Chapter 5

Homological Algebra

5.1 Introduction

Homological Algebra has now reached into almost every corner of modern mathematics. It started with the invasion of algebra into topology at the hands of Emmy Noether. She pointed out that the ranks and "torsion coefficients" computed for various spaces were just the descriptions of finitely generated abelian groups as coproducts of cyclic groups; so, one should instead study these "homology invariants" as homology *groups*. Algebraic topology was born.

In the late 30's through the decade of the 40's, the invasion was reversed and topology invaded algebra. Among the principal names here were Eilenberg, MacLane, Hochschild, Chevalley and Koszul. This created "homological algebra" and the first deeply influential book was in fact called "Homological Algebra" and authored by H. Cartan and S. Eilenberg (1956) [9].

Our study below is necessarily abbreviated, but it will allow the reader access to the major applications as well as forming a good foundation for deeper study in more modern topics and applications.

5.2 Complexes, Resolutions, Derived Functors

From now on, let $\mathcal{A}$ denote an abelian category; think of $\mathcal{M}\mathrm{od}(R)$, where R is a ring, not necessarily commutative. This is not so restrictive an example. The Freyd-Mitchell embedding theorem [15, 40], says that each "reasonable" abelian category admits a full embedding into $\mathcal{M}\mathrm{od}(R)$ for a suitable ring R.

We make a new category, $\mathrm{Kom}(\mathcal{A})$, its objects are sequences of objects and morphisms from $\mathcal{A}$:

$$\cdots \longrightarrow A^{-n} \xrightarrow{d^{-n}} A^{-n+1} \xrightarrow{d^{-n+1}} \cdots \longrightarrow A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \longrightarrow \cdots \longrightarrow A^n \xrightarrow{d^n} A^{n+1} \longrightarrow \cdots,$$

in which $d^{i+1} \circ d^i = 0$, for all i. That is, its objects are complexes from $\mathcal{A}$.

Such a complex is usually denoted by $A^\bullet$ (sometimes, $(A^\bullet, d^\bullet)$). The morphisms of $\mathrm{Kom}(\mathcal{A})$ are more complicated. However, we have the notion of "premorphism": $(A^\bullet, d^\bullet) \xrightarrow{\varphi^\bullet} (B^\bullet, \delta^\bullet)$. This is a sequence, $\varphi^\bullet$, of morphisms from $\mathcal{A}$, where $\varphi^n \colon A^n \to B^n$, and we require that for all n, the diagram

$$\begin{tikzcd} A^n \arrow[r, "d^n"] \arrow[d, "\varphi^n"'] & A^{n+1} \arrow[d, "\varphi^{n+1}"] \\ B^n \arrow[r, "\delta^n"'] & B^{n+1} \end{tikzcd}$$

commutes. Such φ's are called *chain maps*, or *cochain maps*. The collection of complexes and their chain maps forms the category $\mathrm{PreKom}(\mathcal{A})$.

Remarks:

(1) Write $A_n = A^{-n}$. This notation is usually used when $A^\bullet$ stops at A^0 (correspondingly, write d_n for d^{-n}).

(2) A complex is *bounded below* (resp. *bounded above*) iff there is some $N \geq 0$ so that $A^k = (0)$ if $k < -N$ (resp. $A^k = (0)$ if $k > N$). It is *bounded* iff it is bounded above and below. The sub (pre)category of the bounded complexes is denoted $\mathrm{PreKom}^b(\mathcal{A})$.

(3) If $A^k = (0)$ for all $k < 0$, we have a *cohomological complex* (*right complex or co-complex*).

(4) If $A^k = (0)$ for all $k > 0$, then we use lower indices and get a *homological complex* (*left complex*, or just *complex*).

(5) The category $\mathcal{A}$ has a full embedding in $\mathrm{PreKom}(\mathcal{A})$ *via* $A \mapsto A^\bullet$, where $A^k = (0)$ if $k \neq 0$ and $A^0 = A$ and all $d^k \equiv 0$.

(6) Given a sequence, $\{A^n\}_{n=-\infty}^{\infty}$ from $\mathcal{A}$, we get an object of $\mathrm{PreKom}(\mathcal{A})$, namely:

$$\cdots \longrightarrow A^{-n} \xrightarrow{0} A^{-n+1} \xrightarrow{0} \cdots \longrightarrow A^{-1} \xrightarrow{0} A^0 \xrightarrow{0} A^1 \xrightarrow{0} A^2 \longrightarrow \cdots,$$

where all maps are the zero map. Since $\mathrm{Kom}(\mathcal{A})$ and $\mathrm{PreKom}(\mathcal{A})$ will have the same objects, we will drop references to $\mathrm{PreKom}(\mathcal{A})$ when objects only are discussed.

(7) Given $(A^\bullet, d^\bullet)$ in $\mathcal{O}\mathrm{b}(\mathrm{Kom}(\mathcal{A}))$, we make a new object of $\mathrm{Kom}(\mathcal{A})$: $H^\bullet(A^\bullet)$, with

$$H^n(A^\bullet) = \mathrm{Ker}\, d^n / \mathrm{Im}\, d^{n-1} \in \mathcal{O}\mathrm{b}(\mathcal{A}),$$

and with all maps equal to the zero map. The object $H^\bullet(A^\bullet)$ is the *homology* of $(A^\bullet, d^\bullet)$.

Nomenclature. A complex $(A^\bullet, d^\bullet) \in \mathrm{Kom}(\mathcal{A})$ is *acylic* iff $H^\bullet(A^\bullet) \equiv (0)$. That is, the complex $(A^\bullet, d^\bullet)$ is an exact sequence.

Given $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$, a *left (acyclic) resolution* of A is a left complex, $P_\bullet = \{P_n\}_{n=0}^{\infty}$, in $\mathrm{Kom}(\mathcal{A})$ and a map $P_0 \longrightarrow A$ so that the new complex

$$\cdots P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

is acyclic. A *right (acyclic) resolution* of $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$ is the dual of a left acyclic resolution of A considered as an object of $\mathcal{A}^D$.

We shall assume of the category $\mathcal{A}$ that:

(I) *$\mathcal{A}$ has enough projectives (or enough injectives, or enough of both).* That is, given any $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$ there exists some projective object, P_0, (resp. injective object Q^0) and a surjection $P_0 \longrightarrow A$ (resp. an injection $A \longrightarrow Q^0$).

Observe that (I) implies that each $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$ has an acyclic resolution $P_\bullet \longrightarrow A \longrightarrow 0$, with all P_n projective, or an acyclic resolution $0 \longrightarrow A \longrightarrow Q^\bullet$, with all Q^n injective. These are called *projective* (resp. *injective*) resolutions. For $\mathcal{M}\mathrm{od}(R)$, both exist. (For $\mathcal{S}h(X)$, the category of sheaves of abelian groups on the topological space, X, injective resolutions exist.)

(II) *$\mathcal{A}$ possesses finite coproducts (resp. finite products, or both).* This holds for $\mathcal{M}\mathrm{od}(R)$ and $\mathcal{S}h(X)$.

Remark: The following simple fact about projectives will be used in several of the subsequent proofs: *If we have a diagram*

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dotted, "\theta"'] & \\ A \arrow[r, "\varphi"'] & B \arrow[r, "\psi"'] & C \end{tikzcd}$$

in which

(1) P is projective.

(2) The lower sequence is exact (i.e., $\mathrm{Im}\,\varphi = \mathrm{Ker}\,\psi$*).*

(3) $\psi \circ f = 0$,

then there is a map $\theta\colon P \to A$ lifting f (as shown by the dotted arrow above). Indeed, $\psi \circ f = 0$ implies that $\mathrm{Im}\, f \subseteq \mathrm{Ker}\,\psi$; so, we have $\mathrm{Im}\, f \subseteq \mathrm{Im}\,\varphi$, and we are reduced to the usual situation where φ is surjective. Of course, the dual property holds for injectives.

Proposition 5.1 *Suppose we are given an exact sequence*

$$0 \longrightarrow A' \stackrel{\psi}{\longrightarrow} A \stackrel{\varphi}{\longrightarrow} A'' \longrightarrow 0$$

and both A' and A'' possess projective resolutions $P'_\bullet \longrightarrow A' \longrightarrow 0$ and $P''_\bullet \longrightarrow A'' \longrightarrow 0$. Then, there exists a projective resolution of A, denote it $P_\bullet$, and maps of complexes $P'_\bullet \longrightarrow P_\bullet$ and $P_\bullet \longrightarrow P''_\bullet$, so that the diagram

$$\begin{tikzcd} 0 \arrow[r] & P'_\bullet \arrow[r, "\psi_\bullet"] \arrow[d] & P_\bullet \arrow[r, "\varphi_\bullet"] \arrow[d] & P''_\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\psi"] \arrow[d] & A \arrow[r, "\varphi"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

commutes and has exact rows and columns. A similar result holds for injective resolutions.

Proof. We have $0 \longrightarrow P'_n \longrightarrow P_n \longrightarrow P''_n \longrightarrow 0$ if P_n exists and P''_n is projective. So, the sequence would split and $P_n = P'_n \amalg P''_n$. Look at

$$0 \longrightarrow P'_n \stackrel{\psi_n}{\longrightarrow} \underbrace{P'_n \amalg P''_n}_{P_n} \; \underset{\varphi_n}{\overset{i''_n}{\rightleftarrows}} \; P''_n \longrightarrow 0.$$

We have a map $P_n \longrightarrow P_n$ *via* $i''_n \circ \varphi_n$; we also have the map $\mathrm{id} - i''_n \circ \varphi_n$ and

$$\varphi_n \circ (\mathrm{id} - i''_n \circ \varphi_n) = \varphi_n - \varphi_n \circ i''_n \circ \varphi_n = \varphi_n - \mathrm{id}''_n \circ \varphi_n = \varphi_n - \varphi_n \equiv 0.$$

It follows that $\mathrm{id} - i''_n \circ \varphi_n$ factors through ψ_n, i.e.,

$$\mathrm{id} - i''_n \circ \varphi_n \colon P_n \longrightarrow P'_n \stackrel{\psi_n}{\longrightarrow} P_n.$$

So, we may speak of "elements of P_n" as pairs $x_n = (x'_n, x''_n)$, where $x''_n = \varphi_n(x_n)$ and $(\mathrm{id} - i''_n \circ \varphi_n)(x_n) = x_n - i''_n(x''_n) = x'_n$. Therefore,

$$x_n = \text{“}x'_n + i''_n(x''_n)\text{”} = (x'_n, x''_n).$$

This shows that for every n, we should define P_n as $P'_n \amalg P''_n$. We need $d_\bullet$ on $P_\bullet$. The map d_n takes P_n to P_{n-1}. These d_n should make the diagram

$$\begin{tikzcd}
0 \arrow[r] & P'_{n+1} \arrow[r] \arrow[d, "d'_{n+1}"] & P_{n+1} \arrow[r] \arrow[d, "d_{n+1}"] & P''_{n+1} \arrow[r] \arrow[d, "d''_{n+1}"] & 0 \\
0 \arrow[r] & P'_n \arrow[r, "\psi_n"] \arrow[d, "d'_n"] & P_n \arrow[r, "\varphi_n"] \arrow[d, "d_n"] & P''_n \arrow[r] \arrow[d, "d''_n"] & 0 \\
0 \arrow[r] & P'_{n-1} \arrow[r, "\psi_{n-1}"] & P_{n-1} \arrow[r, "\varphi_{n-1}"] & P''_{n-1} \arrow[r] & 0
\end{tikzcd}$$

commute and $d_n \circ d_{n+1} = 0$. In terms of pairs, $x_n = (x'_n, x''_n)$, where $\psi_n(x'_n) = (x'_n, 0)$ and $\varphi_n(x_n) = x''_n$, the commutativity of the lower left square requires

$$d_n(x'_n, 0) = (d'_n x'_n, 0).$$

How about $(0, x''_n)$? Observe that we have $\varphi_{n-1} d_n(0, x''_n) = d''_n(x''_n)$. Write $d_n(0, x''_n) = (\alpha_{n-1}, \beta_{n-1})$; we know that $\varphi_{n-1}(\alpha_{n-1}, \beta_{n-1}) = \beta_{n-1}$, thus,

$$d_n(0, x''_n) = (\alpha_{n-1}, d''_n x''_n).$$

So, *we need a map* $\theta_n \colon P''_n \to P'_{n-1}$; namely $\theta_n(x''_n) = \alpha_{n-1}$, the first component of $d_n(0, x''_n)$. If we know θ_n, then

$$\begin{aligned}
d_n(x_n) &= d_n(x'_n, x''_n) = d_n((x'_n, 0) + (0, x''_n)) \\
&= (d'_n(x'_n), 0) + d_n(0, x''_n) \\
&= (d'_n(x'_n), 0) + (\theta_n(x''_n), d''_n(x''_n)) \\
&= (d'_n(x'_n) + \theta_n(x''_n), d''_n(x''_n)).
\end{aligned}$$

Everything would be OK in one layer from P_n to P_{n-1}, but we need $d_n \circ d_{n+1} = 0$. Since

$$d_{n+1}(x_{n+1}) = d_{n+1}(x'_{n+1}, x''_{n+1}) = (d'_{n+1}(x'_{n+1}) + \theta_{n+1}(x''_{n+1}), d''_{n+1}(x''_{n+1})),$$

we must have

$$\begin{aligned}
d_n \circ d_{n+1}(x_{n+1}) &= (d'_n \circ d'_{n+1}(x'_{n+1}) + d'_n \circ \theta_{n+1}(x''_{n+1}) + \theta_n \circ d''_{n+1}(x''_{n+1}), d''_n \circ d''_{n+1}(x''_{n+1})) \\
&= (d'_n \circ \theta_{n+1}(x''_{n+1}) + \theta_n \circ d''_{n+1}(x''_{n+1}), 0) = 0.
\end{aligned}$$

Therefore, *we need*

$$d'_n \circ \theta_{n+1} + \theta_n \circ d''_{n+1} = 0, \quad \text{for all } n \geq 1. \qquad (\dagger_n)$$

The case $n = 0$ requires commutativity in the diagram

$$\begin{tikzcd}
0 \arrow[r] & P'_1 \arrow[r] \arrow[d, "d'_1"] & P_1 \arrow[r] \arrow[d, "d_1"] & P''_1 \arrow[r] \arrow[d, "d''_1"] & 0 \\
0 \arrow[r] & P'_0 \arrow[r, "\psi_0"] \arrow[d, "\epsilon'"] & P_0 \arrow[r, "\varphi_0"] \arrow[d, "\epsilon"] & P''_0 \arrow[r] \arrow[d, "\epsilon''"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\psi"] \arrow[d] & A \arrow[r, "\varphi"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Since P_0'' is projective, there is a map $\sigma\colon P_0'' \to A$ so that

$$\varphi \circ \sigma = \epsilon''.$$

We can now define ϵ. We have $\epsilon(x_0) = \epsilon((x_0', x_0'')) = \epsilon((x_0', 0)) + \epsilon((0, x_0''))$ and $\epsilon((x_0', 0)) = \psi\epsilon'(x_0')$, as the lower left square commutes. We also have

$$\varphi(\epsilon(0, x_0'')) = \epsilon''(\varphi_0(0, x_0'')) = \epsilon''(x_0'') = \varphi\sigma(x_0'').$$

Consequently, $\epsilon((0, x_0'')) - \sigma(x_0'')$ is killed by φ and it follows that

$$\epsilon((x_0', x_0'')) = \psi\epsilon'(x_0') + \sigma(x_0'').$$

We construct the map θ_n by induction on n and begin with $n = 1$. Note that

$$0 = \epsilon d_1(x_1', x_1'') = \epsilon(d_1'(x_1') + \theta_1(x_1''), d_1''(x_1'')) = \psi\epsilon'(d_1'(x_1') + \theta_1(x_1'')) + \sigma d_1''(x_1'').$$

Therefore, we need to have

$$\psi\epsilon'\theta_1 + \sigma d_1'' = 0. \tag{††}$$

Construction of θ_1: In the diagram

$$\begin{tikzcd} & P_1'' \arrow[dl, dotted, "\theta_1"'] \arrow[d, "-\sigma d_1''"] & & \\ P_0' \arrow[r, "\psi\epsilon'"'] & A \arrow[r] & A'' \arrow[r] & 0 \end{tikzcd}$$

as P_1'' is projective, the map $-\sigma d_1''$ lifts to a map $\theta_1\colon P_1'' \to P_0'$; thus (††) holds.

