products x_iy_j, the equations are

$$z_{i,j}z_{h,k} - z_{i,k}z_{h,j} = 0.$$

The equations mean that all the 2×2 minors of the matrix $X^\top Y$ vanish, where $X = (x_0, \ldots, x_m)$ and $Y = (y_0, \ldots, y_n)$, or equivalently, that the matrix $X^\top Y$ has rank 1. This is indeed equivalent to the fact that $Z = X \otimes Y$. For example, the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ is given by

$$((x_0, y_1), (y_0, y_1)) \mapsto \begin{pmatrix} x_0y_0 & x_0y_1 \\ x_1y_0 & x_1y_1 \end{pmatrix} = \begin{pmatrix} z_{0,0} & z_{0,1} \\ z_{1,0} & z_{1,1} \end{pmatrix}.$$

The image is a nonsingular quadric surface in $\mathbb{P}^3$. By Problem 3, we know that every nonsingualr quadric is equivalent to this quadric. Its real affine trace in $\mathbb{A}^3$ is a *hyperbolic paraboloid*. It is generated by two families of lines, and as such, it is a *ruled surface*.

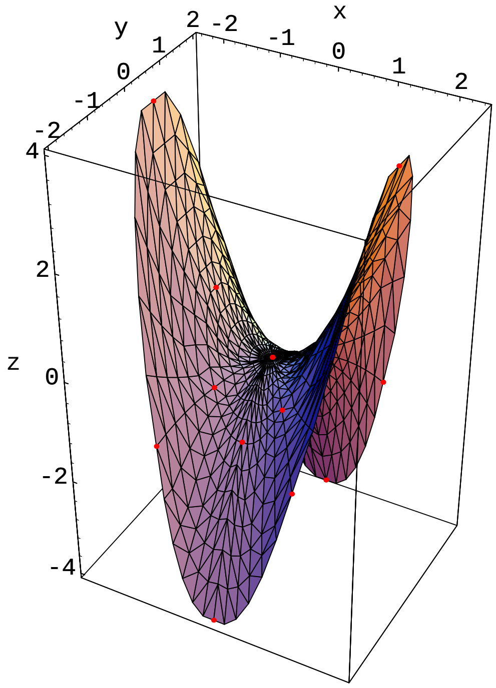

Figure 1.1: An hyperbolic paraboloid

Cones:

Given a variety $X \subseteq \mathbb{P}^n$, we can view the equations of X as defining an affine cone $C(X)$ in $\mathbb{A}^{n+1}$, with vertex $(0, \ldots, 0)$. Indeed, every point $x \in X$ corresponds to the line through the origin in $\mathbb{A}^{n+1}$ whose points (other than the origin) are given by the homogeneous coordinates $(x_0, \ldots, x_n)$ of x. Given any point $p \in \mathbb{P}^{n+1}$, we can also define the cone $C_p(X)$ in $\mathbb{P}^{n+1}$. After some suitable change of frame, we can assume that $p = (0, \ldots, 0, 1) \in \mathbb{P}^{n+1}$. If X is defined by the ideal I whose homogeneous polynomials involve the variables $x_0, \ldots, x_n$, then $C_p(X)$ is also defined by I, viewing the polynomials in I as polynomials in the variables $x_0, \ldots, x_{n+1}$. The relationship between $C(X)$ and $C_p(X)$ is that

$$C_p(X) = C(X) \cup X,$$

since there is a bijection between $\mathbb{P}^{n+1}$ and $\mathbb{A}^{n+1} \cup \mathbb{P}^n$. The cone $C_p(X)$ has the same degree as X and dimension $\dim(X) + 1$.

Projections:

Given a variety $X \subseteq \mathbb{P}^n$, we first assume that the center of projection p does not belong to X. Then, we can define the *projection* $\overline{X}$ *of* X *from* p *to* $\mathbb{P}^{n-1}$. Assuming that $p = (0, \ldots, 0, 1)$, the projection map is given by dropping the last coordinate:

$$(x_0, \ldots, x_{n-1}, x_n) \mapsto (x_0, \ldots, x_{n-1}).$$

Gometrically, $\overline{X}$ is obtained by intersecting every line L connecting p to a point of X with a fixed hyperplane H such that $p \notin H$. This can be made more precise as follows. If $\mathbf{P}(V)$ if the projective space associated with a vector space V and p corresponds to a line $L \subseteq V$, then the projection map can be viewed as the map $\pi\colon V \to V/L$ onto the quotient space V/L. So far, we have only defined $\overline{X}$ as a subset of $\mathbf{P}(H)$ (or $\mathbb{P}^{n-1}$). That $\overline{X}$ is indeed a variety can be proved using resultants (see Section 1.13). If X is given by the homogeneous ideal I, then $\overline{X}$ is the zero locus of

$$\{R(F,G) \mid F, G \in I,\ F, G \text{ homogeneous}\},$$

where $R(F,G)$ is the resultant of F and G.

Exercise: Prove the above assertion.

As an example of a projection, if $p = (0,1,0,0) \in \mathbb{P}^3$ and X is the twisted cubic

$$X = \{(s^3,\ s^2t,\ st^2,\ t^3) \mid (s,t) \in \mathbb{P}^1\},$$

we have $p \notin X$, and the projection $\overline{X}$ of X is the curve in $\mathbb{P}^2$ given by

$$\overline{X} = \{(s^3,\ st^2,\ t^3) \mid (s,t) \in \mathbb{P}^1\}.$$

If the homogeneous coordinates in $\mathbb{P}^2$ are (w,x,y), this is the curve of equation

$$y^2 w = x^3.$$

In the affine patch corresponding to $w \neq 0$, this is the cuspidal cubic of equation $y^2 = x^3$.

Observe that the point $p = (0,1,0,0)$ is on the tangent of the twisted cubic at the origin. Indeed, in the affine patch $x \neq 0$, we have $s \neq 0$, and letting $s = 1$, the affine trace of the twisted cubic is given by

$$t \mapsto (1, t, t^2, t^3).$$

Then, the tangent at t is defined by the vector $(0,1,2t,3t^2)$, and at the origin, i.e. for $t = 0$, we get $(0,1,0,0)$, which corresponds to the point p. This is a general fact. Projecting from a point on any tangent to the twisted cubic X (and not on X) yields a cuspidal cubic. On the other hand, projecting from any other point not on this surface yields a nodal cubic. For example, projecting from the point $p = (1,0,0,1)$ (which is not on X) yields a nodal cubic. Indeed, every line connecting p to a point on X is the set of homogeneous coordinates

$$(\lambda + \mu s^3,\ \mu s^2 t,\ \mu s t^2,\ \lambda + \mu t^3),$$

and the intersection of this line with the plane $w = 0$ (assuming homogeneous coordinates (x,y,z,w)) yields the point such that

$$\lambda + \mu t^3 = 0,$$

that is,

$$(-\mu t^3 + \mu s^3,\ \mu s^2 t,\ \mu s t^2,\ 0) = \mu(s^3 - t^3,\ s^2 t,\ s t^2,\ 0).$$

Thus, we get the curve of equation

$$y^3 - z^3 - xyz = 0.$$

The partials are

$$(-yz,\ 3y^2 - xz,\ -3z^2 - xy).$$

They vanish simultaneously at just one point, namely $(1, 0, 0)$, which is a singular point. This point is a node and not a cusp, as we can see in the affine patch $x \neq 0$. Letting $x = 1$, the affine trace of the cubic is

$$y^3 - z^3 - yz = 0.$$

It is easily seen that the two tangents at the origin are the line $y = 0$ and $z = 0$.

1.11 Topology of Curves

We begin by defining linear systems of hypersurfaces. Given a vector space V, recall that $\mathrm{Sym}^d V$ denote the vector space of symmetric d-tensors (over V), and that $(\mathrm{Sym}^d V)^* \approx \mathrm{Sym}^d V^*$ is isomorphic to the space of homogeneous polynomial maps of degree d over V.

Definition 1.11.1 Let V and W be vector spaces, where $\dim(W) = k$ and $\dim(V) = n+1$, for any integer $d \geq 1$, a *linear system of hypersurfaces in* $\mathbf{P}(V)$ is a linear map

$$L \colon W \to \mathrm{Sym}^d V^*.$$

When $\dim(W) = 2$, the linear system is called a *pencil of hypersurfaces*. When $\dim(W) = 3$, the linear system is called a *net*, and when $\dim(W) = 4$, the linear system is called a *web*.

Of course, when $\dim(W) = 1$, the linear system simply picks out some hypersurface in $\mathbf{P}(V)$. If $(\overrightarrow{e_1}, \ldots, \overrightarrow{e_k})$ is a basis of W, the linear map L is completely determined by the k homogeneous polynomials

$$F_1 = L(\overrightarrow{e_1}), \ldots, F_k = L(\overrightarrow{e_k}),$$

each of degree d. Each F_i defines a hypersurface in $\mathbf{P}(V)$. When $d = 2$, we have a linear system of quadric hypersurfaces. In in addition $n = 2$, then we have a linear system of conics. When $k = 2$, we have a pencil of conics. This pencil consists of all conics of equation

$$\lambda F_1 + \mu F_2 = 0,$$

where $(\lambda, \mu) \in \mathbb{P}^1$. In general, the conic C_1 defined by $F_1 = 0$ and the conic C_2 defined by $F_2 = 0$ intersect in four distinct points, as illustrated in figure 1.2.

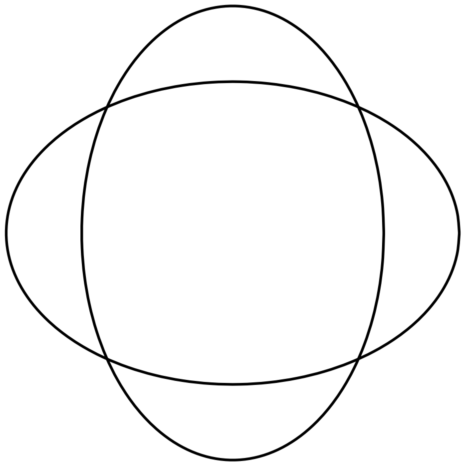

Figure 1.2: Base points of a pencil of conics

Every conic in the pencil passes through these four points. Conversely, the set of all conics passing through these four distinct points is a one-dimensional family, and thus, it is the original pencil of conics.

In general, if a linear system is generated by some hypersurfaces of equations

$$F_1 = 0, \dots, F_k = 0,$$

all hypersurfaces in the linear system contain the set

$$B = \{x \in \mathbf{P}(V) \mid F_1(x) = 0, \dots, F_k(x) = 0\},$$

called the *base locus* of the linear system. In the case of a pencil of conics, the base locus is generally a set of four points called *base points*, and the base locus determines the pencil (there are cases where some of the points are multiple, for example if two of the conics are tangent in one or two points). However, if we consider a net of conics, since it is generated by three conics C_1, C_2, C_3, in general, the base locus is empty (C_1 and C_2 intersect in four points, but C_3 may not contain any of these points).

A useful tool to investigate properties of linear systems is their incidence variety.

Definition 1.11.2 Let V and W be vector spaces, where $\dim(W) = k+1$ and $\dim(V) = n+1$. For any integer $d \geq 1$, given a linear system of hypersurfaces in $\mathbf{P}(V)$

$$L\colon W \to \operatorname{Sym}^d V^*,$$

the *incidence variety* of L is the hypersurface $I \subseteq \mathbf{P}(W) \times \mathbf{P}(V)$ defined by the equation

$$w_0 F_0(x) + \cdots + w_k F_k(x) = 0,$$

where $w_0, \ldots, w_k$ are homogeneous coordinates for points in $\mathbf{P}(W)$. Equivalently,

$$I = \{(w, x) \mid w \in W,\, x \in V,\, L(w)(x) = 0\}.$$

The projective space $\mathbf{P}(W)$ is also denoted as Π, and the projective space $\mathbf{P}(W)$ as P. Then, I is a subvariety of $\Pi \times P$. We also have the two projections $pr_1\colon \Pi \times P \to \Pi$ and $pr_2\colon \Pi \times P \to P$. When restricted to I, these projections yield maps also denoted $pr_1\colon I \to \Pi$ and $pr_2\colon I \to P$. Let us consider some examples.

Example 1: A net of conics. In this case, $\Pi = \mathbb{P}^2$, $P = \mathbb{P}^2$, and $I \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ is defined by a bihomogeneous polynomial of bidegree $(1, 2)$. Since I is a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$, which has dimension 4, we have $\dim(I) = 3$. We can use information on the hypersurface I, such as its Euler characteristic $\chi(I)$, to extract information about the conics in the net. For example, the singular conics in the net defined by I induce a stratification of Π into two strata Δ and Σ, with

$$\Pi \supseteq \Delta \supseteq \Sigma,$$

where Δ consists of pairs of intersecting lines, and Σ of pairs of double lines. However, since pairs of double lines correspond to the image of $\mathbb{P}^2$ under the Veronese embedding in $\mathbb{P}^5$, the variety of pairs of double lines has dimension 2, and thus, codimension 3 in $\mathbb{P}^5$. Since I has dimension 3, this implies that for a general net of conics, $\Sigma = \emptyset$. From now on, we assume that $\Sigma = \emptyset$.

(** Ron, what lemma are we using here? **)

We shall see that Δ is a curve in $\Pi = \mathbb{P}^2$. What is the Euler characteristic of Δ? What is the degree of Δ?

We can use the projections pr_1 and pr_2 together with properties of continuous maps and Euler characteristics. Let us first look at the projection $pr_1\colon I \subseteq \Pi \times \mathbb{P}^2 \to \Pi$. Given any $w \in \Pi = \mathbb{P}^2$, observe that the fiber $pr_1^{-1}(w)$ consists of the subset of $\mathbb{P}^2$ defined by the equation

$$w_0 F_0 + w_1 F_1 + w_2 F_2 = 0,$$

which is just an arbitrary conic in the net (where F_0, F_1, F_3 generate the net of conics).

Let us now look at the projection $pr_2\colon I \subseteq \Pi \times \mathbb{P}^2 \to \mathbb{P}^2$. Given any $p \in \mathbb{P}^2$, observe that the fiber $pr_2^{-1}(p)$ consists of the subset of $\Pi = \mathbb{P}^2$ defined by the equation

$$w_0 F_0(p) + w_1 F_1(p) + w_2 F_2(p) = 0.$$

Thus, unless p is a base point of the net (in which case $F_0(p) = F_1(p) = F_2(p) = 0$), the fiber $pr_2^{-1}(p)$ consists of $\mathbb{P}^1$. When p is a base point, $pr_2^{-1}(p)$ is the entire $\mathbb{P}^2$. Since in general, the base locus of a net is empty, for the rest of the discussion, we assume that this is the case.

Let us now compute the Euler characteristic $\chi(\Delta)$ of Δ by computing $\chi(I)$ in two different ways. For more information on Euler characteristics and the topology of surfaces, we recommend Fulton [10], Massey [23, 24], and Shafarevich [35].

First, we have $\chi(\mathbb{P}^n) = n + 1$. To prove this, observe that

$$\mathbb{P}^n \approx \mathbb{A}^n \cup \mathbb{P}^{n-1},$$

with $\mathbb{P}^0$ a one point set, and use induction.