Next, we construct θ_2: Consider the diagram

$$\begin{tikzcd} & P_2'' \arrow[dl, dotted, "\theta_2"'] \arrow[d, "-\theta_1 d_2''"] & & \\ P_1' \arrow[r, "d_1'"'] & P_0' \arrow[r, "\epsilon'"'] & A' \arrow[r] & 0. \end{tikzcd}$$

If we know that $\epsilon'(-\theta_1 d_2'') = 0$, we can lift our map and get θ_2, as shown. But, apply ψ, then by (††), we get

$$\psi\epsilon'\theta_1 d_2'' = \sigma d_1'' d_2'' = 0.$$

Yet, ψ is an injection, so $\epsilon'\theta_1 d_2'' = 0$. Thus, the map θ_2 exists and we have $d_1'\theta_2 = -\theta_1 d_2''$, i.e. $(\dagger_1)$ holds.

Finally, consider the case $n > 1$ and assume the θ_r are constructed for $r \le n$ and $(\dagger_k)$ holds for all $k \le n - 1$. By the induction hypothesis,

$$-d_{n-1}'\theta_n d_{n+1}'' = \theta_{n-1} d_n'' d_{n+1}'' = 0.$$

We have the diagram

$$\begin{tikzcd} & P_{n+1}'' \arrow[dl, dotted, "\theta_{n+1}"'] \arrow[d, "-\theta_n d_{n+1}''"] & \\ P_n' \arrow[r, "d_n'"'] & P_{n-1}' \arrow[r, "d_{n-1}'"'] & P_{n-2}' \end{tikzcd}$$

in which P''_{n+1} is projective, $-d'_{n-1}\theta_n d''_{n+1} = 0$ and the lower sequence is exact. Therefore, $-\theta_n d''_{n+1}$ lifts to θ_{n+1} so that
$$d'_n\theta_{n+1} = -\theta_n d''_{n+1},$$
which is $(\dagger_n)$. The case of injectives follows from the dual category. $\square$

Definition 5.1 Say

$$\cdots \longrightarrow X^{-n} \xrightarrow{d_X^{-n}} X^{-n+1} \xrightarrow{d_X^{-n+1}} \cdots \longrightarrow X^{-1} \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^0} X^1 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{d_X^n} X^{n+1} \longrightarrow \cdots$$

and

$$\cdots \longrightarrow Y^{-n} \xrightarrow{d_Y^{-n}} Y^{-n+1} \xrightarrow{d_Y^{-n+1}} \cdots \longrightarrow Y^{-1} \xrightarrow{d_Y^{-1}} Y^0 \xrightarrow{d_Y^0} Y^1 \longrightarrow \cdots \longrightarrow Y^n \xrightarrow{d_Y^n} Y^{n+1} \longrightarrow \cdots$$

are objects of $\mathrm{Kom}(\mathcal{A})$. A *homotopy* between two maps $f^\bullet, g^\bullet \colon X^\bullet \longrightarrow Y^\bullet$ is a sequence, $\{s^n\}$, of maps $s^n \colon X^n \to Y^{n-1}$ so that
$$f^n - g^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n, \quad \text{for all } n,$$
as illustrated in the diagram below:

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d, "\Delta^{n-1}"'] & X^n \arrow[r, "d_X^n"] \arrow[dl, "s^n"'] \arrow[d, "\Delta^n"'] & X^{n+1} \arrow[r] \arrow[dl, "s^{n+1}"'] \arrow[d, "\Delta^{n+1}"] & \cdots \\
\cdots \arrow[r] & Y^{n-1} \arrow[r, "d_Y^{n-1}"'] & Y^n \arrow[r, "d_Y^n"'] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

where $\Delta^n = f^n - g^n$.

Remark: From $f^\bullet$ and $g^\bullet$ we get two maps on homology:
$$\begin{aligned} H^\bullet(f^\bullet) \colon H^\bullet(X^\bullet) &\longrightarrow H^\bullet(Y^\bullet) \\ H^\bullet(g^\bullet) \colon H^\bullet(X^\bullet) &\longrightarrow H^\bullet(Y^\bullet). \end{aligned}$$

But, *when $f^\bullet$ and $g^\bullet$ are homotopic, these maps on homology are* **equal**. Indeed,
$$\begin{aligned} H^\bullet(f^\bullet - g^\bullet) &= H^\bullet(s^{\bullet+1}d^\bullet) + H^\bullet(d^{\bullet-1}s^\bullet) \\ &= H^\bullet(s^{\bullet+1})H^\bullet(d^\bullet) + H^\bullet(d^{\bullet-1})H^\bullet(s^\bullet). \end{aligned}$$

As $H^\bullet(d^\bullet) = 0$ and $H^\bullet(d^{\bullet-1}) = 0$, we get
$$H^\bullet(f^\bullet) - H^\bullet(g^\bullet) = H^\bullet(f^\bullet - g^\bullet) = 0,$$
as claimed.

Now, based on this, we define the category $\mathrm{Kom}(\mathcal{A})$ by changing the morphisms in $\mathrm{PreKom}(\mathcal{A})$.

Definition 5.2 $\mathrm{Kom}(\mathcal{A})$ is *the category whose objects are the chain complexes from $\mathcal{A}$ and whose morphisms are the homotopy classes of chain maps of the complexes.*

Theorem 5.2 *Under the usual assumptions on $\mathcal{A}$, suppose $P^\bullet(A) \longrightarrow A \longrightarrow 0$ is a projective resolution of A and $X^\bullet(A') \longrightarrow A' \longrightarrow 0$ is an acyclic resolution of A'. If $\xi \colon A \to A'$ is a map in $\mathcal{A}$, it lifts uniquely to a morphism $P^\bullet(A) \longrightarrow X^\bullet(A')$ in* $\mathrm{Kom}(\mathcal{A})$. [*If $0 \longrightarrow A \longrightarrow Q^\bullet(A)$ is an injective resolution of A and $0 \longrightarrow A' \longrightarrow Y^\bullet(A')$ is an acyclic resolution of A', then any map $\xi \colon A' \to A$ lifts uniquely to a morphism $Y^\bullet(A') \longrightarrow Q^\bullet(A)$ in* $\mathrm{Kom}(\mathcal{A})$.]

Proof. We begin by proving the existence of the lift, stepwise, by induction. Since we have morphisms $\epsilon\colon P_0(A) \to A$ and $\xi\colon A \to A'$, we get a morphism $\xi \circ \epsilon\colon P_0(A) \to A'$ and we have the diagram

$$\begin{tikzcd}
& P_0(A) \arrow[dl, dotted, "f_0"'] \arrow[d, "\xi\circ\epsilon"] & \\
X_0(A') \arrow[r] & A' \arrow[r] & 0.
\end{tikzcd}$$

As $P_0(A)$ is projective, the map $f_0\colon P_0(A) \to X_0(A')$ exists and makes the diagram commute. Assume the lift exists up to level n. We have the diagram

$$\begin{tikzcd}
P_{n+1}(A) \arrow[r, "d^P_{n+1}"] & P_n(A) \arrow[r, "d^P_n"] \arrow[d, "f_n"] & P_{n-1}(A) \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
X_{n+1}(A') \arrow[r, "d^X_{n+1}"'] & X_n(A') \arrow[r, "d^X_n"'] & X_{n-1}(A') \arrow[r] & \cdots,
\end{tikzcd} \tag{$\dagger$}$$

so we get a map $f_n \circ d^P_{n+1}\colon P_{n+1}(A) \to X_n(A')$ and a diagram

$$\begin{tikzcd}
& P_{n+1}(A) \arrow[dl, dotted, "f_{n+1}"'] \arrow[d, "f_n\circ d^P_{n+1}"] & \\
X_{n+1}(A') \arrow[r] & X_n(A') \arrow[r, "d^X_n"'] & X_{n-1}(A').
\end{tikzcd}$$

But, by commutativity in ($\dagger$), we get

$$d^X_n \circ f_n \circ d^P_{n+1} = f_{n-1} \circ d^P_n \circ d^P_{n+1} = 0.$$

Now, $P_{n+1}(A)$ is projective and the lower row in the above diagram is exact, so there is a lifting $f_{n+1}\colon P_{n+1}(A) \to X_{n+1}(A')$, as required.

Now, we prove uniqueness (in $\mathrm{Kom}(\mathcal{A})$). Say we have two lifts $\{f_n\}$ and $\{g_n\}$. Construct the homotopy $\{s_n\}$, by induction on n.

For the base case, we have the diagram

$$\begin{tikzcd}
& P_0(A) \arrow[r, "\epsilon"] \arrow[dl, dotted, "s_0"'] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] & A \arrow[r] \arrow[d, "\xi"] & 0 \\
X_1(A') \arrow[r, "d^X_1"'] & X_0(A') \arrow[r, "\epsilon'"'] & A' \arrow[r] & 0.
\end{tikzcd}$$

As $\epsilon'(f_0 - g_0) = (\xi - \xi)\epsilon = 0$, the lower row is exact and $P_0(A)$ is projective, we get our lifting $s_0\colon P_0(A) \to X_1(A')$ with $f_0 - g_0 = d^X_1 s_0$.

Assume, for the induction step, that we already have $s_0, \ldots, s_{n-1}$. Write $\Delta_n = f_n - g_n$, then we get the diagram

$$\begin{tikzcd}
& P_n(A) \arrow[r, "d^P_n"] \arrow[d, "\Delta_n"'] & P_{n-1}(A) \arrow[r] \arrow[dl, "s_{n-1}"'] \arrow[d, "\Delta_{n-1}"] & P_{n-2}(A) \arrow[r] \arrow[d, "\Delta_{n-2}"] & \cdots \\
X_{n+1}(A') \arrow[r] & X_n(A') \arrow[r, "d^X_n"'] & X_{n-1}(A') \arrow[r] & X_{n-2} \arrow[r] & \cdots
\end{tikzcd} \tag{$\dagger\dagger$}$$

There results a map $\Delta_n - s_{n-1} \circ d_n^P \colon P_n(A) \longrightarrow X_n(A')$ and a diagram

$$\begin{tikzcd} & P_n(A) \arrow[d, "\Delta_n - s_{n-1}\circ d_n^P"] & \\ X_{n+1}(A') \arrow[r, "d_{n+1}^X"'] & X_n(A') \arrow[r, "d_n^X"'] & X_{n-1}(A'). \end{tikzcd}$$

As usual, if we show that $d_n^X \circ (\Delta_n - s_{n-1} \circ d_n^P) = 0$, then there will be a lift $s_n \colon P_n(A) \to X_{n+1}(A')$ making the diagram commute. Now, by the commutativity of (††), we have $d_n^X \circ \Delta_n = \Delta_{n-1} \circ d_n^P$; so

$$d_n^X \circ (\Delta_n - s_{n-1} \circ d_n^P) = \Delta_{n-1} \circ d_n^P - d_n^X \circ s_{n-1} \circ d_n^P.$$

By the induction hypothesis, $\Delta_{n-1} = f_{n-1} - g_{n-1} = s_{n-2} \circ d_{n-1}^P + d_n^X \circ s_{n-1}$, and therefore

$$\Delta_{n-1} \circ d_n^P - d_n^X \circ s_{n-1} \circ d_n^P = d_n^X \circ s_{n-1} \circ d_n^P + s_{n-2} \circ d_{n-1}^P \circ d_n^P - d_n^X \circ s_{n-1} \circ d_n^P = 0.$$

Hence, s_n exists and we are done. The case of injective resolutions follows by duality. □

Corollary 5.3 *Say $\xi \colon A \to A'$ is a morphism in $\mathcal{A}$ and P, P' are respective projective resolutions of A and A'. Then, ξ extends uniquely to a morphism $P \longrightarrow P'$ of* Kom($\mathcal{A}$). *(A similar result holds for injective resolutions.)*

Corollary 5.4 *If P and P' are two projective resolutions of the same object, A, of $\mathcal{A}$, then in* Kom($\mathcal{A}$), *P is uniquely isomorphic to P'. (Similarly for injective resolutions.)*

Proof. We have the identity morphism, $\mathrm{id} \colon A \to A$, so we get unique lifts, f and g in Kom($\mathcal{A}$), where $f \colon P \to P'$ and $g \colon P' \to P$ (each lifting the identity). But then, $f \circ g$ and $g \circ f$ lift the identity to endomorphisms of P' and P respectively. Yet, the identity on each is also a lift; by the theorem we must have $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$ *in* Kom($\mathcal{A}$). □

Using the same methods and no new ideas, we can prove the following important proposition. The proof will be omitted–it provides nothing new and has many messy details.