(** Ron, do we need to say more? **)

Since we assumed that the net has no base points, the map $pr_2 \colon I \subseteq \Pi \times \mathbb{P}^2 \to \mathbb{P}^2$ is such that every fiber is $\mathbb{P}^1$, and thus,

$$\chi(I) = \chi(\mathbb{P}^2)\chi(\mathbb{P}^1) = 3 \cdot 2 = 6.$$

(** Ron, what lemma are we using here? **)

On the other hand, we have

$$\chi(I) = \chi(\Pi - \Delta)\chi(\bigcirc) + \chi(\Delta)\chi(\times),$$

where $\bigcirc$ stands for a round conic (a nonsingular conic) and $\times$ for a singular conic consisting of a pair of intersecting lines. Since a round conic is isomorphic to $\mathbb{P}^1$ (use a projection from a point in $\mathbb{P}^2$ not on the conic, to $\mathbb{P}^1$), we have

$$\chi(\bigcirc) = 2,$$

and

$$\chi(\times) = \chi(\mathbb{P}^1) + \chi(\mathbb{P}^1) - \chi(\text{point}) = 2 + 2 - 1 = 3,$$

we get

$$\chi(I) = 2\chi(\Pi - \Delta) + 3\chi(\Delta) = 2(\chi(\Pi) - \chi(\Delta)) + 3\chi(\Delta) = 2\chi(\Pi) + \chi(\Delta) = 6 + \chi(\Delta).$$

Since $\chi(I) = 6$, we get

$$\chi(\Delta) = 0.$$

Since $\Delta \subseteq \Pi$ is defined by a single equation (equation stating that a conic is singular), $\dim(\Delta) = 1$. Thus, Δ is a curve in $\mathbb{P}^2$. The degree of Δ is equal to the degree of the hypersurface of singular conics of the form $\times$, namely 3, since such a conic is defined by the equation

$$\det(A) = 0$$

where A is the symmetric 3×3 matrix defining the conic. Since $\chi(\Delta) = 0$, topologically, Δ is a torus. The genus g of a torus is $g = 1$, since

$$\chi = 2 - 2g$$

for a compact orientable surface.

Example 2: A pencil of conics. In this case, $\Pi = \mathbb{P}^1$, and $I \subseteq \mathbb{P}^1 \times \mathbb{P}^2$. We have $\dim(I) = 2 = 1 + 2 - 1$. As in the previous example, the fibers of $pr_1\colon I \subseteq \Pi \times \mathbb{P}^2 \to \mathbb{P}^1$ are the conics in the pencil. The subvariety $\Delta \subseteq \Pi$ corresponding to the singular conics in the pencil is a finite set corresponding to the base points of the pencil. In general, Δ consists of three points. Indeed, in general, the pencil of conics is a set B of four base points, and these four points define three pairs of lines, as illustrated in figure 1.3.

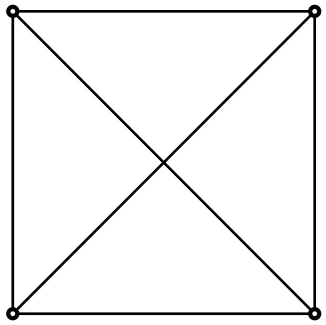

Figure 1.3: Singular conics of a pencil of conics

Since we are assuming that the base points are all distinct, the singular conics are pairs of distinct lines. Then, we can compute $\chi(I)$:

$$\chi(I) = \chi(\mathbb{P}^1 - \{3 \text{ points}\})\chi(\bigcirc) + \chi(\{3 \text{ points}\}) = -1 \cdot 2 + 3 \cdot 3 = 7.$$

For any $p \in \mathbb{P}^2$, the fiber $pr_2^{-1}(p)$ of $pr_2\colon I \subseteq \Pi \times \mathbb{P}^2 \to \mathbb{P}^2$ is either a point when p is not a base point, or $\mathbb{P}^1$ when $p \in B$ (p is a base point). Letting $B = \{p_1, p_2, p_3, p_4\}$ and $L_i = pr_2^{-1}(p_i)$, we can write

$$I = I - (L_1 \cup L_2 \cup L_3 \cup L_4) \cup (L_2 \cup L_2 \cup L_3 \cup L_4),$$

and pr_2 maps $I - (L_1 \cup L_2 \cup L_3 \cup L_4)$ bijectively onto $\mathbb{P}^2 - B$, and $(L_2 \cup L_2 \cup L_3 \cup L_4)$ onto B. Thus, we can compute $\chi(I)$ again:

$$\chi(I) = \chi(I - \bigcup L_i) + 4\chi(L_i) = \chi(\mathbb{P}^2 - B)\chi(\text{point}) + 4\chi(\mathbb{P}^1) = -1 \cdot 1 + 4 \cdot 2 = 7.$$

1.12 Genus of a Plane Curve and Hurwitz's Formula

The following question is obviously of interest: given a nonsingular (irreducible) plane curve $C \subseteq \mathbb{P}^2$, assuming that C is of degree d, compute the genus of C. By genus, we mean the topological genus of the compact surface corresponding to C. It can indeed be shown that a nonsingular irreducible plane curve is a complex 1-dimensional analytical manifold, and thus, a compact real 2-manifold.

** Ron, do you want to say more? **

It turns out that

$$g = \frac{(d-1)(d-2)}{2}.$$

One way to prove this formula is to consider a projection of the curve C from a point $p \in \mathbb{P}^2$ not on C to a line not containing p (a copy of $\mathbb{P}^1$).

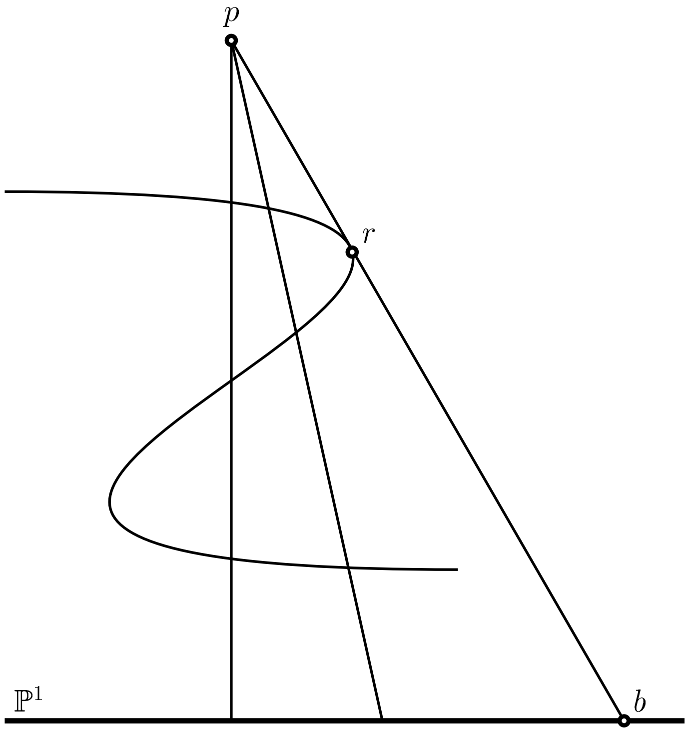

Figure 1.4: Projection of a curve onto a line

In general, a line through p intersects C in d points, except when it is tangent to the curve C. In that case, the point of contact is called a *ramification point*, and the corresponding projection is called a *branch point* (for example, the points r and b in figure 1.4). It would seem that the projection map π has (topological) degree d (which means that every fiber consists of d points), but this is not true because of branch points. Instead, denoting the set of branch points as Branch, we have

$$\chi(C) = d\chi(\mathbb{P}^1 - \text{Branch}) + (d-1)\chi(\text{Branch}) = d\chi(\mathbb{P}^1) - \text{Branch} = 2d - |\text{Branch}|.$$

Thus

$$\chi(C) = 2d - |\text{Branch}|.$$

** Ron, is this known as *Hurwitz's formula*? **

Note that the argument requires that we pick a center of projection p such that branch points have degree exactly $d-1$, but this is indeed possible.

** Ron, is the above OK? **

To compute the number |Branch| of branch points, we observe that |Branch| is the number of tangents to C from a general point. However, this is the degree of the dual curve.

** Ron, justification? Is it obvious?**

The following references are recommended for more information on algebraic curves: Griffith and Harris [13], Griffith [12], Fulton [9], Mumford [27], Hartshorne [16], Miranda [26], and Clemens [4]. Some older books such as Walker [36] (1950) and Coolidge [5] (1931), although a bit old fashion, are still worth consulting. Knapp [19] is really about elliptic curves, but it contains a nice and easy introduction to plane algebraic curves.

Problem 5. Prove the correctness of the genus formula for a nonsingular plane curve (in $\mathbb{P}^2$) of degree d by three methods.

(1) Topologically.

(2) Algebraically.

(3) Geometrically.