Proposition 5.5 *Suppose we have a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"'] & A \arrow[r] \arrow[d, "f"'] & A'' \arrow[r] \arrow[d, "f''"'] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0. \end{tikzcd}$$

(We call such a diagram a "small commutative diagram.") Given objects, $X^{'\bullet}, X^{\bullet}$, etc. of Kom($\mathcal{A}$) *as below, an exact sequence*

$$0 \longrightarrow X^{'\bullet} \longrightarrow X^{\bullet} \longrightarrow X^{''\bullet} \longrightarrow 0$$

over the A-sequence and an exact sequence

$$0 \longrightarrow Y^{'\bullet} \longrightarrow Y^{\bullet} \longrightarrow Y^{''\bullet} \longrightarrow 0$$

over the B-sequence, assume $X^{''\bullet}$ and $Y^{''\bullet}$ are projective resolutions, while $X^{'\bullet}$ and $Y^{'\bullet}$ are acyclic resolutions. Suppose further we have maps $\Phi' \colon X^{'\bullet} \to Y^{'\bullet}$ and $\Phi'' \colon X^{''\bullet} \to Y^{''\bullet}$ over f' and f''. Then, there exists a unique $\Phi \colon X^{\bullet} \to Y^{\bullet}$ (over f) in Kom($\mathcal{A}$) *so that the "big diagram" of augmented complexes commutes and $X^{\bullet}$ and $Y^{\bullet}$ are acyclic.*

Definition 5.3 If T is a functor (resp. cofunctor) on $\mathcal{A}$ to another abelian category $\mathcal{B}$, the *left derived functors* of T are the functors, L_nT, given by

$$(L_nT)(A) = H_n(T(P_\bullet(A))),$$

where $P_\bullet(A)$ is any projective resolution of A (resp., when T is a cofunctor, the *right derived functors of* T are the functors, R^nT, given by $(R^nT)(A) = H^n(T(P_\bullet(A))))$.

If T is a functor, its *right derived functors* are the functors, R^nT, given by

$$(R^nT)(A) = H^n(T(Q^\bullet(A))),$$

where $Q^\bullet(A)$ is any injective resolution of A (when T is a cofunctor, the *left derived functors of* T, written $(L_nT)(A)$, are given by $(L_nT)(A) = H_n(T(Q^\bullet(A))))$.

The definition of derived functors is somewhat complicated and certainly unmotivated. Much of the complication disappears when one observes that the values of either right or left derived functors are just the homology objects of a complex; that, no matter whether T is a functor or a cofunctor, *right (resp. left) derived functors are the homology of a right (resp. left) complex* (homology of a right complex is usually called *cohomology*). Thus, for a functor, T, an injective resolution will yield a right complex and so is used to compute right derived functors of T. *Mutatis mutandis* for projective resolutions; for cofunctors, T, simply reverse all arrows. Of course, what we are investigating here is the *effect of* T *on a resolution*. We *always get a complex*, but acyclicity is in general not preserved and *the deviation from acyclicity is measured by the derived functors*.

As for motivation, the concept arose from experience first from algebraic topology later from homological methods applied to pure algebra. Indeed the notion of derived functor took a long time to crystallize from all the gathered examples and results of years of work. Consider, for example, a group G and the abelian category of G-modules. On this category, we have already met the left exact functor $M \rightsquigarrow M^G$ with values in $\mathcal{A}$b. Our notation for this functor was $H^0(G, M)$. Now, in Chapters 1 and 4, we constructed a sequence of functors of M, namely $H^n(G, M)$. An obvious question is: Are the functors $H^n(G, -)$ the right derived functors of $H^0(G, -)$? We will answer this question below by characterizing the derived functors of a given functor, T.

Further remarks:

(1) The definition makes sense, i.e., derived functors are independent of the resolution chosen. Use Corollary 5.4 to see this.

(2) Suppose T is a functor and A is a projective object of $\mathcal{A}$ (resp. an injective object of $\mathcal{A}$), then $(L_nT)(A) = (0)$ for $n > 0$ (resp. $(R^nT)(A) = (0)$ for $n > 0$). If T is a cofunctor, interchange conclusions. (A is its own resolution in either case; so, remark (1) provides the proof.)

(3) If T is exact, then L_nT and R^nT are (0) for $n > 0$ (the homology of an acyclic complex is zero).

Proposition 5.6 *If T is any functor, there are always maps of functors $T \longrightarrow R^0T$ and $L_0T \longrightarrow T$. If Q is injective and P projective, then $T(Q) \longrightarrow (R^0T)(Q)$ and $(L_0T)(P) \longrightarrow T(P)$ are isomorphisms. When T is a cofunctor interchange P and Q. For either a functor or a cofunctor, T, the zeroth derived functor R^0T is always left-exact while L_0T is always right-exact. A necessary and sufficient condition that T be left-exact (resp. right-exact) is that $T \longrightarrow R^0T$ be an isomorphism of functors (resp. $L_0T \longrightarrow T$ be an isomorphism of functors). Finally, the functor map $T \longrightarrow R^0T$ induces an isomorphism of functors $R^nT \longrightarrow R^nR^0T$ for all $n \geq 0$ and similarly there is an isomorphism of functors $L_nL_0T \longrightarrow L_nT$.*

Proof. Most of this is quite trivial. The existence of the maps $T \longrightarrow R^0T$ and $L_0T \longrightarrow T$ follows immediately from the definition (and the strong uniqueness of Corollary 5.4 as applied in Remark (1) above). That R^0T is left exact is clear because it is a kernel and because the exact sequence of resolutions lifting a given exact sequence can always be chosen as split exact at each level. Similarly, L_0T is right exact as a cokernel. Of course, if T is isomorphic to R^0T it must be left exact, while if T is left exact, the terms in the augmented complex outlined by the braces form an exact sequence:

$$\underbrace{0 \longrightarrow T(A) \longrightarrow TQ^0(A) \xrightarrow{T(d^0)} TQ^1(A)} \longrightarrow \cdots .$$

Thus, the canonical map $T(A) \longrightarrow (R^0T)(A) = \mathrm{Ker}\ T(d^0)$ is an isomorphism. Similarly for right exactness and L_0.

Should Q be injective, the sequence

$$0 \longrightarrow Q \xrightarrow{\mathrm{id}} Q \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

is an injective resolution of Q and it shows that $T(Q)$ is equal to $(R^0T)(Q)$. Similarly for P and for cofunctors. But now if A is arbitrary and $Q^\bullet(A)$ is an injective resolution of A, the diagram

$$\begin{tikzcd}
0 \arrow[r] & T(A) \arrow[r] \arrow[d] & T(Q^0(A)) \arrow[r] \arrow[d] & T(Q^1(A)) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & (R^0T)(A) \arrow[r] & (R^0T)Q^0(A) \arrow[r] & (R^0T)Q^1(A) \arrow[r] & \cdots
\end{tikzcd}$$

in which the vertical arrows except the leftmost are isomorphisms shows immediately that $R^nT \longrightarrow R^n(R^0T)$ is an isomorphism for all $n \geq 0$. Similarly for $L_n(L_0T) \longrightarrow L_nT$. $\square$

The point of the above is that *right derived functors belong with left exact functors and similarly if we interchange left and right*.

There are two extremely important examples of derived functors—they appear over and over in many applications.

Definition 5.4 If $\mathcal{A}$ is any abelian category and $\mathcal{B} = \mathcal{A}\mathrm{b}$ (abelian groups), write $T_B(A) = \mathrm{Hom}_{\mathcal{A}}(A, B)$, for fixed B. (This is a left-exact cofunctor, so we want its right derived functors R^nT_B). Set

$$\mathrm{Ext}^n_{\mathcal{A}}(A, B) = (R^nT_B)(A). \tag{$*$}$$

If $\mathcal{A} = \mathcal{M}\mathrm{od}(R^{\mathrm{op}})$ and $\mathcal{B} = \mathcal{A}\mathrm{b}$, set $S_B(A) = A \otimes_R B$, for fixed B. (This is a right-exact functor, so we want its left-derived functors L_nS_B). Set

$$\mathrm{Tor}^R_n(A, B) = (L_nS_B)(A). \tag{$**$}$$

To be more explicit, in order to compute $\mathrm{Ext}^\bullet_{\mathcal{A}}(A, B)$, we take a projective resolution of A

$$P^\bullet \longrightarrow A \longrightarrow 0$$

apply $\mathrm{Hom}_{\mathcal{A}}(-, B)$ and compute the cohomology of the (right) complex $\mathrm{Hom}_{\mathcal{A}}(P^\bullet, B)$. For the tensor product, we similarly take a projective resolution of the R^{op} module, A,

$$P^\bullet \longrightarrow A \longrightarrow 0$$

apply $-\otimes_R B$ and compute the homology of the complex $P^\bullet \otimes_R B$. Because $\mathrm{Hom}_{\mathcal{A}}(-, B)$ is left exact and $-\otimes_R B$ is right exact, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(A, B) &= \mathrm{Ext}^0_{\mathcal{A}}(A, B) \\ A \otimes_R B &= \mathrm{Tor}^R_0(A, B). \end{aligned}$$

The following proposition, which we will call the *basic lemma*, will give us a chief property of derived functors and help us characterize the sequence of derived functors of a given functor.

Proposition 5.7 *(Long (co)homology sequence) Suppose $X^\bullet, Y^\bullet, Z^\bullet$ are complexes and*

$$0 \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow 0$$

is exact in pre-$\mathrm{Kom}(\mathcal{A})$ *(also OK in* $\mathrm{Kom}(\mathcal{A})$*). Then, there exists a long exact sequence of homology (or cohomology)*

$$\begin{tikzcd}
 & \cdots \arrow[r] & H^{n-1}(Z^\bullet) \arrow[dll, rounded corners] \\
H^n(X^\bullet) \arrow[r] & H^n(Y^\bullet) \arrow[r] & H^n(Z^\bullet) \arrow[dll, rounded corners] \\
H^{n+1}(X^\bullet) \arrow[r] & H^{n+1}(Y^\bullet) \arrow[r] & H^{n+1}(Z^\bullet) \arrow[dll, rounded corners] \\
H^{n+2}(X^\bullet) \arrow[r] & \cdots &
\end{tikzcd}$$

(for all n). The maps $\delta^n \colon H^n(Z^\bullet) \to H^{n+1}(X^\bullet)$ are called connecting homomorphisms.

Proof. Look at the diagram

$$\begin{tikzcd}
0 \arrow[r] & X^{n-1} \arrow[r] \arrow[d] & Y^{n-1} \arrow[r] \arrow[d] & Z^{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^n \arrow[r] \arrow[d, "d_X^n"'] & Y^n \arrow[r] \arrow[d, "d_Y^n"'] & Z^n \arrow[r] \arrow[d, "d_Z^n"'] & 0 \\
0 \arrow[r] & X^{n+1} \arrow[r] \arrow[d] & Y^{n+1} \arrow[r] \arrow[d] & Z^{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^{n+2} \arrow[r] & Y^{n+2} \arrow[r] & Z^{n+2} \arrow[r] & 0
\end{tikzcd}$$

and apply the snake lemma to the rows n and $n+1$. We get

$$0 \longrightarrow \mathrm{Ker}\ d_X^n \longrightarrow \mathrm{Ker}\ d_Y^n \longrightarrow \mathrm{Ker}\ d_Z^n \xrightarrow{\delta} \mathrm{Coker}\ d_X^n \longrightarrow \mathrm{Coker}\ d_Y^n \longrightarrow \mathrm{Coker}\ d_Z^n \longrightarrow 0.$$

If we look at $\mathrm{Im}\ d_X^{n-1}$ and apply the map $X^\bullet \longrightarrow Y^\bullet$, we land in $\mathrm{Im}\ d_Y^{n-1}$, etc. Thus, at *every* level we get that

$$H^n(X^\bullet) \longrightarrow H^n(Y^\bullet) \longrightarrow H^n(Z^\bullet) \quad \text{is exact.}$$

Now, the connecting map, δ, of the snake lemma maps $\mathrm{Ker}\ d_Z^n$ to $H^{n+1}(X^\bullet)$. But, clearly, $\mathrm{Im}\ d_Z^{n-1}$ goes to zero under δ (because every element of $\mathrm{Im}\ d_Z^{n-1}$ comes from some element in Y^{n-1}). So, we get the connecting homomorphism

$$\delta^n \colon H^n(Z^\bullet) \longrightarrow H^{n+1}(X^\bullet).$$

A diagram chase proves exactness (DX). $\square$

Corollary 5.8 *Given a commutative diagram of complexes*

$$\begin{tikzcd}
0 \arrow[r] & X^\bullet \arrow[r] \arrow[d] & Y^\bullet \arrow[r] \arrow[d] & Z^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{X}^\bullet \arrow[r] & \widetilde{Y}^\bullet \arrow[r] & \widetilde{Z}^\bullet \arrow[r] & 0
\end{tikzcd}$$

we have the big diagram of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(X^\bullet) \arrow[r] \arrow[d] & H^n(Y^\bullet) \arrow[r] \arrow[d] & H^n(Z^\bullet) \arrow[r] \arrow[d] & H^{n+1}(X^\bullet) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^n(\widetilde{X}^\bullet) \arrow[r] & H^n(\widetilde{Y}^\bullet) \arrow[r] & H^n(\widetilde{Z}^\bullet) \arrow[r] & H^{n+1}(\widetilde{X}^\bullet) \arrow[r] & \cdots
\end{tikzcd} \tag{$**$}$$

which commutes.

Proof. Chase the diagram in the usual way. $\square$

Suppose T is a right-exact functor on $\mathcal{A}$ and

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence in $\mathcal{A}$. Resolve this exact sequence (as we have shown is possible, cf. Proposition 5.1) to get

$$\begin{tikzcd}
0 \arrow[r] & P^\bullet(A) \arrow[r] \arrow[d, "\epsilon_A"'] & P^\bullet(B) \arrow[r] \arrow[d, "\epsilon_B"'] & P^\bullet(C) \arrow[r] \arrow[d, "\epsilon_C"'] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Then, as L_nT is the homology of the $TP^\bullet$ complexes (still horizontaly exact on the complex level, as our objects are projectives and the horizontal complex sequences split!), from the basic lemma, we get the long exact sequence (of derived functors)

$$\begin{tikzcd}
\cdots \arrow[r] & L_nT(A) \arrow[r] & L_nT(B) \arrow[r] & L_nT(C) \arrow[dll] \\
 & L_{n-1}T(A) \arrow[r] & \cdots & \cdots \arrow[dll] \\
 & \cdots & \cdots \arrow[r] & L_1T(C) \arrow[dll] \\
 & T(A) \arrow[r] & T(B) \arrow[r] & T(C) \arrow[r] & 0
\end{tikzcd}$$

Moreover, we have a commutative diagram corresponding to $(**)$:

$$\begin{tikzcd}
\cdots \arrow[r] & (L_nT)(A) \arrow[r] \arrow[d] & (L_nT)(B) \arrow[r] \arrow[d] & (L_nT)(C) \arrow[r] \arrow[d] & (L_{n-1}T)(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & (L_nT)(\widetilde{A}) \arrow[r] & (L_nT)(\widetilde{B}) \arrow[r] & (L_nT)(\widetilde{C}) \arrow[r] & (L_{n-1}T)(\widetilde{A}) \arrow[r] & \cdots
\end{tikzcd}$$

stemming from Proposition 5.5.

We can abstract this behavior of the sequences $\{L_nT\}_{n=0}^{\infty}$ and $\{R^nT\}_{n=0}^{\infty}$ according to the following definition:

Definition 5.5 A *δ-functor* (resp. *∂-functor*) is a sequence, $\{T^n\}$, (resp. a sequence, $\{T_n\}$) of functors so that for every exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$, we have a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & T^0(A) \arrow[r] & T^0(B) \arrow[r] & T^0(C) \arrow[dll, rounded corners] \\ & T^1(A) \arrow[r] & T^1(B) \arrow[r] & T^1(C) \arrow[dll, rounded corners] \\ & T^2(A) \arrow[r] & \cdots & \end{tikzcd}$$

and it is functorial in morphisms of exact sequences (similarly for ∂-functors, but reverse the arrows). [This means that for every commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0, \end{tikzcd}$$

where the rows are exact, the induced diagram

$$\begin{tikzcd} 0 \arrow[r] & T^0(A) \arrow[r] \arrow[d] & T^0(B) \arrow[r] \arrow[d] & T^0(C) \arrow[r] \arrow[d] & T^1(A) \cdots \arrow[r] \arrow[d] & T^{n-1}(C) \arrow[r] \arrow[d] & T^n(A) \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & T^0(A') \arrow[r] & T^0(B') \arrow[r] & T^0(C') \arrow[r] & T^1(A') \cdots \arrow[r] & T^{n-1}(C') \arrow[r] & T^n(A') \arrow[r] & \cdots \end{tikzcd}$$

is also commutative.]

Of course, our results on derived functors give us

Proposition 5.9 *The sequence of derived functors $\{R^nT\}$ (resp. $\{L_nT\}$) for a left-exact (resp. right-exact) functor, T, forms a δ-functor (resp. ∂-functor).*

Another δ-functor is the sequence $\{H^n(G,-)\}_{n=0}^{\infty}$ defined on the category of G-modules. To get at the characterization of derived functors, we need

Definition 5.6 A δ-functor $\{T^n\}$ is *universal* iff for all δ-functors, $\{S^n\}$, given a map (of functors), $f^0\colon T^0 \to S^0$, there exists a *unique* extension of f^0 to a map $\{f^n\colon T^n \to S^n\}$ **of δ-functors**. *Similarly for ∂-functors, but reverse the directions of the arrows.*

Remark: *Say $\{T^n\}$ and $\{S^n\}$ are universal δ-functors and $f^0\colon T^0 \cong S^0$ is an isomorphism of functors. Then, there is a unique isomorphism $\{f^n\colon T^n \to S^n\}$ of δ-functors lifting f^0. That is, universal δ-functors are determined by their components in dimension 0.*

Proof. Since $f^0 \colon T^0 \cong S^0$ is an isomorphism of functors, there is a map of functors, $g^0 \colon S^0 \to T^0$ so that $f_0 g_0 = \mathrm{id}$ and $g_0 f_0 = \mathrm{id}$. Universality implies that there exist unique $f^n \colon T^n \to S^n$ and $g^n \colon S^n \to T^n$ lifting f^0 and g^0. But, $f^n g^n$ and $g^n f^n$ lift $f^0 g^0$ and $g^0 f^0$, i.e., lift id. Yet, id lifts id in both cases. By uniqueness, $f^n g^n = \mathrm{id}$ and $g^n f^n = \mathrm{id}$. $\square$

Theorem 5.10 *(Uniqueness I; Weak effaceability criterion) Say $\{T^n\}$ is a δ-functor on $\mathcal{A}$ and suppose for every $n > 0$ there is some functor, $E_n \colon \mathcal{A} \to \mathcal{A}$, which is exact and for which there is a monomorphism of functors* $\mathrm{id} \longrightarrow E_n$ *[i.e., for every object A in $\mathcal{O}\mathrm{b}(\mathcal{A})$ and all $n > 0$, we have an injection $A \longrightarrow E_n(A)$ functorially in A and E_n is exact] so that the map $T^n(A) \longrightarrow T^n(E_n(A))$ is the zero map for every* $n > 0$. *Then, $\{T^n\}$ is a universal δ-functor. Hence, $\{T^n\}$ is uniquely determined by T^0.*

Proof. Construct the liftings by induction on n. The case $n = 0$ is trivial since the map $f^0 \colon T^0 \to S^0$ is given. Assume the lifting exists for all $r < n$. We have the exact sequence

$$0 \longrightarrow A \longrightarrow E_n(A) \longrightarrow \mathrm{cok}_A \longrightarrow 0$$

and so, we have a piece of the long exact diagram

$$\begin{tikzcd}
T^{n-1}(E_n(A)) \arrow[r] \arrow[d, "f_{n-1}"'] & T^{n-1}(\mathrm{cok}_A) \arrow[r, "\delta"] \arrow[d, "f_{n-1}"'] & T^n(A) \arrow[r, "0"] & T^n(E_n(A)) \\
S^{n-1}(E_n(A)) \arrow[r] & S^{n-1}(\mathrm{cok}_A) \arrow[r, "\delta"] & S^n(A) &
\end{tikzcd} \tag{††}$$

where the left square commutes and the rows are exact. Hence, by a simple argument, there is a unique $f_n \colon T^n(A) \to S^n(A)$ that makes the diagram commute. This construction is functorial since E_n is an exact functor; when we are done, all the diagrams commute.

Now, we need to prove uniqueness. Say we have two extensions $\{f_n\}$ and $\{g_n\}$ of f_0. We use induction to prove that $f_n = g_n$ for all n. This is obviously true for $n = 0$. Assume that uniqueness holds for all $r < n$. Write (††) again:

$$\begin{tikzcd}
T^{n-1}(E_n(A)) \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & T^{n-1}(\mathrm{cok}_A) \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & T^n(A) \arrow[r, "0"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & T^n(E_n(A)) . \\
S^{n-1}(E_n(A)) \arrow[r] & S^{n-1}(\mathrm{cok}_A) \arrow[r] & S^n(A) &
\end{tikzcd}$$

As $f_{n-1} = g_{n-1}$ on all arguments, the above diagram implies $f_n = g_n$ on A. As A is arbitrary, $f_n = g_n$ and the proof is complete. $\square$

Corollary 5.11 *Say $E_n = E$ for all n (E functorial and exact) and E satisfies the hypotheses of Theorem 5.10. (For example, this happens when $E(A)$ is $\{T^n\}$-acyclic for all A (i.e., $T^n(E(A)) = (0)$ for all A and all $n > 0$).) Then, $\{T^n\}$ is universal.*

We can apply Corollary 5.11 to the sequence $\{H^n(G, -)\}$, because $E(A) = \mathrm{Map}(G, A)$ satisfies all the hypotheses of that Corollary according to Proposition 4.54. Hence, we obtain the important

Corollary 5.12 *The sequence of functors $\{H^n(G, -)\}$ is a universal δ-functor from the category G-mod to* $\mathcal{A}\mathrm{b}$.

Corollary 5.13 *If $E_n(A)$ is functorial and exact for every $n > 0$, and $E_n(Q)$ is T^n-acyclic for each n and for every injective Q, then every injective object of $\mathcal{A}$ is $\{T^n\}$-acyclic.*

Proof. Pick Q injective, then we have an exact sequence

$$0 \longrightarrow Q \longrightarrow E_n(Q) \longrightarrow \mathrm{cok}_Q \longrightarrow 0.$$

Since Q is injective, the sequence splits and so,

$$T^n(E_n(Q)) = T^n(Q) \amalg T^n(\mathrm{cok}_Q).$$

By assumption, the left hand side is zero; thus, $T^n(Q) = (0)$. $\square$

Theorem 5.14 *(Uniqueness II) Say $\{T^n\}$ and $\{S^n\}$ are δ-functors on $\mathcal{A}$ and $\{f_n : T^n \to S^n\}$ is a map of δ-functors. If for all injectives, Q, the map $f_n(Q) : T^n(Q) \to S^n(Q)$ is an isomorphism (all n), then $\{f_n\}$ is an isomorphism of δ-functors. The same statement holds for ∂-functors and projectives.*

Proof. (Eilenberg) Of course, we use induction on n. First, we consider the case $n = 0$.

Step 1. I claim that $f_0 : T^0(A) \to S^0(A)$ is a monomorphism for all A.

Since $\mathcal{A}$ has enough injectives, we have an exact sequence

$$0 \longrightarrow A \longrightarrow Q \longrightarrow \mathrm{cok}_A \longrightarrow 0,$$

for some injective, Q. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T^0(A) \arrow[r] \arrow[d, "f_0"] & T^0(Q) \arrow[d, "\theta_{Q,0}"] \\
0 \arrow[r] & S^0(A) \arrow[r] & S^0(Q)
\end{tikzcd}$$

where $\theta_{Q,0} : T^0(Q) \to S^0(Q)$ is an isomorphism, by hypothesis. It follows that f_0 is injective.

Step 2. The map f_0 is an isomorphism, for all A.

We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & T^0(A) \arrow[r] \arrow[d] & T^0(Q) \arrow[r] \arrow[d, "\theta_{Q,0}"] & T^0(\mathrm{cok}_A) \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & S^0(A) \arrow[r] & S^0(Q) \arrow[r] & S^0(\mathrm{cok}_A),
\end{tikzcd}$$

where the rightmost vertical arrow is injective by step 1 and $\theta_{Q,0}$ is an isomorphism. By the five lemma, the middle arrow is surjective, and thus bijective.

Next, consider the induction step.

Step 3. The map f_n is injective for all A.

Consider the commutative diagram

$$\begin{tikzcd}
T^{n-1}(Q) \arrow[r] \arrow[d, "\theta_{Q,n-1}"'] & T^{n-1}(\mathrm{cok}_A) \arrow[r] \arrow[d, "f_{n-1}"'] & T^n(A) \arrow[r] \arrow[d, "f_n"] & T^n(Q) \arrow[r] \arrow[d, "\theta_{Q,n}"] & T^n(\mathrm{cok}_A) \arrow[d] \\
S^{n-1}(Q) \arrow[r] & S^{n-1}(\mathrm{cok}_A) \arrow[r] & S^n(A) \arrow[r] & S^n(Q) \arrow[r] & S^n(\mathrm{cok}_A).
\end{tikzcd}$$

By the induction hypothesis, f_{n-1} is injective; moreover, $\theta_{Q,n-1}$ and $\theta_{Q,n}$ are bijective, by assumption, so the five lemma implies that $f_n : T^n(A) \to S^n(A)$ is injective.

Step 4. The map f_n is an isomorphism for all n.

By step 3, the righthand vertical arrow is an injection and by the induction hypothesis, f_{n-1} is an isomorphism. As $\theta_{Q,n}$ and $\theta_{Q,n-1}$ are isomorphisms, by the five lemma, again, f_n is surjective and thus bijective. $\square$

Theorem 5.15 *(Uniqueness III) Given a δ-functor $\{T^n\}$ on $\mathcal{A}$, suppose that for any $A \in \mathcal{A}$, any injective Q and any exact sequence*

$$0 \longrightarrow A \longrightarrow Q \longrightarrow \mathrm{cok}_A \longrightarrow 0,$$

the sequence

$$T^{n-1}(Q) \longrightarrow T^{n-1}(\mathrm{cok}_A) \longrightarrow T^n(A) \longrightarrow 0 \quad \textit{is exact, if } n > 0.$$

Under these conditions, $\{T^n\}$ is a universal δ-functor. (Similarly for ∂-functors and projectives).

Proof. We proceed by induction. Given another δ-functor, $\{S^n\}$, and a morphism of functors $f_0\colon T^0 \to S^0$, suppose f_0 is already extended to a morphism $f_r\colon T^r \to S^r$, for all $r \leq n-1$. Since $\mathcal{A}$ has enough injectives, we have the exact sequence

$$0 \longrightarrow A \longrightarrow Q \longrightarrow \mathrm{cok}_A \longrightarrow 0$$

and we get the diagram

$$\begin{tikzcd}
T^{n-1}(Q) \arrow[r] \arrow[d, "f_{n-1}"] & T^{n-1}(\mathrm{cok}_A) \arrow[r] \arrow[d, "f_{n-1}"] & T^n(A) \arrow[r] \arrow[d, dotted, "\varphi_Q"] & 0 \\
S^{n-1}(Q) \arrow[r] & S^{n-1}(\mathrm{cok}_A) \arrow[r] & S^n(A) & .
\end{tikzcd}$$

By a familiar argument, there exists only one map, φ_Q, making the diagram commute. Note that φ_Q might depend on Q. To handle dependence on Q and functoriality, take some $\widetilde{A}$ and its own exact sequence

$$0 \longrightarrow \widetilde{A} \longrightarrow \widetilde{Q} \longrightarrow \mathrm{cok}_{\widetilde{A}} \longrightarrow 0$$

and say we have a map $g\colon A \to \widetilde{A}$. Since $\widetilde{Q}$ is injective, there exist θ and $\overline{\theta}$ making the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "g"] & Q \arrow[r] \arrow[d, "\theta"] & \mathrm{cok}_A \arrow[r] \arrow[d, "\overline{\theta}"] & 0 \\
0 \arrow[r] & \widetilde{A} \arrow[r] & \widetilde{Q} \arrow[r] & \mathrm{cok}_{\widetilde{A}} \arrow[r] & 0.
\end{tikzcd}$$

We have the diagram:

$$
\begin{tikzcd}[column sep=small]
& T^{n-1}(Q) \arrow[rr] \arrow[dl] \arrow[dd, "f_{n-1}" near start] & & T^{n-1}(\mathrm{cok}_A) \arrow[rr] \arrow[dl] \arrow[dd, "f_{n-1}" near start] & & T^n(A) \arrow[rr] \arrow[dl, "T^n(g)"'] \arrow[dd, "\varphi_Q" near start] & & 0 \\
T^{n-1}(\widetilde{Q}) \arrow[rr, crossing over] \arrow[dd, "f_{n-1}"] & & T^{n-1}(\mathrm{cok}_{\widetilde{A}}) \arrow[rr, crossing over] & & T^n(\widetilde{A}) \arrow[rr, crossing over] & & 0 & \\
& S^{n-1}(Q) \arrow[rr] \arrow[dl] & & S^{n-1}(\mathrm{cok}_A) \arrow[rr] \arrow[dl] & & S^n(A) \arrow[dl, "S^n(g)"] & & \\
S^{n-1}(\widetilde{Q}) \arrow[rr] & & S^{n-1}(\mathrm{cok}_{\widetilde{A}}) \arrow[rr] \arrow[from=uu, crossing over, "f_{n-1}" near end] & & S^n(\widetilde{A}) \arrow[from=uu, crossing over, "\varphi_{\widetilde{Q}}" near end] & & &
\end{tikzcd}
$$

All squares at top and bottom commute and the two left hand vertical squares also commute by the induction hypothesis. It follows that the righthand vertical square commutes (DX), i.e.:

$$\varphi_{\widetilde{Q}} \circ T^n(g) = S^n(g) \circ \varphi_Q.$$

If we set $g = \mathrm{id}$ (perhaps for different Q and $\widetilde{Q}$), we see that

$$\varphi_{\widetilde{Q}} = \varphi_Q,$$

so φ is independent of Q. Moreover, for any g, the righthand vertical diagram gives functoriality.