For the topological method, compute the Euler characteristic $\chi(C)$ of the curve C using linear systems. Consider the projection $\pi\colon C \to \mathbb{P}^1$ from a well chosen point not on C, and establish a relationship between $\chi(C)$ and the number of ramification points of $C =$ the degree of the dual curve C^*.

Another method is to consider the incidence relation

$$I = \{(p,q) \mid p \in C,\, q \in T_pC\}.$$

Yet another way is to consider the correspondence $\mathcal{I} \subseteq C \times \mathbb{P}^{2^*}$ defined such that

$$\mathcal{I} = \{(p,l) \mid p \in C \cap l\}.$$

For the algebraic method, use resultants, as explained next. Given any two polynomials $f, g \in \mathbb{C}[x]$, there is a polynomial $R(f,g)$ in the coefficients of f and g such that f and g have a common root (in $\mathbb{C}$) iff $R(f,g) = 0$. Now, given $f, g \in \mathbb{C}[x,y]$, we can compute $R_x(f,g)$, by eliminating x from $f(x,y)$ and $g(x,y)$, and we obtain a polynomial in $\mathbb{C}[y]$. Then, for every $y \in \mathbb{C}$, $R_x(f,g) = 0$ gives us the finite set of $x \in \mathbb{C}$ such that (x,y) is on both curves $f(x,y) = 0$ and $g(x,y) = 0$, i.e., the intersection points of these two curves having the same y-coordinate.

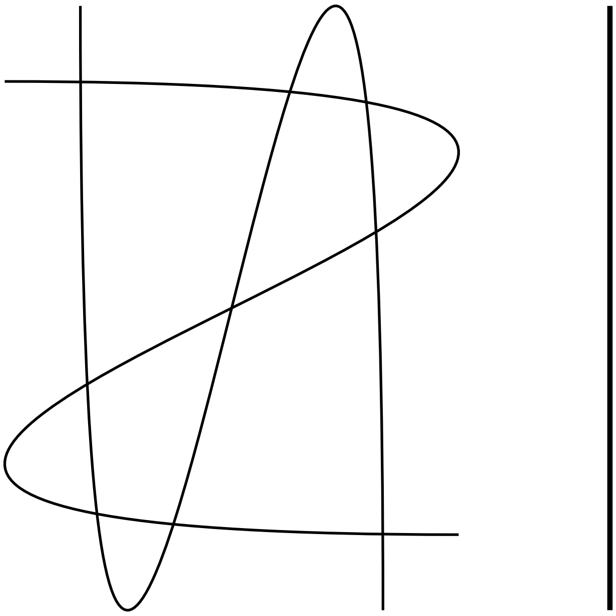

Figure 1.5: Common zeros of $f(x, y)$ and $g(x, y)$

We can also form the resultant $R(f, f')$ of a polynomial $f \in \mathbb{C}[x]$ and its derivative. This is called the *discriminant* of f, and it is denoted as $Disc(f)$. The discriminant has the property that $Disc(f) = 0$ iff f has a double root. Similarly, we can compute $Disc_x(f)$where $f \in \mathbb{C}[x, y]$. The pairs (x, y) for which $Disc_x(f) = 0$ correspond to points on the curve $f(x, y) = 0$ where the curve has a self-intersection or where the tangent to the curve is parallel to the y-axis. Use the above to find the degree μ of the dual curve (called the *class* of the curve), by taking a pencil of lines through an appropriate point at infinity.

For the geometric proof, use line bundles and the so-called adjunction formula. Recall that a line bundle is a vector bundle of rank 1, i.e., the fibers are one-dimensional subspaces. For example, if X is a nonsingular variety, T_X is a vector bundle. The dual T_X^* of T_X is called the *cotangent bundle*. The cotangent bundle is also denoted as Ω^1. We let $\omega = \det(\Omega^1)$ be the *canonical bundle*. When X is a smooth curve,

$$\omega = \Omega^1 = T^*.$$

A line bundle on a curve has a degree $\deg(\omega)$ defined as the number of points of X at which a holomorphic 1-form vanishes. It turns out that

$$\chi(X) = \deg(T) = -\deg(T^*) = -\deg(\omega).$$

Prove that if Y is a nonsingular projective variety and if $X \subseteq Y$ is a nonsingular hypersurface, then

$$\omega_X = \omega_{Y|X} \otimes N_{Y/X}.$$

In the above *adjunction formula*, $N_{Y/X}$ is the *normal bundle*. It is the quotient $T_{Y|X}/T_X$ appearing in the exact sequence

$$0 \longrightarrow T_X \longrightarrow T_{Y|X} \longrightarrow N_{Y/X} \longrightarrow 0.$$

** Ron, I hope this is OK, I don't know what I'm talking about!!! **

1.13 Resultants

We now give an explicit formula for the resultant of two polynomials. Actually, since we also need resultants of polynomials f, g in several variables, we have to be a little careful. Indeed, we can view a polynomial $f \in \mathbb{C}[x, y]$ as a polynomial in y with coefficient in $\mathbb{C}[x]$, but $\mathbb{C}[x]$ is not a field. Fortunately, $\mathbb{C}[x]$ is a UFD (unique factorization domain), and that is exactly what is needed for the proof. Therefore, following Griffith [12], we state the main results about resultants for a UFD D. Given two polynomials $f, g \in D[x]$, where f has degree $m \geq 1$ and g has degree $n \geq 1$, letting

$$f = f_0 x^m + f_1 x^{m-1} + \cdots + f_m \quad \text{and} \quad g = g_0 x^n + g_1 x^{n-1} + \cdots + g_n,$$

(with $f_0 \neq 0$ and $g_0 \neq 0$), we first prove a simple Lemma characterizing when f and g have a common nontrivial factor (by nontrivial, we mean a factor of degree at least 1).

Lemma 1.13.1 *Given a UFD D, two polynomials $f, g \in D[x]$ with $\deg(f) = m \geq 1$ and $\deg(g) = n \geq 1$ have a common nontrivial factor iff there exist polynomials $p, q \in D[x]$ such that*

$$fp = gq,$$

where $\deg(p) \leq n - 1$ and $\deg(q) \leq m - 1$.

Proof. If f and g have a common nontrivial factor h, then we have $f = hf_1$, $g = hg_1$, with $\deg(f_1) \leq m - 1$ and $\deg(g_1) \leq n - 1$. Clearly,

$$fg_1 = gf_1 = f_1 g_1 h.$$

Conversely, assume that

$$fp = gq,$$

where $\deg(p) \leq n - 1$ and $\deg(q) \leq m - 1$. Since D is a UFD, by Gauss's Theorem, so is $D[x]$. Since $\deg(f) \geq 1$, we can write

$$f = \alpha f_1^{n_1} \cdots f_k^{n_k},$$

where $\alpha \in D$ and the f_i are irreducible polynomials. By the unique factorization property, each $f_i^{n_i}$ is one of the factors of gq, and since $\deg(q) \leq m - 1$ and $\deg(f) = m$, one of the f_i's must be a factor of g. $\square$

We can use Lemma 1.13.1 to give the following explicit criterion stating when two polynomials share a common nontrivial factor. This criterion uses a quantity known as the resultant of two polynomials.

Lemma 1.13.2 *Given a UFD D, two polynomials $f, g \in D[x]$ with $\deg(f) = m \geq 1$ and $\deg(g) = n \geq 1$ have a common nontrivial factor iff the following determinant $R(f, g)$ known as the resultant of f and g is null:*

$$R(f,g) = \begin{vmatrix} f_0 & f_1 & \cdots & \cdots & f_m & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & f_0 & f_1 & \cdots & \cdots & f_m & 0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & f_0 & f_1 & \cdots & \cdots & f_m \\ g_0 & g_1 & \cdots & \cdots & \cdots & \cdots & \cdots & g_n & 0 & \cdots & 0 \\ 0 & g_0 & g_1 & \cdots & \cdots & \cdots & \cdots & \cdots & g_n & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & g_0 & g_1 & \cdots & \cdots & \cdots & \cdots & \cdots & g_n \end{vmatrix} = 0.$$

Proof. Let V_n denote the vector space of polynomials of degree at most $n-1$. The map $T\colon V_n \oplus V_m \to V_{m+n}$ given by

$$T(p, q) = fp - gq$$

is a linear map. By Lemma 1.13.2, f and g have a common nontrivial factor iff the linear map T is not injective. Since D is a UFD, the linear map T is not injective iff $\det(T) = 0$. Picking the basis

$$x^{n-1}, x^{n-2}, \ldots, 1$$

for V_n and similarly for V_m and V_{m+n}, the matrix of the linear map T is the transpose of the underlying matrix of the determinant $R(f, g)$, and thus, f and g have a common root iff $R(f, g) = 0$. $\square$

Remarks:

(i) According to Eisenbud [6], the notion of resultant is due to Euler, and the expression for the resultant given in Lemma 1.13.2 is due to Sylvester (1840) (see Eisenbud [6], Chapter 14, Section 14.1). However, Sylvester simply asserted Lemma 1.13.2, and a proof was later given by Cauchy.