It remains to show commutativity with the connecting homomorphisms. Given an exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

begin the resolution of A' by injectives, i.e., consider an exact sequence

$$0 \longrightarrow A' \longrightarrow Q' \longrightarrow \mathrm{cok}' \longrightarrow 0.$$

We obtain the diagram below in which θ and $\overline{\theta}$ exist making the diagram commute:

$$
\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, "\theta"] & A'' \arrow[r] \arrow[d, "\overline{\theta}"] & 0 \\
0 \arrow[r] & A' \arrow[r] & Q' \arrow[r] & \mathrm{cok}' \arrow[r] & 0.
\end{tikzcd}
$$

Consequently, we get the diagram below in which all top and bottom diagrams commute and the left vertical cube commutes:

$$
\begin{tikzcd}
 & T^{n-1}(A) \arrow[rr] \arrow[dl] \arrow[dd, "f_{n-1}"] & & T^{n-1}(A'') \arrow[rr, "\delta_T"] \arrow[dl] \arrow[dd, "f_{n-1}"] & & T^n(A') \arrow[dl, equal] \arrow[dd, "f_n"] \\
T^{n-1}(Q') \arrow[rr] \arrow[dd, "f_{n-1}"] & & T^{n-1}(\mathrm{cok}') \arrow[rr] \arrow[dd, "f_{n-1}"] & & T^n(A') \arrow[dd, "f_n"] & \\
 & S^{n-1}(A) \arrow[rr] \arrow[dl] & & S^{n-1}(A'') \arrow[rr, "\delta_S"] \arrow[dl] & & S^n(A') \arrow[dl, equal] \\
S^{n-1}(Q') \arrow[rr] & & S^{n-1}(\mathrm{cok}') \arrow[rr] & & S^n(A') &
\end{tikzcd}
$$

If we use the rightmost horizontal equalities, a diagram chase shows

$$
\begin{tikzcd}
T^{n-1}(A'') \arrow[r, "\delta_T"] \arrow[d, "f_{n-1}"'] & T^n(A') \arrow[d, "f_n"] \\
S^{n-1}(A'') \arrow[r, "\delta_S"] & S^n(A')
\end{tikzcd}
$$

commutes (DX). $\square$

Corollary 5.16 *The right derived (resp. left derived) functors of T are universal δ-functors (resp. universal ∂-functors). A necessary and sufficient condition that the δ-functor $\{T^n\}$ be isomorphic to the δ-functor $\{R^nT^0\}$ is that $\{T^n\}$ be universal. Similarly for ∂-functors and the sequence $\{L_nT_0\}$.*

Corollary 5.17 *For any group, G, the δ-functor $\{H^n(G, -)\}$ is isomorphic to the δ-functor $\{R^nH^0(G, -)\}$.*

5.3 Various (Co)homological Functors

There are many homological and cohomological functors all over mathematics. Here, we'll give a sample from various areas and some simple applications. By these samples, some idea of the ubiquity of (co)homological functors may be gleaned.

First of all, the functors $\mathrm{Ext}^{\bullet}_{\mathcal{A}}(A, B)$ and $\mathrm{Tor}^{R}_{\bullet}(A, B)$ have been defined in an asymmetric manner: We resolved A, not B. We'll investigate now what happens if we resolve B.

Pick any $B \in \mathcal{A}$ and write

$$T_B(-) = \mathrm{Hom}_{\mathcal{A}}(-, B).$$

[Remember, $(R^n T_B)(A) = \mathrm{Ext}^n_{\mathcal{A}}(A, B)$.]

If $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ is exact and P is projective, we get the exact sequence

$$0 \longrightarrow T_{B'}(P) \longrightarrow T_B(P) \longrightarrow T_{B''}(P) \longrightarrow 0.$$

[Recall, P is projective iff $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact.]

Resolve A: $P_\bullet \xrightarrow{\epsilon} A \longrightarrow 0$. We get the commutative diagram

$$\begin{tikzcd}
 & {} & {} & {} & \\
0 \arrow[r] & T_{B'}(P_n) \arrow[r] \arrow[u] & T_B(P_n) \arrow[r] \arrow[u] & T_{B''}(P_n) \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \\
0 \arrow[r] & T_{B'}(P_0) \arrow[r] \arrow[u] & T_B(P_0) \arrow[r] \arrow[u] & T_{B''}(P_0) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & T_{B'}(A) \arrow[r] \arrow[u] & T_B(A) \arrow[r] \arrow[u] & T_{B''}(A) \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Applying cohomology,[1] we get the long exact sequence of (co)homology:

$$\begin{tikzcd}
 & \cdots \arrow[r] & R^{n-1}T_{B''}(A) \arrow[dll] \\
R^n T_{B'}(A) \arrow[r] & R^n T_B(A) \arrow[r] & R^n T_{B''}(A) \arrow[dll] \\
R^{n+1}T_{B'}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

[1]The locution "apply (co)homology" always means make the long exact sequence arising from the given short one.

Therefore, we have the exact sequence

$$\begin{aligned}
\cdots \longrightarrow{} & \mathrm{Ext}^{n-1}_{\mathcal{A}}(A, B'') \longrightarrow \\
\longrightarrow{} & \mathrm{Ext}^{n}_{\mathcal{A}}(A, B') \longrightarrow \mathrm{Ext}^{n}_{\mathcal{A}}(A, B) \longrightarrow \mathrm{Ext}^{n}_{\mathcal{A}}(A, B'') \longrightarrow \\
\longrightarrow{} & \mathrm{Ext}^{n+1}_{\mathcal{A}}(A, B') \longrightarrow \cdots
\end{aligned}$$

Consequently, we find that:

(1) $\mathrm{Ext}^{\bullet}_{\mathcal{A}}(A, B)$ is a functor of A and B, actually a co-functor of A and a functor of B.

(2) $\{\mathrm{Ext}^{\bullet}_{\mathcal{A}}(A, -)\}_{n=0}^{\infty}$ is a δ-functor (functorial in A).

(3) $\{\mathrm{Ext}^{\bullet}_{\mathcal{A}}(-, B)\}_{n=0}^{\infty}$ is a universal δ-functor (functorial in B).

Now, write $\widetilde{\mathrm{Ext}}^{\bullet}_{\mathcal{A}}(A, B)$ for what we get by resolving the righthand variable B (using injective resolutions). We obtain analogs of (1), (2), (3); call them $(\widetilde{1})$, $(\widetilde{2})$ and $(\widetilde{3})$. Note that

$$\widetilde{\mathrm{Ext}}^{0}_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathcal{A}}(A, B) = \mathrm{Ext}^{0}_{\mathcal{A}}(A, B).$$

Now, $\widetilde{\mathrm{Ext}}^{\bullet}_{\mathcal{A}}(A, -)$ is a universal δ-functor and $\mathrm{Ext}^{\bullet}_{\mathcal{A}}(A, -)$ is a δ-functor. Thus, there is a unique extension

$$\widetilde{\mathrm{Ext}}^{n}_{\mathcal{A}}(A, B) \xrightarrow{\varphi_n} \mathrm{Ext}^{n}_{\mathcal{A}}(A, B),$$

which is a map of δ-functors. When B is injective, the left hand side is (0) (as derived functors vanish on injectives). Moreover, in this case, $\mathrm{Hom}_{\mathcal{A}}(-, B)$ is exact, and so,

$$R^n\mathrm{Hom}_{\mathcal{A}}(A, B) = (0), \quad \text{for all } n > 0 \text{ and all } A.$$

By Uniqueness II (Theorem 5.14), we conclude

Theorem 5.18 *The derived functor* $\mathrm{Ext}^{\bullet}_{\mathcal{A}}$ *can be computed by resolving either variable. The same result holds for* $\mathrm{Tor}^{R}_{\bullet}$ *(in* $\mathcal{M}\mathrm{od}(R)$*).*

There is a technique by which the value of $R^nT(A)$ can be computed from $R^{n-1}T(\widetilde{A})$ for a suitable $\widetilde{A}$. This is known as *décalage*[2] or *dimension shifting*. Here is how it goes for a left exact functor, T, or left exact cofunctor, S.

For T, consider A and embed it in an acyclic object for R^nT, e.g., an injective

$$0 \longrightarrow A \longrightarrow Q \longrightarrow \mathrm{cok}_A \longrightarrow 0.$$

Now apply cohomology:

$$\begin{aligned}
0 \longrightarrow T(A) \longrightarrow T(Q) \longrightarrow T(\mathrm{cok}_A) \longrightarrow R^1T(A) \longrightarrow 0 \longrightarrow R^1T(\mathrm{cok}_A) \longrightarrow \\
\longrightarrow R^2T(A) \longrightarrow 0 \longrightarrow R^2T(\mathrm{cok}_A) \longrightarrow \cdots \longrightarrow 0 \longrightarrow R^{n-1}T(\mathrm{cok}_A) \longrightarrow \\
\longrightarrow R^nT(A) \longrightarrow 0 \longrightarrow \cdots
\end{aligned}$$

[2]The French word means a shift in space and is also used for time.

We find that

$$\begin{aligned} R^{n-1}T(\mathrm{cok}_A) &\xrightarrow{\sim} R^nT(A), \quad n \geq 2 \\ \mathrm{cok}(T(Q) \longrightarrow T(\mathrm{cok}_A)) &\xrightarrow{\sim} R^1T(A), \end{aligned}$$

so the suitable $\widetilde{A}$ is just cok_A.

For the cofunctor, S, project an acyclic object (for R^nS), e.g., a projective, onto A:

$$0 \longrightarrow \mathrm{Ker}_A \longrightarrow P \longrightarrow A \longrightarrow 0.$$

Just as above, we find

$$\begin{aligned} R^{n-1}S(\mathrm{Ker}_A) &\xrightarrow{\sim} R^nS(A) \\ \mathrm{cok}(S(P) \longrightarrow S(\mathrm{Ker}_A)) &\xrightarrow{\sim} R^1S(A). \end{aligned}$$

Similar statements hold for right exact functors or cofunctors and their left derived functors.

There is a very important interpretation of $\mathrm{Ext}^1_{\mathcal{A}}(A, B)$; indeed this interpretation is the origin of the word "Ext" for the derived functor of Hom. To keep notation similar to that used earlier for modules in Chapter 2, we'll replace A by M'' and B by M' and consider $\mathrm{Ext}^1_{\mathcal{A}}(M'', M')$.

Say

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{E}$$

is an extension of M'' by M'. Equivalence is defined as usual: In the diagram below, the middle arrow, g, is an isomorphism that makes the diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "g"] & M'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M' \arrow[r] & \widetilde{M} \arrow[r] & M'' \arrow[r] & 0 \end{tikzcd}$$

Apply to (E) the functor $\mathrm{Hom}_{\mathcal{A}}(M'', -)$. We get

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(M'', M') \longrightarrow \mathrm{Hom}_{\mathcal{A}}(M'', M) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(M'', M'') \xrightarrow{\delta_{(E)}} \mathrm{Ext}^1_{\mathcal{A}}(M'', M').$$

So, $\delta_{(E)}(\mathrm{id})$ is a canonical element in $\mathrm{Ext}^1_{\mathcal{A}}(M'', M')$; it is called the *characteristic class of the extension* (E), denoted $\chi(E)$. Note: $\chi(E) = 0$ iff (E) splits.

Now, given $\xi \in \mathrm{Ext}^1_{\mathcal{A}}(M'', M')$, resolve M' by injectives:

$$0 \longrightarrow M' \longrightarrow Q^0 \longrightarrow Q^1 \longrightarrow Q^2 \longrightarrow \cdots.$$

If we apply $\mathrm{Hom}_{\mathcal{A}}(M'', -)$, we get

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(M'', M') \longrightarrow \mathrm{Hom}_{\mathcal{A}}(M'', Q^0) \xrightarrow{d_0} \mathrm{Hom}_{\mathcal{A}}(M'', Q^1) \xrightarrow{d_1} \mathrm{Hom}_{\mathcal{A}}(M'', Q^2) \longrightarrow \cdots,$$

and we have $\mathrm{Ext}^1(M'', M') = \mathrm{Ker}\ d_1/\mathrm{Im}\ d_0$. Consequently, ξ comes from some $f \in \mathrm{Hom}_{\mathcal{A}}(M'', Q^1)$ and $d_1(f) = 0$.

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] & Q^0 \arrow[r, "d_0"] & Q^1 \arrow[r, "d_1"] & Q^2 \\ & & & M'' \arrow[u, "f"] \arrow[ur, "d_1(f)=0"'] & \end{tikzcd}$$

Thus,

$$\mathrm{Im}\ f \subseteq \mathrm{Ker}\ d_1 = \mathrm{Im}\ d_0 = X,$$

and so, f is a map $M'' \longrightarrow X \subseteq Q^1$. We get

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] & Q^0 \arrow[r, "d_0"] & X \arrow[r] & 0 \\ & & & M'' \arrow[u, "f"'] & \end{tikzcd} \tag{$*$}$$

Taking the pullback of $(*)$ by f, we find

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "g"] & M'' \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & M' \arrow[r] & Q^0 \arrow[r] & X \arrow[r] & 0, \end{tikzcd} \tag{E}$$

i.e., we get an extension, (E). One checks that (E) is independent of f, but depends only on ξ. This involves two steps (DX): (E) does not change if f is replaced by $f + d_0(h)$; (E) does not change if we use another injective resolution. Hence, we've proved

Theorem 5.19 *There is a one-to-one correspondence*

$$(E) \mapsto \chi(E)$$

between equivalence classes of extensions of modules of M'' by M' and elements of $\mathrm{Ext}^1_{\mathcal{A}}(M'', M')$.