(ii) Note that the resultant is homogeneous of degree m in the f_i's and homogeneous of degree n in the g_j's.

The following Lemma is quite useful:

Lemma 1.13.3 *Given a UFD D, for any two polynomials $f, g \in D[x]$ with $\deg(f) = m \geq 1$ and $\deg(g) = n \geq 1$, there exist polynomials A and B with $\deg(A) \leq n-1$ and $\deg(B) \leq m-1$ such that the resultant R of f and g is given by*

$$R = Af + Bg.$$

Proof. Multiplying

$$f = f_0x^m + f_1x^{m-1} + \cdots + f_m$$

by x^i for $1 \leq i \leq n-1$ and multiplying

$$g = g_0x^n + g_1x^{n-1} + \cdots + g_n$$

by x^j for $1 \leq j \leq m-1$, we get

$$\begin{aligned}
f &= f_0x^m + f_1x^{m-1} + \cdots + f_m \\
xf &= \quad f_0x^{m+1} + f_1x^m + \cdots + f_mx \\
&\cdots\cdots\cdots\cdots \\
x^{n-1}f &= \qquad f_0x^{m+n-1} + f_1x^{m+n-2} + \cdots + f_mx^{n-1} \\
g &= g_0x^n + g_1x^{n-1} + \cdots + g_n \\
xg &= \quad g_0x^{n+1} + g_1x^n + \cdots + g_nx \\
&\cdots\cdots\cdots\cdots \\
x^{m-1}g &= \qquad g_0x^{m+n-1} + g_1x^{m+n-2} + \cdots + g_nx^{m-1}
\end{aligned}$$

Letting $A_1, \ldots, A_{m+n}$ be the cofactors of the first column of R, if we multiply the ith equation by A_i and add up all the equations for $i = 1, \ldots, m+n$, we get

$$(A_1 + A_2x + \cdots + A_nx^{n-1})f + (A_{n+1} + A_{n+2}x + \cdots + A_{m+n}x^{m-1})g = R.$$

Letting $A = (A_1 + A_2x + \cdots + A_nx^{n-1})$ and $B = (A_{n+1} + A_{n+2}x + \cdots + A_{m+n}x^{m-1})$, the result follows. $\square$

For more on resultants and elimination theory, see Lang [20], Chapter IX, Sections 3,4, pages 389-412.

Let us give a few simple examples. If $m = n = 1$, we can write $f = ax + b$ and $g = cx + d$, and

$$R(f, g) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The condition $ad - bc = 0$ does assert that f and g have the same root.

If $m = 1$ and n is arbitrary, by expanding the resultant according to its last row, we get

$$R(f, g) = f_0^n g_n - f_0^{n-1} f_1 g_{n-1} + \cdots + (-1)^n f_1^n g_0 = f_0^n g\left(-\frac{f_1}{f_0}\right).$$

Since

$$f = f_0x + f_1 \quad \text{and} \quad g = g_0x^n + g_1x^{n-1} + \cdots + g_n,$$

the condition $R(f, g) = 0$ says that $-f_1/f_0$ is a root of g, as desired!

Now, for an example of a discriminant, if $f = ax^2 + bx + c$, since $f' = 2ax + b$, we get

$$D = \begin{vmatrix} a & b & c \\ 2a & b & 0 \\ 0 & 2a & b \end{vmatrix} = a(4ac - b^2).$$

Thus, $ax^2 + bx + c = 0$ has a double root iff $b^2 - 4ac = 0$, a well known condition!

1.14 Morphisms Of Varieties And Blowing-Up

As we saw in Section 1.4, given a topological space X, a manifold structure on X is given by a sheaf of rings. For example, in the case of continuous manifolds, it is the sheaf of algebras of continuous functions. In the case of smooth manifolds, it is the algebra of C^∞-functions, etc. Also, a morphism f of ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a continuous map between the underlying topological spaces that "pulls back" the sheaf of functions on Y to the sheaf of functions on X.

The definition of a map between ringed spaces can be applied to algebraic varieties. Given a projective variety $X \subseteq \mathbb{P}^n$, let us first review the definition of a regular function.

Definition 1.14.1 Given a projective variety $X \subseteq \mathbb{P}^n$, for every point $p \in X$, we say that a function $f\colon X \to \mathbb{C}$ is *regular at p* if there is an open subset $U \subseteq X$ with $p \in U$, and some homogeneous polynomials $G, H \in \mathbb{C}[x_0, \ldots, x_n]$ of the same degree, such that H is nowhere null on U and

$$f = G/H \quad \text{on} \quad U.$$

We say that *f is regular* if it is regular for all $p \in X$. The *sheaf $\mathcal{O}_X$ of regular functions on X* is defined such that

$$\mathcal{O}_X(U) = \{f \mid \quad f \text{ is regular on } U\}$$

for every open subset $U \subseteq X$.

Obviously, Definition 1.14.1 also applies to a function $f\colon U \to \mathbb{C}$, where $U \subseteq X$ is any nonempty open subset of X. It is immediately checked that a regular function is continuous. Furthermore, if X is irreducible, then every nonempty open subset $U \subseteq X$ is dense in X, and if f, g are two regular functions such that $f = g$ on U, then $f = g$ (see Hartshorne [16], Chapter I, Section 3). We define a morphism of algebraic varieties as follows.

Definition 1.14.2 Given two algebraic projective varieties X, Y, a continuous map $f\colon X \to Y$ is a *morphism of varieties* if for every open subset $V \subseteq Y$,

$$\text{if} \quad h \in \mathcal{O}_Y(V) \quad \text{then} \quad h \circ f \in \mathcal{O}_X(f^{-1}(V)),$$

where $\mathcal{O}_X$ and $\mathcal{O}_Y$ are the sheaves of regular functions on X and Y.

Let us give some examples of morphisms of varieties. If $X = Y = \mathbb{P}^1$, we can choose some affine coordinate $z = z_1/z_0$ on X and some affine coordinate $w = w_1/w_0$ on Y (where $(z_0, z_))$ and (w_0, w_1) are homogeneous coordinates on X and Y), and we can specify a morphism $f \colon X \to Y$ by a function

$$z \mapsto w = f(z).$$

For example, the map

$$w = z$$

corresponds to the identity. The map

$$w = z^2$$

is a morphism that "covers" $\mathbb{P}^1$ twice. The map

$$w = \frac{1}{z}$$

is a morphism, although this is more easily seen using homogeneous coordinates:

$$\begin{aligned} w_1 &= z_0, \\ w_0 &= z_1. \end{aligned}$$

On the other hand, the maps

$$z \mapsto \sqrt{z}$$

and

$$z \mapsto |z|$$

are not morphisms.

Let us now assume that $X = \mathbb{P}^1$ with homogeneous coordinates (x_0, x_1) and $Y = \mathbb{P}^2$ with homogeneous coordinates (x, y, z). The following are morphisms:

$$\begin{aligned} x &= x_0, \\ y &= x_1, \\ z &= 0, \end{aligned}$$

an injection of $\mathbb{P}^1$ into $\mathbb{P}^2$,

$$\begin{aligned} x &= x_0^2, \\ y &= x_1^2, \\ z &= 0, \end{aligned}$$

and

$$\begin{aligned} x &= x_0^2, \\ y &= x_1^2, \\ z &= x_0 x_1. \end{aligned}$$

The map

$$(x_0, x_1) \mapsto (x_0^3,\, x_0x_1^2,\, x_1^3)$$

defines a curve in $\mathbb{P}^2$. Indeed, the image of the map is the null set of the equation

$$y^3 = xz^2.$$

In the affine patch $x \neq 0$, setting $x = 1$, we have the *cuspidal cubic* of equation

$$y^3 = x^2.$$

In the affine patch $y \neq 0$, we have the cubic of equation

$$xz^2 = 1,$$

and in the affine patch $z \neq 0$, we have the cubic of equation

$$y^3 = x.$$

The above map is injective, since

$$\frac{x_1}{x_0} = \frac{z}{y}.$$

Thus, there is a bijection between $\mathbb{P}^1$ and the cuspidal cubic, and as consequence, the surface corresponding to the cuspidal cubic $y^3 = xz^2$ has Euler characteristic 2. It is a sphere with a pointy end!

The map

$$(x_0, x_1) \mapsto ((x_1^2 - x_0^2)x_0,\, (x_1^2 - x_0^2)x_1,\, x_0^3)$$

defines another cubic in $\mathbb{P}^2$. Indeed, its image satisfies the equation

$$y^2z = x^2(x + z).$$

In the affine patch $z \neq 0$, setting $z = 1$, we get

$$y^2 = x^2(x + 1).$$

Letting $u = y/x$, we have $y = ux$, and we get

$$u^2x^2 = x^2(x + 1),$$

from which we get the parameterization

$$\begin{aligned} x &= u^2 - 1, \\ y &= u(u^2 - 1). \end{aligned}$$

This is a *nodal cubic*, with a double point (node) at the origin. This time, the map is not injective. Indeed, from

$$y^2z = x^2(x + z),$$

we get

$$\frac{x_1}{x_0} = \frac{y}{x},$$

except at $(0, 0, 1)$, where there are two possible values

$$\frac{x_1}{x_0} = \pm 1.$$

Thus, the surface corresponding to the nodal cubic is a sphere with two points pinched together. As a consequence, the Euler characteristic of the nodal cubic is 1 (since the Euler characteristic of a sphere is 2, minus 2 points deleted, plus one point added).

Sometimes, the above method does not yield a morphism. Sometimes, the map can be extended, but sometimes it can't. For example, the map

$$(x_0, x_1) \mapsto (x_0^3 x_1,\, x_0^4,\, x_0^2 x_1^2)$$

yields $(0, 0, 0)$ at $(1, 0)$, but $(0, 0, 0)$ does not correspond to any point of the projective space $\mathbb{P}^2$. If we observe that the homogenous coordinates

$$(x_0^3 x_1,\, x_0^4,\, x_0^2 x_1^2) \quad \text{and} \quad (x_0 x_1,\, x_0^2,\, x_1^2)$$

are equivalent, i.e., define the same point, we can extend the above map at $(1, 0)$ by assigning the value $(0, 1, 0)$.

On the other hand, the map from $\mathbb{P}^2$ to $\mathbb{P}^1$ given by

$$(x, y, z) \mapsto (x, y)$$

is undefined at $(0, 0, 1)$. This map is a projection from $\mathbb{P}^2 - \{(0, 0, 1)\}$ to $\mathbb{P}^1$. It has no continuous extension to all of $\mathbb{P}^2$, because the limit of this map when p tends to $(0, 0, 1)$ in $\mathbb{P}^2$ depends on the direction in which $(0, 0, 1)$ is approached.

As a preparation for the discussion of the "blowing-up" process, we define the *graph* of a morphism $f\colon X \to Y$. The graph of f is the subvariety $\Gamma_f \subseteq X \times Y$ defined by the following equations: for every function y defined on an open subset V of Y, the function $f^* y$ is a function x defined on $f^{-1}(V) \subseteq X$, and for each such y, we have the equation

$$\pi_1^* x - \pi_2^* y = 0,$$

where $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ are the projection maps. We also need to define rational maps.

Definition 1.14.3 Given two algebraic projective varieties X, Y, we define an equivalence relation on pairs (U, f) and (V, g), where $U, V \subseteq X$ are nonempty open subsets of X, f is regular on U, and g is regular on V, by requiring that f and g agree on $U \cap V$. A *rational map* from X to Y is any equivalence class of pairs (U, f) as above. A rational map φ from X to Y is denoted as $\varphi\colon X \dashrightarrow Y$.

Obviously, we say that two rational maps $\varphi\colon X \dashrightarrow Y$ and $\psi\colon X \dashrightarrow Y$ are equal iff they are equivalent. A rational map $\varphi\colon X \dashrightarrow Y$ is defined by any of its representatives (U, f):

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & Y \\ \uparrow & \nearrow_f & \\ U & & \end{array}$$

The graph of a rational map f is the closure of the graph of any of its representatives (U, f_0). For example, going back to the example of the projection $\pi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ given by

$$(x, y, z) \mapsto (x, y),$$

letting $U = \mathbb{P}^2 - \{(0,0,1)\}$, we see that π is a rational map defined by $(U, \pi|U)$. The graph $\Gamma_{\pi|U}$ of $\pi|U$ is isomorphic to U. However, the closure Γ_π of $\Gamma_{\pi|U}$ is

$$\Gamma_\pi = \Gamma_{\pi|U} \coprod \{(0,0,1)\} \times \mathbb{P}^1.$$

We also denote Γ_π as $\widetilde{\mathbb{P}^2}$, or as $\mathrm{Bl}_{(0,0,1)}\mathbb{P}^2$, and call it the *blowing-up of* $\mathbb{P}^2$ *at the point* $(0,0,1)$.

Geometrically, we can view $\mathrm{Bl}_{(0,0,1)}\mathbb{P}^2$ as a subset of $\mathbb{P}^2 \times \mathbb{P}^1$ consisting of U and a copy of $\mathbb{P}^1$ standing above the point being blown up. This copy of $\mathbb{P}^1$ is called the *exceptional curve, or exceptional divisor*.

If instead of blowing up the origin in $\mathbb{P}^2$ we blow up the origin in $\mathbb{A}^2$, we use the map $\pi\colon \mathbb{A}^2 \to \mathbb{P}^1$ given by

$$(x, y) \mapsto (x, y),$$

where (x, y) stands both for the coordinates of a point in $\mathbb{A}^2 - \{(0,0)\}$ and for the homogeneous coordinates of a point in $\mathbb{P}^1$. Then, $\mathrm{Bl}_{(0,0)}\mathbb{P}^2$ is a subset of $\mathbb{A}^2 \times \mathbb{P}^1$ consisting of $U = \mathbb{A}^2 - \{(0,0)\}$ and a copy of $\mathbb{P}^1$. However, $\mathbb{P}^1$ is in bijection with the set of lines through the origin in $\mathbb{A}^2$, that is, the set of directions in $\mathbb{A}^2$. Thus, we can view $\mathbb{P}^1$ as a set of lines parallel to U meeting the vertical line above the origin and forming a helicoid in $\mathbb{A}^3$. The resulting picture for $\mathrm{Bl}_{(0,0)}\mathbb{A}^2$ is quite pretty. You can see it on the cover of Shafarevich [34], and there is a similar picture in Hartshorne [16], Chapter I, Section 4.

More generally, given a variety Y and a subvariety $X \subseteq Y$, we would like to define $\widetilde{Y} = \mathrm{Bl}_X Y$, the blowing-up of Y at the subvariety X, so that the following properties hold:

$$\begin{array}{ccc} \widetilde{X} & \subseteq & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \subseteq & Y \end{array}$$

(1) $\widetilde{Y} - \widetilde{X}$ is isomorphic to $Y - X$.

(2) $\widetilde{X}$ has codimension 1 in $\widetilde{Y}$.