An interpretation of $\mathrm{Ext}^n_{\mathcal{A}}(M'', M')$ for $n \geq 2$ will be left for the exercises. The cohomological functor $\mathrm{Ext}^{\bullet}_{\mathcal{A}}(A, B)$ is the most important of the various cohomological functors because many cohomological functors are special cases of it. The same holds for $\mathrm{Tor}^R_{\bullet}(A, B)$ with respect to homological functors. Here are several examples of these considerations:

We begin with groups. Recall that we proved the δ-functor $\{H^n(G, A)\}$ coincided with the right-derived functors of the functor $A \rightsquigarrow A^G$. (Of course, here G is a group and A is a G-module.) We form the group ring $R = \mathbb{Z}[G]$[3]; every G-module is an R-module and conversely—in particular, every abelian group is an R-module with trivial action by G. Consider $\mathbb{Z}$ as R-module with trivial G-action and for any G-module introduce the functor

$$A \rightsquigarrow \mathrm{Hom}_R(\mathbb{Z}, A).$$

It is left exact and its derived functors are $\mathrm{Ext}^{\bullet}_R(\mathbb{Z}, A)$. But, a homomorphism $f \in \mathrm{Hom}_R(\mathbb{Z}, A)$ is just an element of A, namely $f(1)$. And, as $\mathbb{Z}$ has trivial G-action, our element, $f(1)$, is fixed by G. Therefore

$$\mathrm{Hom}_R(\mathbb{Z}, A) \xrightarrow{\sim} A^G,$$

and so we find

Proposition 5.20 *If G is any group and A is any G-module, there is a canonical isomorphism*

$$\mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A) \xrightarrow{\sim} H^n(G, A), \quad \textit{all} \quad n \geq 0.$$

[3] Recall that $\mathbb{Z}[G]$ is the free $\mathbb{Z}$-module on the elements of G. Multiplication is defined by $\sigma \otimes \tau \mapsto \sigma\tau$, where $\sigma, \tau \in G$ and we extend by linearity.

As for group homology, first consider the exact sequence of G-modules

$$0 \longrightarrow I \longrightarrow \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0, \tag{$*$}$$

in which ϵ takes the element $\sum a_\sigma \cdot \sigma$ to $\sum a_\sigma$; that is, ϵ sends each group element to 1. The ideal I is by definition Ker ϵ; one sees easily that I is freely generated by the elements $\sigma - 1$ (for $\sigma \in G$, $\sigma \neq 1$) as a $\mathbb{Z}$-module. A little less obvious is the following:

Proposition 5.21 *The mapping* $\log(\sigma) = (\sigma - 1) (\bmod I^2)$ *is an isomorphism of abelian groups*

$$\log\colon G/[G,G] \xrightarrow{\sim} I/I^2.$$

Proof. The operations on the two sides of the claimed isomorphism, log, are the group multiplication abelianized and addition respectively. Clearly, $\log(\sigma) = (\sigma - 1)(\bmod I^2)$ is well-defined and

$$(\sigma\tau - 1) = (\sigma - 1) + (\tau - 1) + (\sigma - 1)(\tau - 1)$$

shows it's a homomorphism. Of course we then have $\log(\sigma^{-1}) = -(\sigma - 1)$, but this is easy to see directly. It follows immediately that $[G,G]$ lies in the kernel of log; so we do get a map

$$\log\colon G/[G,G] \longrightarrow I/I^2.$$

As I is the free $\mathbb{Z}$-module on the elements $(\sigma - 1)$, as σ ranges over G ($\sigma \neq 1$), we can define

$$\exp\colon I \longrightarrow G/[G,G],$$

via

$$\exp\left(\sum_{\sigma\neq 1} n_\sigma(\sigma - 1)\right) = \prod_{\sigma\neq 1} \sigma^{n_\sigma} \bmod [G,G]$$

and considerations entirely simililar to those above for log show that exp is a homomorphism from I to $G/[G,G]$ and that I^2 is killed by exp. It should be obvious that log and exp are mutually inverse, so we're done. $\square$

If A is a G-module, we can tensor exact sequence $(*)$ over $\mathbb{Z}[G]$ with A; this gives

$$I \otimes_{\mathbb{Z}[G]} A \longrightarrow A \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}[G]} A \longrightarrow 0.$$

Of course, this shows

$$A/(IA) \xrightarrow{\sim} \mathbb{Z} \otimes_{\mathbb{Z}[G]} A.$$

The functor $A \rightsquigarrow A/IA$ is a right-exact functor from G-modules to $\mathcal{A}$b and its left derived functors, $H_n(G,A)$, are the *homology groups of* G *with coefficients in* A. The isomorphism we've just observed (together with the usual arguments on universal ∂-functors) allows us to conclude

Proposition 5.22 *If* G *is a group and* A *is any* G*-module, there is a canonical isomorphism (of* ∂*-functors)*

$$H_n(G,A) \xrightarrow{\sim} \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, A), \quad \textit{all} \quad n \geq 0.$$

We first introduced and computed group cohomology *via* an explicit chain complex, is there a similar description for group homology? There is indeed, and while we can be quite direct and give it, perhaps it is better to make a slight detour which is necessary anyway if one is to define (co)homology of algebras in a direct manner.

Write K for a commutative ring and R for a (possibly non-commutative) K-algebra. In the case of groups, K will be $\mathbb{Z}$ and R will be $\mathbb{Z}[G]$, while for other purposes K will be a field and R the polynomial ring $K[T_1, \ldots, T_n]$; there will be still other purposes.

For an integer, $n \geq -1$, write $C_n(R)$ for the $(n+2)$-fold tensor product of R with itself over K:

$$\begin{aligned} C_n(R) &= \underbrace{R \otimes_K R \otimes_K \cdots \otimes_K R}_{n+2} \\ C_{-1}(R) &= R \\ C_{n+1}(R) &= R \otimes_K C_n(R). \end{aligned}$$

Next, introduce the module $R \otimes_K R^{\mathrm{op}}$. We want to make $R \otimes_K R^{\mathrm{op}}$ into a K-algebra by the multiplication

$$(\rho \otimes \sigma^{\mathrm{op}})(r \otimes s^{\mathrm{op}}) = \rho r \otimes \sigma^{\mathrm{op}} s^{\mathrm{op}} = \rho r \otimes (s\sigma)^{\mathrm{op}}$$

and *for this we must have K in the center of R*. To see this, pick $\lambda \in K$, set $\rho = s = 1$, set $\sigma = \lambda$, and compute

$$\begin{aligned} r\lambda \otimes_K 1^{\mathrm{op}} &= (r \otimes_K \lambda^{\mathrm{op}} 1^{\mathrm{op}}) = r \otimes_K \lambda^{\mathrm{op}} \\ &= (1 \otimes_K \lambda^{\mathrm{op}})(r \otimes_K 1^{\mathrm{op}}) \\ &= (\lambda \otimes_K 1^{\mathrm{op}})(r \otimes_K 1^{\mathrm{op}}) = \lambda r \otimes_K 1^{\mathrm{op}}. \end{aligned}$$

As r is arbitrary, we are done. So, **from now on, we shall assume K is in the center of R**. The algebra $R \otimes_K R^{\mathrm{op}}$ is called the *enveloping algebra of R over K*; it is usually denoted R^e. Now, there is a map $R^e \longrightarrow R$ *via*

$$r \otimes s^{\mathrm{op}} \mapsto rs.$$

This map is **not** a map of K-algebras, only a map of R^e-modules. (R^e acts on R *via* $(r \otimes s^{\mathrm{op}})(m) = rms$; in general, two-sided R-modules are just R^e-modules (as well as $(R^e)^{\mathrm{op}}$-modules).) It will be a K-algebra map if R is commutative.

We should also note that the map

$$r \otimes s^{\mathrm{op}} \mapsto s^{\mathrm{op}} \otimes r$$

is a K-isomorphism of K-algebras $R^e \xrightarrow{\sim} (R^e)^{\mathrm{op}}$. (DX)

It will be best to use "homogeneous notation" for elements of $C_n(R)$: $r_0 \otimes r_1 \otimes \cdots \otimes r_{n+1}$. Then $C_n(R)$ is a left R^e-module under the rule

$$(s \otimes t^{\mathrm{op}})(r_0 \otimes r_1 \otimes \cdots \otimes r_n \otimes r_{n+1}) = (sr_0) \otimes r_1 \otimes \cdots \otimes r_n \otimes (r_{n+1}t).$$

Now we'll make $\{C_n(R)\}_{n=0}^{\infty}$ into an acyclic left complex. The boundary map is

$$\partial_n(r_0 \otimes r_1 \otimes \cdots \otimes r_{n+1}) = \sum_{i=0}^{n} (-1)^i \, r_0 \otimes \cdots \otimes (r_i r_{i+1}) \otimes \cdots \otimes r_{n+1},$$

it is an R^e-homomorphism $C_n(R) \longrightarrow C_{n-1}(R)$. In particular, ∂_0 is the R^e-module map discussed above,

$$\partial_0(r_0 \otimes r_1) = r_0 r_1$$

and

$$\partial_1(r_0 \otimes r_1 \otimes r_2) = (r_0 r_1) \otimes r_2 - r_0 \otimes (r_1 r_2).$$

From these, we see $\partial_0\partial_1 = 0$ precisely because R is associative. To prove $\{C_n(R)\}$ is a complex and acyclic, introduce the map
$$\sigma_n\colon C_n(R) \longrightarrow C_{n+1}(R) \quad \textit{via} \quad \sigma_n(\xi) = 1 \otimes_K \xi.$$
The map σ_n is only an R^{op}-module map but it is injective because there is a map $\tau_n\colon C_{n+1}(R) \to C_n(R)$ given by $\tau_n(r_0 \otimes (\text{rest})) = r_0(\text{rest})$ and we have $\tau_n\sigma_n = \mathrm{id}$. Moreover, $\mathrm{Im}\,(\sigma_n)$ generates $C_{n+1}(R)$ as R-module! It is easy to check the relation
$$\partial_{n+1}\sigma_n + \sigma_{n-1}\partial_n = \mathrm{id} \quad \text{on} \quad C_n(R), \quad \text{for} \quad n \geq 0. \tag{$\dagger$}$$
Now use induction to show $\partial_{n-1}\partial_n = 0$ as follows: Above we showed it for $n = 1$, assume it up to n and apply ∂_n (on the left) to ($\dagger$), we get
$$\partial_n\partial_{n+1}\sigma_n + \partial_n\sigma_{n-1}\partial_n = \partial_n.$$
However, $\partial_n\sigma_{n-1} = \mathrm{id}_{n-1} - \sigma_{n-2}\partial_{n-1}$, by ($\dagger$) at $n-1$. So,
$$\partial_n\partial_{n+1}\sigma_n + \partial_n - \sigma_{n-2}\partial_{n-1}\partial_n = \partial_n,$$
that is, $\partial_n\partial_{n+1}\sigma_n = 0$ (because $\partial_{n-1}\partial_n = 0$). But, the image of σ_n generates C_{n+1} as R-module; so $\partial_n\partial_{n+1} = 0$, as needed. Now, notice that ∂_0 takes $C_0(R) = R^e$ onto $C_{-1}(R) = R$, and so
$$\cdots \longrightarrow C_n(R) \xrightarrow{\partial_n} C_{n-1}(R) \longrightarrow \cdots \longrightarrow C_0(R) \xrightarrow{\partial_0} R \longrightarrow 0$$
is an acyclic resolution of R as R^e-module.

Since $C_n(R) = R \otimes_K (\underbrace{R \otimes_K \cdots \otimes_K R}_{n}) \otimes_K R$, we find
$$C_n(R) = R^e \otimes_K C_n[R],$$
where
$$C_n[R] = R \otimes_K \cdots \otimes_K R, \quad n\text{-times}$$
and
$$C_0[R] = K.$$

Several things follow from this description of $C_n(R)$: First, we see exactly how $C_n(R)$ is an R^e-module and also see that it is simply the base extension of $C_n[R]$ from K to R^e. Next, we want a projective resolution, so we want to insure that $C_n(R)$ is indeed projective even over R^e. For this we prove

Proposition 5.23 *Suppose R is a K-algebra and R is projective as a K-module (in particular this holds if R is K-free, for example when K is a field). Then*

(1) $C_n[R]$ is K-projective for $n \geq 0$,

(2) $R \otimes_K C_n[R]$ is R-projective for $n \geq 0$,

(3) $C_n(R)$ is R^e-projective for $n \geq 0$.

Proof. This is a simple application of the ideas in Chapter 2, Section 2.6. Observe that (2) and (3) follow from (1) because we have
$$\mathrm{Hom}_R(R \otimes_K C_n[R], T) \xrightarrow{\sim} \mathrm{Hom}_K(C_n[R], T) \tag{$\dagger$}$$
and
$$\mathrm{Hom}_{R^e}(R^e \otimes_K C_n[R], T) \xrightarrow{\sim} \mathrm{Hom}_K(C_n[R], T) \tag{$\dagger\dagger$}$$

where T is an R-module in (†) and an R^e-module in (††). An exact sequence of R-modules (resp. R^e-modules) is exact as sequence of K-modules and (1) shows that the right sides of (†) and (††) are exact as functors of T. Such exactness characterizes projectivity; so, (2) and (3) do indeed follow from (1).

To prove (1), use induction on n and Proposition 2.47 which states in this case

$$\mathrm{Hom}_K(R \otimes_K C_{n-1}[R], T) \xrightarrow{\sim} \mathrm{Hom}_K(C_{n-1}[R], \mathrm{Hom}_K(R, T)). \tag{$*$}$$

Now, $T \rightsquigarrow \mathrm{Hom}_K(R, T)$ is exact by hypothesis; so, the right hand side of $(*)$ is an exact functor of T by induction hypothesis. Consequently, $(*)$ completes the proof. $\square$

Corollary 5.24 *If the K-algebra, R, is K-projective, then*

$$\cdots \longrightarrow C_n(R) \xrightarrow{\partial_n} C_{n-1}(R) \longrightarrow \cdots \longrightarrow C_0(R) \xrightarrow{\partial_0} R \longrightarrow 0$$

is an R^e-projective resolution of the R^e-module R.

The resolution of Corollary 5.24 is called the *standard* (or *bar*) *resolution of R.* We can define the homology and cohomology groups of the K-algebra R with coefficients in the *two-sided R-module*, M, as follows:

Define the functors

$$H_0(R, -)\colon M \rightsquigarrow M/M\mathfrak{J}$$

and

$$H^0(R, -)\colon M \rightsquigarrow \{m \in M \mid rm = mr, \quad \text{all } r \in R\} = M^R$$

to the category of K-modules. Here, the (left) ideal, $\mathfrak{J}$, of R^e is defined by the exact sequence

$$0 \longrightarrow \mathfrak{J} \longrightarrow R^e \xrightarrow{\partial_0} R \longrightarrow 0, \tag{$**$}$$

and is called the *augmentation ideal of* R^e. It's easy to check that $M \rightsquigarrow M/M\mathfrak{J}$ is right exact and $M \rightsquigarrow M^R$ is left exact. We make the definition

Definition 5.7 *The n-th homology group of R with coefficients in the two-sided R-module, M*, is

$$H_n(R, M) = (L^n H_0)(M)$$

and *the nth cohomology group with coefficients in M* is

$$H^n(R, M) = (R^n H^0)(M).$$

We'll refer to these groups as the *Hochschild homology* and *cohomology groups of R* even though our definition is more general than Hochschild's–he assumed K is a field and gave an explicit (co)cycle description. We'll recover this below and for this purpose notice that

The augmententation ideal, $\mathfrak{J}$, is generated (as left R^e-ideal) by the elements $r \otimes 1 - 1 \otimes r^{\mathrm{op}}$ for $r \in R$.