For example, consider the case where $Y = \mathbb{A}^n$ and $X = \{O\}$, the origin in $\mathbb{A}^n$. We use the projection map $\pi\colon \mathbb{A}^n \dashrightarrow \mathbb{P}^{n-1}$, and as in the case $n = 2$, we define the blowing-up $\widetilde{Y} = \mathrm{Bl}_O Y$ of Y at O as the graph of the rational map $\pi\colon \mathbb{A}^n \dashrightarrow \mathbb{P}^{n-1}$. The blowing-up of the origin O is a copy of $\mathbb{P}^{n-1}$ called the exceptional divisor, and also denoted as E. The exceptional divisor is a subvariety of codimension 1:

$$\begin{array}{ccccc} \widetilde{Y} - E & \subseteq & \widetilde{Y} & \supseteq & E \approx \mathbb{P}^{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ Y - \{O\} & \subseteq & Y & \supseteq & O \end{array}$$

If $\varphi\colon \widetilde{Y} \to \mathbb{A}^n$ is the restriction of the first projection $\pi_1\colon \mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$, then $E = \varphi^{-1}(O)$, and there is an isomorphism from $\widetilde{Y} - E$ to $Y - \{O\}$.

Let us check that $\widetilde{Y}$ is indeed an algebraic variety, in the case $n = 3$. We need to cover $\widetilde{Y}$ by affine charts, and find the equations defining $\widetilde{Y}$ as a subvariety of $\mathbb{A}^3 \times \mathbb{P}^2$. Let (x, y, z) be coordinates on $\mathbb{A}^3$ and (U, V, W) be homogeneous coordinates on $\mathbb{P}^2$. The projection $\pi\colon \mathbb{A}^3 \dashrightarrow \mathbb{P}^2$ is a rational map such that

$$(x, y, z) \mapsto (U, V, W),$$

which means that $(U, V, W) = \lambda(x, y, z)$ for some $\lambda \neq 0$. Thus, the equations are those stating that $(U, V, W) = \lambda(x, y, z)$ for some $\lambda \neq 0$, i.e.,

$$\begin{aligned} xV &= yU, \\ xW &= zU, \\ yW &= zV. \end{aligned}$$

We have three charts corresponding to $x \neq 0$, $y \neq 0$, and $z \neq 0$. Consider the chart $z \neq 0$. Then, $W \neq 0$. On this chart, we have well defined functions

$$x,\ y,\ z,\ u = \frac{U}{W},\ v = \frac{V}{W}.$$

Since we are assuming that $z \neq 0$, we can express x and y in terms of z, u, v, since the equations

$$\begin{aligned} xV &= yU, \\ xW &= zU, \\ yW &= zV, \end{aligned}$$

yield

$$\begin{aligned} xv &= yu, \\ x &= zu, \\ y &= zv \end{aligned}$$

after division by W (since $z \neq 0$, we also have $W \neq 0$). On this chart, the equation $xv = yu$ is a consequence of the other two equations $x = zu$ and $y = zu$ (since $z \neq 0$). Therefore, (z, u, v) are natural coordinates on this patch, and this affine patch is isomorphic to $(\mathbb{A} - \{0\}) \times \mathbb{A}^2)$, which is isomorphic to $\mathbb{A}^3 - \{O\}$. Since a similar argument applies to the other two patches, $\widetilde{\mathbb{A}^3}$ can be covered by three affine patches, each isomorphic to $\mathbb{A}^3 - \{O\}$. Observe that the exceptional divisor $E = \mathbb{P}^2$ corresponds to $z = 0$, or equivalently $x = 0$, or $y = 0$. Indeed, $E = \varphi^{-1}(O)$. The previous argument showed that $\widetilde{\mathbb{A}^3} - \varphi^{-1}(O)$ is isomorphic to $\mathbb{A}^3 - \{O\}$. For more more on blowing-up, consult Hartshorne [16], Chapter I, Section 4, and in particular, Example 4.9.1.

In order to consider a more general blowing-up process, we need to take a closer look at tangent spaces.

1.15 Zariski Tangent Spaces and Tangent Cones

The problem is to define the tangent space at at singular point of a variety. For example, consider the affine nodal cubic X of equation

$$y^2 = x^2(x + 1).$$

At the origin $p = (0, 0)$, the curve has a self-intersection. Should the tangent space T_pX at p be the entire affine plane $\mathbb{A}^2$, or should it be the set of the two tangent lines defined by the equation $y^2 - x^2 = 0$?

Both approaches have been explored. The first approach is to define the so-called *Zariski tangent space*. Given an algebraic variety X, possibly singular, rather than giving a formal definition of the Zariski tangent space T_pX to X at $p \in X$, we state some properties that actually characterize it uniquely.

(1) T_pX is a vector space.

(2) $\dim(T_pX) \geq \dim(X)$.

(3) $\dim(T_pX) = \dim(X)$ iff p is nonsingular.

(4) If $X \subseteq Y$ is a subvariety of Y with $p \in X$, and X is defined by the equations

$$f_1 = 0, \ldots, f_k = 0,$$

then $T_pX \subseteq T_pY$ is the subspace defined by the linear equations

$$df_1(p) = 0, \ldots, df_k(p) = 0.$$

The second approach is to define the *tangent cone $\mathcal{C}_pX$ to X at p.* This time, assuming that Y is a nonsingular variety and that $X \subseteq Y$ is a subvariety, possibly singular, if X is defined by the equations

$$f_1 = 0, \ldots, f_k = 0,$$

then the tangent cone $\mathcal{C}_pX$ is defined by the equations

$$f_1^{d_1} = 0, \ldots, f_k^{d_k} = 0,$$

where $f_i^{d_i}$ is the leading term (nonnull term of lowest degree) in the Taylor expansion of f_i at p.

In general,

$$\mathcal{C}_pX \subseteq T_pX \subseteq T_pY.$$

This is because the equations of T_pX are the linear equations among the equations of $\mathcal{C}_pX$. It turns out that T_pX is the span of $\mathcal{C}_pX$, in the sense of schemes (see later).

Let us give some examples.

Example 1. Let $Y = \mathbb{A}^2$, $X = \{p\}$, and $p = (0,0)$. The equations of X are

$$x = 0,\ y = 0.$$

The Taylor expansions of x and y at p have leading terms x and y, and thus, $T_pX = \mathcal{C}_pX = \{p\}$. Equivalently, T_pX is defined by

$$dx = dy = 0.$$

Example 2. Let $Y = \mathbb{A}^2$, $X = \{(x,y) \mid x^2 = y\}$, and $p = (0,0)$. The curve X is a parabola. We have

$$f_1 = y - x^2.$$

The leading term in the Taylor expansion is y, and $T_pX = \mathcal{C}_pX$ are both the line of equation $y = 0$. Equivalently, T_pX is defined by

$$dy = 0.$$

Example 3. Let $Y = \mathbb{A}^2$, $X = \{(x,y) \mid y^2 = x(x+1)\}$, and $p = (0,0)$. The curve X is a nodal cubic. We have

$$f_1 = y^2 - x^2 - x^3,$$

and the leading term in the Taylor expansion is $y^2 - x^2$. Since there are no linear terms, $T_pX = T_p\mathbb{A}^2$, which can be identified with $\mathbb{A}^2$. However, the equation of $\mathcal{C}_pX$ is

$$y^2 = x^2 = 0,$$

and $\mathcal{C}_pX$ consists of the two lines $y = x$ and $y = -x$. These are indeed the two tangents at the node.

Example 4. Let $Y = \mathbb{A}^2$, $X = \{(x, y) \mid y^2 = x^3\}$, and $p = (0, 0)$. The curve X is a cuspidal cubic. We have

$$f_1 = y^2 - x^3,$$

and the leading term in the Taylor expansion is y^2. Since there are no linear terms, $T_pX = T_p\mathbb{A}^2$, which can be identified with $\mathbb{A}^2$. However, the equation of $\mathcal{C}_pX$ is

$$y^2 = 0,$$

and $\mathcal{C}_pX$ consists of the "double line" of equation $y = 0$.

One should observe that the above definitions depend on the equations defining X. This is not quite right. One needs to pick "enough" generators of the ideal associated with X. We will do this in more detail later on.