To see this, observe that $\sum_i r_i \otimes s_i^{\mathrm{op}} \in \mathfrak{J}$ iff we have $\sum_i r_i s_i = 0$. But then

$$\sum_i r_i \otimes s_i^{\mathrm{op}} = \sum_i r_i \otimes s_i^{\mathrm{op}} - \sum_i r_i s_i \otimes 1 = \sum_i (r_i \otimes 1)(1 \otimes s_i^{\mathrm{op}} - s_i \otimes 1).$$

Now, to apply this, tensor our exact sequence $(**)$ with M:

$$M \otimes_{R^e} \mathfrak{J} \longrightarrow M \xrightarrow{1 \otimes \partial_0} M \otimes_{R^e} R \longrightarrow 0,$$

so we find

$$H_0(R,M) = M/M\mathfrak{J} \xrightarrow{\sim} M \otimes_{R^e} R.$$

It follows immediately that we have an isomorphism

$$H_n(R,M) \xrightarrow{\sim} \mathrm{Tor}_n^{R^e}(M,R).$$

Similarly, we take $\mathrm{Hom}_{R^e}(-,M)$ of $(**)$ and get

$$0 \longrightarrow \mathrm{Hom}_{R^e}(R,M) \longrightarrow M \xrightarrow{\theta} \mathrm{Hom}_{R^e}(\mathfrak{J},M).$$

The isomorphism

$$\mathrm{Hom}_{R^e}(R^e,M) \xrightarrow{\sim} M$$

is just

$$f \mapsto f(1),$$

thus if $f \in \mathrm{Hom}_{R^e}(R^e,M)$ and $m = f(1)$, we find for $\xi \in \mathfrak{J}$ that

$$(\theta(f))(\xi) = f(\xi) = \xi m.$$

Therefore, f is in Ker θ iff $\xi m = 0$ for all $\xi \in \mathfrak{J}$, where $m = f(1)$. But, by the above, such ξ are generated by $r \otimes 1 - 1 \otimes r^{\mathrm{op}}$, and so $m \in \mathrm{Ker}\ \theta$ when and only when $(r \otimes 1)m = (1 \otimes r^{\mathrm{op}})m$; i.e., exactly when $rm = mr$, for all $r \in R$. We have proved that there is an isomorphism (of K-modules)

$$\mathrm{Hom}_{R^e}(R,M) \xrightarrow{\sim} M^R = H^0(R,M).$$

Once again we obtain an isomorphism

$$\mathrm{Ext}^n_{R^e}(R,M) \xrightarrow{\sim} H^n(R,M).$$

Our discussion above proves the first two statements of

Theorem 5.25 *If R is a K-algebra (with K contained in the center of R), then for any two-sided R-module, M, we have canonical, functorial isomorphisms*

$$H_n(R,M) \xrightarrow{\sim} \mathrm{Tor}_n^{R^e}(M,R)$$

and

$$H^n(R,M) \xrightarrow{\sim} \mathrm{Ext}^n_{R^e}(R,M).$$

If R is K-projective, then homology can be computed from the complex

$$M \otimes_K C_n[R]$$

with boundary operator

$$\partial_n(m \otimes r_1 \otimes \cdots \otimes r_n) = mr_1 \otimes r_2 \otimes \cdots \otimes r_n + \sum_{i=1}^{n-1}(-1)^i m \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n + (-1)^n r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1};$$

while cohomology can be computed from the complex

$$\mathrm{Hom}_K(C_n[R],M)$$

with coboundary operator

$$(\delta_n f)(r_1 \otimes \cdots \otimes r_n \otimes r_{n+1}) = r_1 f(r_2 \otimes \cdots \otimes r_{n+1}) + \sum_{i=1}^{n}(-1)^i f(r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_{n+1})$$
$$(-1)^{n+1} f(r_1 \otimes \cdots \otimes r_n) r_{n+1}.$$

Proof. Only the statements about the explicit complex require proof. Since homology and cohomology are given by specific Tor's and Ext's, and since the standard resolution *is* an R^e-projective resolution of R, we can use the latter to compute these Tor's and Ext's. Here, it will be important to know the R^e-module structure of $C_n(R)$ and the fact that the map

$$r \otimes s^{\mathrm{op}} \mapsto s^{\mathrm{op}} \otimes r$$

establishes a K-algebra isomorphism of R^e and $(R^e)^{\mathrm{op}}$.

Now, consider the map

$$\Theta\colon M \otimes_{R^e} C_n(R) = M \otimes_{R^e} (R^e \otimes_K C_n[R]) \xrightarrow{\sim} M \otimes_K C_n[R].$$

Observe that M is treated as an $(R^e)^{\mathrm{op}}$-module, the action being

$$m(r \otimes s^{\mathrm{op}}) = smr.$$

Thus,

$$\begin{aligned}
\Theta\colon m \otimes_{R^e} (r_0 \otimes \cdots \otimes r_{n+1}) &= m \otimes_{R^e} (r_0 \otimes r_{n+1}^{\mathrm{op}}) \otimes_K (r_1 \otimes \cdots \otimes r_n) \\
&\mapsto [m \cdot (r_0 \otimes r_{n+1}^{\mathrm{op}})] \otimes_K (r_1 \otimes \cdots \otimes r_n) \\
&= (r_{n+1} m r_0) \otimes_K (r_1 \otimes \cdots \otimes r_n) \in M \otimes_K C_n[R].
\end{aligned}$$

We now just have to see the explicit form of the boundary map induced on $M \otimes_K C_n[R]$ by the diagram

$$\begin{tikzcd}
M \otimes_{R^e} C_n(R) \arrow[d, "1\otimes\partial_n"] & M \otimes_K C_n[R] \arrow[l, "\Theta^{-1}"'] \\
M \otimes_{R^e} C_{n-1}(R) \arrow[r, "\Theta"] & M \otimes_K C_{n-1}[R]
\end{tikzcd}$$

This goes as follows:

$$\begin{aligned}
m \otimes_K (r_1 \otimes \cdots \otimes r_n) &\xrightarrow{\Theta^{-1}} m \otimes_{R^e} (1 \otimes 1) \otimes_K (r_1 \otimes \cdots \otimes r_n) \\
&= m \otimes_{R^e} 1 \otimes r_1 \otimes \cdots \otimes r_n \otimes 1 \\
&\xrightarrow{1\otimes\partial_n} m \otimes_{R^e} r_1 \otimes \cdots \otimes r_n \otimes 1 \\
&\quad + \sum_{i=1}^{n-1} (-1)^i m \otimes_{R^e} (1 \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n \otimes 1) \\
&\quad + (-1)^n m \otimes_{R^e} (1 \otimes r_1 \otimes \cdots \otimes r_n) \\
&\xrightarrow{\Theta} m r_1 \otimes r_2 \otimes \cdots \otimes r_n + \sum_{i=1}^{n-1} (-1)^i m \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n \\
&\quad + (-1)^n r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1},
\end{aligned}$$

exactly the formula of the theorem. For cohomology we proceed precisely the same way, but remember that here M is treated as an R^e-module. Details are left as a (DX). □

When K is a field, the explicit (co)chain descriptions of $H_n(R, M)$ and $H^n(R, M)$ apply; these are Hochschild's original descriptions for the (co)homology of K-algebras, Hochschild [25, 26].

By now, it should be clear that there is more than an analogy between the (co)homology of algebras and that for groups. This is particularly evident from comparison of the original formula (Chapter 1, Section 1.4) for cohomology of groups and Hochschild's formula for the cohomology of the K-algebra, R. If we use just analogy then R will be replaced by $\mathbb{Z}[G]$ and K by $\mathbb{Z}$; R is then free (rank $= \#(G)$) over K. But, M is just a left G-module (for cohomology) and for K-algebras, R, we assumed M was a two-sided R-module. This is easily fixed: *Make* $\mathbb{Z}[G]$ *act trivially on the right*. Then, $H^0(\mathbb{Z}[G], M)$ is our old M^G and the coboundary formula becomes (it is necessary only to compute on $\sigma_1 \otimes \cdots \otimes \sigma_{n+1}$ as such tensors generate):

$$\begin{aligned}(\delta_n f)(\sigma_1 \otimes \cdots \otimes \sigma_{n+1}) = {} & \sigma_1 f(\sigma_2 \otimes \cdots \otimes \sigma_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(\sigma_1 \otimes \cdots \otimes \sigma_i\sigma_{i+1} \otimes \cdots \otimes \sigma_{n+1}) \\ & + (-1)^{n+1} f(\sigma_1 \otimes \cdots \otimes \sigma_n),\end{aligned}$$

as in Chapter 1. Therefore, keeping the analogy, for homology, where we have a right G-module, we should make $\mathbb{Z}[G]$ *act trivially on the left*, and get the explicit formula:

$$\begin{aligned}\partial_n(m \otimes \sigma_1 \otimes \cdots \otimes \sigma_n) = {} & m\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_n \\ & + \sum_{i=1}^{n-1} (-1)^i m \otimes \sigma_1 \otimes \cdots \otimes \sigma_i\sigma_{i+1} \otimes \cdots \otimes \sigma_n \\ & + (-1)^n m \otimes \sigma_1 \otimes \cdots \otimes \sigma_{n-1},\end{aligned} \tag{$*$}$$

which formula we had in mind at the beginning of this discussion several pages ago.

The ideal $\mathfrak{J}$ is generated by $\sigma \otimes 1 - 1 \otimes \sigma^{\mathrm{op}}$ as σ ranges over G ($\sigma \neq 1$). Thus $M\mathfrak{J}$ is the submodule generated by $\{m - m\sigma \mid \sigma \neq 1\}$. Now the formula

$$\sigma^{-1}m = m\sigma \qquad \text{(special for groups)}$$

turns M into a left $\mathbb{Z}[G]$-module and shows that $M\mathfrak{J}$ is exactly our old IM and therefore proves the Hochschild $H_0(\mathbb{Z}[G], M)$ is our old $H_0(G, M)$.

However, all this is heuristic, it does not prove the Hochschild groups for $\mathbb{Z}[G]$ on our *one-sided* modules are the (co)homology groups for G. For one thing, we are operating on a subcategory: The modules with trivial action on one of their sides. For another, the Hochschild groups are $\mathrm{Tor}_\bullet^{\mathbb{Z}[G]^e}(-, \mathbb{Z}[G])$ and $\mathrm{Ext}^\bullet_{\mathbb{Z}[G]^e}(\mathbb{Z}[G], -)$ not $\mathrm{Tor}_\bullet^{\mathbb{Z}[G]}(-, \mathbb{Z})$ and $\mathrm{Ext}^\bullet_{\mathbb{Z}[G]}(\mathbb{Z}, -)$. We do know that everything is correct for cohomology because of a previous argument made about universal δ-functors. Of course, it is perfectly possible to prove that the groups

$$\widetilde{H}_n(G, M) = \mathrm{Ker}\ \partial_n / \mathrm{Im}\ \partial_{n+1}$$

for ∂_n given by ($*$) above form a universal ∂-functor—they clearly form a ∂-functor and universality will follow from the effaceability criterion (Theorem 5.10). The effaceing module will be $M \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ in analogy with $\mathrm{Map}(G, M)$ (which is $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], M)$). Here, details are best left as an exercise.

Instead, there is a more systematic method that furthermore illustrates a basic principle handy in many situations. We begin again with our K-algebra, R, and we *assume there is a K-algebra homomorphism* $\epsilon\colon R \to K$. Note that this is the same as saying all of (DX)

(i) K is an R-module (and R contains K in its center),

(ii) There is an R-module map $R \xrightarrow{\epsilon} K$,

(iii) The composition $K \longrightarrow R \xrightarrow{\epsilon} K$ is the identity.

Examples to keep in mind are: $K = \mathbb{Z}$, $R = \mathbb{Z}[G]$ and $\epsilon(\sigma) = 1$, all $\sigma \in G$; K arbitrary (commutative), $R = K[T_1, \ldots, T_n]$ or $K\langle T_1, \ldots, T_n\rangle$ and $\epsilon(T_j) = 0$, all j.

In general, there will be no such homomorphism. However *for* **commutative** *K-algebras, R, we can arrange a "section"*[4]*, ϵ, after base extension*. Namely, we pass to $R \otimes_K R$ and set $\epsilon(r \otimes s) = rs \in R$; so now R plays the role of K and $R \otimes_K R$ the role of R. (The map $\epsilon\colon R \to K$ is also called an *augmentation of R as K-algebra*.) Hence, the basic principle is that *after base extension* (at least for commutative R) *our K-algebra has a section; we operate assuming a section and then try to use descent* (cf. Chapter 2, Section 2.8).

This technique doesn't quite work with non-commutative R, where we base extend to get $R^e = R \otimes_K R^{\mathrm{op}}$ and try $\partial_0\colon R^e \to R$ for our ϵ. We certainly find that R is an R^e-module, that ∂_0 is an R^e-module map, that the composition $R \xrightarrow{i} R^e \longrightarrow R$ ($i(r) = r \otimes 1$) is the identity; but, R is **not** in the center of R^e and R^e (with the multiplication we've given it) is not an R-algebra.