We now return to blowing-up. Assume that X, Y are varieties, with $X \subseteq Y \subseteq \mathbb{A}^n$, and let $p \in X$. We define the *proper transform* $\widetilde{X}$ *of* X *in* $\widetilde{Y} = Bl_pY$ as the closure of $\varphi^{-1}(X - \{p\})$, where $\varphi\colon \widetilde{Y} \to \mathbb{A}^n$ is the restriction of the first projection $\pi_1\colon \mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$.

Question: What is $\widetilde{X}$?

$$\begin{array}{ccccccc}
\widetilde{X} & \subseteq & \widetilde{Y} - E & \subseteq & \widetilde{Y} & \supseteq & E \approx \mathbb{P}^{n-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X - \{p\} & \subseteq & Y - \{p\} & \subseteq & Y & \supseteq & p
\end{array}$$

The answer is that $\widetilde{X}$ is the projectivization $\mathbf{P}(\mathcal{C}_pX)$ of the cone $\mathcal{C}_pX$.

Remark: Given an open subset $U \subseteq X \subseteq \mathbb{A}^n$ of a variety X, for any point $p \in U$, we can

(1) blow up X at p and restrict to U;

(2) blow up U at p.

It can be shown that (1) and (2) are equivalent.

We observed that when $p \in X$ is a nonsingular point, then

$$\widetilde{X} = (X - \{p\}) \coprod \mathbb{P}^{n-1}.$$

Let us now see what is the result of blowing-up a singular point on a curve.

Example 5. Let X be the nodal cubic of equation $y^2 = x^2(x + 1)$, and let $p = (0, 0)$. By definition,

$$\widetilde{\mathbb{A}^2} \subseteq \mathbb{A}^2 \times \mathbb{P}^1.$$

Let (x, y) be coordinates on $\mathbb{A}^2$ and (u, v) be homogeneous coordinates on $\mathbb{P}^1$. We can cover $\widetilde{\mathbb{A}^2}$ by the two affine open subsets U_1, U_2 of $\mathbb{A}^2 \times \mathbb{P}^1$ corresponding to $u \neq 0$ and $v \neq 0$. The projective line $\mathbb{P}^1$ is also covered by the two affine open sets U_1^0 and U_2^0 corresponding to $u \neq 0$ and $v \neq 0$. Thus, we have

$$\mathbb{A}^2 \times \mathbb{P}^1 = (\mathbb{A}^2 \times U_1^0) \cup (\mathbb{A}^2 \times U_2^0).$$

We can use the coordinates $x, y, v/u$, on $\mathbb{A}^2 \times U_1^0$, and $x, y, u/v$, on $\mathbb{A}^2 \times U_2^0$. The open set U_1 is defined inside $\mathbb{A}^2 \times U_1^0$ by the equation

$$y = x\frac{v}{u},$$

and the open set U_2 is defined inside $\mathbb{A}^2 \times U_2^0$ by the equation

$$x = y\frac{v}{u}.$$

Letting $z = v/u$ and $w = u/v$, we can use the coordinates x, z in U_1 and the coordinates y, w in U_2.

In U_1, where $y = xz$, the equation

$$y^2 = x^2 + x^3$$

becomes

$$x^2(z^2 - x - 1) = 0.$$

In U_2, where $x = yw$, the equation

$$y^2 = x^2 + x^3$$

becomes

$$y^2(w^2 + yw^3 - 1) = 0.$$

The curves $x = 0$ and $y = 0$ are the traces in U_1 and U_2 of the exceptional curve. Actually, the exceptional curve has multiplcity 2, which is the multiplicity of the singular point. The curve $x = z^2 - 1$ is a parabola. It intersects the exceptional curve in two points. The other curve of equation

$$y = \frac{1 - w^3}{w^2}$$

also intersects the exceptional curve in two points. We see that the singular point is blown up into two points. This is indeed the projectivization of the two tangent lines at the singular point.

Example 6. Let X be the cuspidal cubic of equation $y^2 = x^3$ and let $p = (0, 0)$. We can use the same affine open sets U_1 and U_2. In U_1, where $y = xz$, the equation

$$y^2 = x^3$$

becomes

$$x^2(z^2 - x) = 0.$$

In U_2, where $x = yw$, the equation

$$y^2 = x^3$$

becomes

$$y^2(yw^3 - 1) = 0.$$

The curve $x = z^2$ is the parabola tangent to the exceptional curve $x = 0$. Again, the exceptional curve has multiplicity 2, but this time, the singular point is blown up into a double point.

Example 7. Let X be the cubic of equation $y^3 = x^3$ and let $p = (0, 0)$. This curve corresponds to three lines through p. The triple point p gets blown up into three points meeting the exceptional divisor (line). Indeed, in U_1, where $y = xz$, the equation

$$y^3 = x^3$$

becomes

$$x^3(z^3 - 1) = 0,$$

and in U_2, where $x = yw$, the equation

$$y^3 = x^3$$

becomes

$$y^3(w^3 - 1) = 0.$$

In each case, we get the exceptional divisor (of multiplicity 3) and three parallel lines meeting it.

Example 8. Let X be the quartic of equation $y^2 = x^4$ and let $p = (0, 0)$. In U_1, where $y = xz$, the equation

$$y^2 = x^4$$

becomes

$$x^2(z^2 - x^2) = 0,$$

and in U_2, where $x = yw$, the equation

$$y^2 = x^4$$

becomes

$$y^2(1 - y^2w^4) = 0.$$

In both cases, the curves $z^2 - x^2 = 0$ and $y^2w^4 - 1 = 0$ are singular at p. In order to resolve the singularity, we need to blow up one more time.

Generally, given a plane curve, its singularities can be removed by performing a finite number of blowing-up's. For a proof, among other sources, see Fulton [9].

Observe that the resolution of a singularity (on a plane curve) can be viewed as the inverse of a projection of a space curve from a point on the space curve to a suitable plane. For example, the nodal cubic and the cuspidal cubic can be obtained from a suitable projection of the twisted cubic.

So far, we only have considered examples of blowing-up involving a curve. We can also blow up a point on a surface. For example, we can blow up the vertex of the cone

$$z^2 = x^2 + y^2.$$

What happens is that we get the curve (in $\mathbb{P}^2$) of equation $z^2 = x^2 + y^2$ as exceptional divisor. More generally, given a variety X and a subvariety $Y \subseteq X \subseteq \mathbb{A}^m$, we would like to define $\widetilde{X} = \mathrm{Bl}_Y X$, the blowing-up of X at the subvariety Y, so that

$$\begin{array}{ccccc}
\widetilde{X} - \widetilde{Y} & \subseteq & \widetilde{X} & \supseteq & \widetilde{Y} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
X - Y & \subseteq & Y & \supseteq & Y
\end{array}$$

where $\widetilde{X} - \widetilde{Y}$ is isomorphic to $X - Y$, and $\widetilde{Y}$ has codimension 1 in $\widetilde{X}$.

We can reduce the problem to the case where $X = \mathbb{A}^m$. Let $\{f_1, \ldots, f_n\}$ be a set of generators of the ideal associated with Y. Then, we have a rational map $\pi\colon \mathbb{A}^m \dashrightarrow \mathbb{P}^{n-1}$ defined such that

$$(x_1, \ldots, x_m) \mapsto (f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)).$$

This map is defined on $\mathbb{A}^m - Y$. The blowing-up $\widetilde{Y}$ of Y is the closure of the graph of π in $\mathbb{A}^m \times \mathbb{P}^{n-1}$.

The blowing-up has the following universal property:

For every rational map $f\colon X \dashrightarrow \mathbb{P}^{n-1}$, if $\widetilde{X}$ is the blowing-up of the indeterminacy locus of f, then there is a unique map $\widetilde{f}$ from $\widetilde{X}$ to $\mathbb{P}^{n-1}$ extending f, and $\widetilde{f}$ is a morphism.

$$\begin{array}{ccc}
X & \overset{f}{\dashrightarrow} & \mathbb{P}^{n-1} \\
\big\uparrow & \nearrow \widetilde{f} & \\
\widetilde{X} & &
\end{array}$$

To be really correct, the above statement requires working in the framework of schemes.