Notwithstanding this cautionary remark, we can do a descent-like comparison in the non-commutative case provided R possesses a section $\epsilon\colon R \to K$. In the first place, the section gives K a special position as R-module. We write $I = \mathrm{Ker}\,\epsilon$, this is a two-sided ideal of R called *the augmentation ideal*. Further, consider the augmentation sequence

$$0 \longrightarrow I \longrightarrow R \xrightarrow{\epsilon} K \longrightarrow 0; \tag{$\dagger$}$$

by using condition (iii) above, we see that, as K-modules, $R \cong I \amalg K$. The special position of K as R-module leads to the consideration of the ∂-functor and δ-functor:

$$\begin{array}{rclr} \{\overline{H}_n(R, M) & = & \mathrm{Tor}_n^R(M, K)\} & \qquad (M \text{ an } R^{\mathrm{op}}\text{-module}) \\ \{\overline{H}^n(R, M) & = & \mathrm{Ext}_R^n(K, M)\} & \qquad (M \text{ an } R\text{-module}) \end{array}$$

which, as usual, are the derived functors of

$$M \rightsquigarrow M/MI$$

and

$$M \rightsquigarrow \{m \in M \mid (\forall \xi \in I)(\xi m = 0)\},$$

respectively. (Here, M is an R^{op}-module for the first functor and an R-module for the second.) You should keep in mind the case: $K = \mathbb{Z}$, $R = \mathbb{Z}[G]$, $\epsilon(\sigma) = 1$ (all σ) throughout what follows. *The idea is to compare the Hochschild groups $H_n(R, M)$ and $H^n(R, M)$ with their "bar" counterparts.*

Secondly, we make precise the notion of giving a two-sided R-module, M, "trivial action" on one of its sides[5]. Given M, a two-sided R-module, we make $\epsilon^* M$ and $\epsilon^{*\mathrm{op}} M$ which are respectively an R^{op}-module ("trivial action" on the left) and an R-module ("trivial action" on the right) as follows:

$$\text{For } m \in \epsilon^* M \text{ and } \lambda^{\mathrm{op}} \in R^{\mathrm{op}}, \quad \lambda^{\mathrm{op}} \cdot m = m\lambda \quad \text{and for } \lambda \in R, \quad \lambda \cdot m = \epsilon(\lambda) m$$

and

$$\text{For } m \in \epsilon^{*\mathrm{op}} M \text{ and } \lambda \in R, \quad \lambda \cdot m = \lambda m \quad \text{and for } \lambda^{\mathrm{op}} \in R^{\mathrm{op}}, \quad \lambda^{\mathrm{op}} \cdot m = m\epsilon(\lambda).$$

Clearly, these ideas can be used to promote one-sided R-modules to two-sided ones (i.e., to R^e-modules), *viz*:

[4] The term "section" is geometric: We have the "structure map" $\mathrm{Spec}\, R \longrightarrow \mathrm{Spec}\, K$ (corresponding to $K \longrightarrow R$) and ϵ gives a continuous map: $\mathrm{Spec}\, K \longrightarrow \mathrm{Spec}\, R$ so that $\mathrm{Spec}\, K \xrightarrow{\epsilon} \mathrm{Spec}\, R \longrightarrow \mathrm{Spec}\, K$ is the identity.

[5] Our earlier, heuristic, discussion was sloppy. For example, in the group ring case and for trivial action on the right, we stated that $\mathbb{Z}[G]$ acts trivially on the right. But, $n \cdot 1 = n \in \mathbb{Z}[G]$ and $m \cdot n \neq m$ if $n \neq 1$; so, our naive idea must be fixed.

Any R, R^{op}, or R^e-module is automatically a K-module and, as K is commutative so that $K^{\mathrm{op}} = K$, we see that $\lambda \cdot m = m \cdot \lambda$ for $\lambda \in K$ in any of these cases. Now if we have an R-module, M, we make R^e operate by

$$(r \otimes s^{\mathrm{op}}) \cdot m = rm\epsilon(s) = r\epsilon(s)m = \epsilon(s)rm,$$

and similarly for R^{op}-modules, M, we use the action

$$(r \otimes s^{\mathrm{op}}) \cdot m = \epsilon(r)ms = m\epsilon(r)s = ms\epsilon(r).$$

When we use the former action and pass from an R-module to an R^e-module, we denote that R^e-module by $\epsilon_*^{\mathrm{op}}(M)$; similarly for the latter action, we get the R^e-module $\epsilon_*(M)$. And so we have pairs of functors

$$\begin{cases} \epsilon^* \colon R^e\text{-mod} \rightsquigarrow R^{\mathrm{op}}\text{-mod} \\ \epsilon_* \colon R^{\mathrm{op}}\text{-mod} \rightsquigarrow R^e\text{-mod} \end{cases}$$

and

$$\begin{cases} \epsilon^{*\mathrm{op}} \colon R^e\text{-mod} \rightsquigarrow R\text{-mod} \\ \epsilon_*^{\mathrm{op}} \colon R\text{-mod} \rightsquigarrow R^e\text{-mod} \end{cases}$$

As should be expected, each pair above is a pair of adjoint functors, the upper star is left adjoint to the lower star and we get the following (proof is (DX)):

Proposition 5.26 *If R is a K-algebra with a section $\epsilon\colon R \to K$, then ϵ^* is left-adjoint to ϵ_* and similarly for $\epsilon^{*\mathrm{op}}$ and ϵ_*^{op}. That is, if M is any R^e-module and T and T' are respectively arbitrary R^{op} and R-modules, we have*

$$\begin{aligned} \mathrm{Hom}_{R^{\mathrm{op}}}(\epsilon^* M, T) &\cong \mathrm{Hom}_{R^e}(M, \epsilon_* T) \\ \mathrm{Hom}_R(\epsilon^{*\mathrm{op}} M, T') &\cong \mathrm{Hom}_{R^e}(M, \epsilon_*^{\mathrm{op}} T'). \end{aligned}$$

Lastly, we come to the comparison of the Hochschild groups with their "bar" counterparts. At first, it will be simpler conceptually and notationally (fewer tensor product signs) to pass to a slightly more general case: *R and $\widetilde{R}$ are merely rings and K and $\widetilde{K}$ are chosen modules over R and $\widetilde{R}$ respectively*. In addition we are given module *surjections* $R \xrightarrow{\epsilon} K$ and $\widetilde{R} \xrightarrow{\widetilde{\epsilon}} \widetilde{K}$. By a *map of the pair* $(\widetilde{R}, \widetilde{K})$ *to* (R, K), we understand a ring homomorphism $\varphi\colon \widetilde{R} \to R$ so that $\varphi(\mathrm{Ker}\, \widetilde{\epsilon}) \subseteq \mathrm{Ker}\, \epsilon$. Of course, $\mathrm{Ker}\, \epsilon$ and $\mathrm{Ker}\, \widetilde{\epsilon}$ are just left ideals and we obtain a map of groups, $\overline{\varphi}\colon \widetilde{K} \to K$ and a commutative diagram

$$\begin{tikzcd} \widetilde{R} \arrow[r, "\varphi"] \arrow[d, "\widetilde{\epsilon}"'] & R \arrow[d, "\epsilon"] \\ \widetilde{K} \arrow[r, "\overline{\varphi}"] & K. \end{tikzcd}$$

Now the ring map $\varphi\colon \widetilde{R} \to R$ makes every R-module an $\widetilde{R}$-module (same for R^{op}-modules). So, K *is an $\widetilde{R}$-module, and the diagram shows $\overline{\varphi}$ is an $\widetilde{R}$-module map.*

Suppose $\widetilde{P}_\bullet \longrightarrow \widetilde{K} \longrightarrow 0$ is an $\widetilde{R}$-projective resolution of $\widetilde{K}$ and $P_\bullet \longrightarrow K \longrightarrow 0$ is an R-projective resolution of K. We form $R \otimes_{\widetilde{R}} \widetilde{K}$, then we get an R-module map

$$\theta\colon R \otimes_{\widetilde{R}} \widetilde{K} \longrightarrow K$$

via

$$\theta(r \otimes_{\widetilde{R}} \widetilde{k}) = r\overline{\varphi}(\widetilde{k}).$$

(Note that as $\overline{\varphi}$ is an $\widetilde{R}$-module, this makes sense.) Now the complex $R \otimes_{\widetilde{R}} \widetilde{P}_\bullet$ is R-projective and surjects to $R \otimes_{\widetilde{R}} \widetilde{K}$.

By a slight generalization of Theorem 5.2, our R-module map lifts uniquely in Kom(R-mod) to a map

$$\Theta\colon R\otimes_{\widetilde{R}}\widetilde{P}_\bullet \longrightarrow P_\bullet$$

(over θ, of course). Thus, if M is an R^{op}-module, we get the map on homology

$$H_\bullet(M\otimes_{\widetilde{R}}\widetilde{P}_\bullet)=H_\bullet(M\otimes_R(R\otimes_{\widetilde{R}}\widetilde{P}_\bullet)\longrightarrow H_\bullet(M\otimes_R P_\bullet),$$

while if M is an R-module, we get the map on cohomology

$$H^\bullet(\mathrm{Hom}_R(P_\bullet,M))\longrightarrow H^\bullet(\mathrm{Hom}_R(R\otimes_{\widetilde{R}}\widetilde{P}_\bullet,M))=H^\bullet(\mathrm{Hom}_{\widetilde{R}}(\widetilde{P}_\bullet,M)).$$

But, $H_\bullet(M\otimes_{\widetilde{R}}\widetilde{P}_\bullet)$ computes $\mathrm{Tor}_\bullet^{\widetilde{R}}(M,\widetilde{K})$ (where, M is an $\widetilde{R}^{\mathrm{op}}$-module through φ) and $H_\bullet(M\otimes_R P_\bullet)$ computes $\mathrm{Tor}_\bullet^R(M,K)$. This gives the map of ∂-functors

$$\mathrm{Tor}_\bullet^{\widetilde{R}}(M,\widetilde{K})\longrightarrow \mathrm{Tor}_\bullet^R(M,K).$$

Similarly, in cohomology we get the map of δ-functors

$$\mathrm{Ext}_R^\bullet(K,M)\longrightarrow \mathrm{Ext}_{\widetilde{R}}^\bullet(\widetilde{K},M).$$

Our arguments give the first statement of

Theorem 5.27 *If $\varphi\colon(\widetilde{R},\widetilde{K})\to(R,K)$ is a map of pairs, then there are induced maps of ∂ and δ–functors*

$$H_\bullet(M,\varphi)\colon \mathrm{Tor}_\bullet^{\widetilde{R}}(M,\widetilde{K})\longrightarrow \mathrm{Tor}_\bullet^R(M,K)$$

(for $M\in R^{\mathrm{op}}$-mod), and

$$H^\bullet(M,\varphi)\colon \mathrm{Ext}_R^\bullet(K,M)\longrightarrow \mathrm{Ext}_{\widetilde{R}}^\bullet(\widetilde{K},M)$$

(for $M\in R$-mod).

Moreover, the following three statements are equivalent:

(1) $\begin{cases} \text{a) } \theta\colon R\otimes_{\widetilde{R}}\widetilde{K}\to K \text{ is an isomorphism, and} \\ \text{b) } \mathrm{Tor}_n^{\widetilde{R}}(R,\widetilde{K})=(0) \text{ for } n>0, \end{cases}$

(2) Both maps $H_\bullet(M,\varphi)$ and $H^\bullet(M,\varphi)$ are isomorphisms for all M,

(3) The map $H_\bullet(M,\varphi)$ is an isomorphism for all M.

Proof. (1) $\Longrightarrow$ (2). Write $\widetilde{P}_\bullet\longrightarrow\widetilde{K}\longrightarrow 0$ for a projective resolution of $\widetilde{K}$. Then $R\otimes_{\widetilde{R}}\widetilde{P}_\bullet\longrightarrow R\otimes_{\widetilde{R}}\widetilde{K}\longrightarrow 0$ is an R-projective complex over $R\otimes_{\widetilde{R}}\widetilde{K}$. By (1)b), it is acyclic and by (1)a) we obtain an R-projective resolution of K. Thus, we may choose as R-projective resolution of K the acyclic complex $R\otimes_{\widetilde{R}}\widetilde{P}_\bullet$. But then, Θ is the identity and (2) follows.

(2) $\Longrightarrow$ (3). This is a tautology.

(3) $\Longrightarrow$ (1). We apply the isomorphism $H_\bullet(M,\varphi)$ for $M=R$. This gives us the isomorphism

$$\mathrm{Tor}_\bullet^{\widetilde{R}}(R,\widetilde{K})\xrightarrow{\sim}\mathrm{Tor}_\bullet^R(R,K).$$

We get (1)a) from the case 0 and (1)b) from $n>0$. $\square$

Corollary 5.28 *If $\varphi\colon(\widetilde{R},\widetilde{K})\to(R,K)$ is a map of pairs and conditions (1)a) and b) of Theorem 5.27 hold, then for any $\widetilde{R}$-projective resolution of $\widetilde{K}$, say $\widetilde{P}_\bullet\longrightarrow\widetilde{K}\longrightarrow 0$, the complex $R\otimes_{\widetilde{R}}\widetilde{P}_\bullet$ is an R-projective resolution of K.*

Proof. This is exactly what we showed in (1) $\Longrightarrow$ (2). $\square$

We apply these considerations to the comparison of the Hochschild groups and their bar counterparts. The idea is to cast R^e in the role of $\widetilde{R}$ (and, since $(R^e)^{\mathrm{op}}$ is K-isomorphic to R^e by the map $\tau\colon s^{\mathrm{op}}\otimes r \mapsto r\otimes s^{\mathrm{op}}$, cast $(R^e)^{\mathrm{op}}$ as $\widetilde{R}$, too). The role of $\widetilde{K}$ is played by R for R^e and by R^{op} for $(R^e)^{\mathrm{op}}$. Then R and K are just themselves and, in the op-case, we use R^{op} and K.

Now $\partial_0\colon R^e \to R$, resp. $\partial_0\colon (R^e)^{\mathrm{op}} \to R^{\mathrm{op}}$, by $\partial_0(r\otimes s^{\mathrm{op}}) = rs$, resp. $\partial_0(s^{\mathrm{op}}\otimes r) = s^{\mathrm{op}}r^{\mathrm{op}} = (rs)^{\mathrm{op}}$, is an R^e-module map, resp. an $(R^e)^{\mathrm{op}}$-module map. Moreover, the diagram

$$
\begin{tikzcd}
(R^e)^{\mathrm{op}} \arrow[r, "\sim", "\tau"'] \arrow[d, "\partial_0"'] & R^e \arrow[d, "\partial_0"] \\
R^{\mathrm{op}} \arrow[r, "\mathrm{op}", "="'] & R
\end{tikzcd}
$$

commutes for our formulae for ∂_0. So, we cast ∂_0 as $\widetilde{\epsilon}$. But *we need the map of pairs and this is where our section, ϵ, is essential*. Define $\varphi\colon R^e \to R$ (resp. $(R^e)^{\mathrm{op}} \longrightarrow R^{\mathrm{op}}$) by

$$
\varphi(r\otimes s^{\mathrm{op}}) = r\epsilon(s) \qquad (\text{resp.}\quad \varphi(s^{\mathrm{op}}\otimes r) = s^{\mathrm{op}}\epsilon(r)).
$$

Clearly, φ is a ring homomorphism and as $\mathrm{Ker}\,\widetilde{\epsilon}$ is generated by $r\otimes 1 - 1\otimes r^{\mathrm{op}}$ (resp. $r^{\mathrm{op}}\otimes 1 - 1\otimes r$), we find $\varphi(\mathrm{Ker}\,\widetilde{\epsilon}) \subseteq \mathrm{Ker}\,\epsilon$. There results the commutative diagram of the map of pairs:

$$
\begin{tikzcd}
(R^e)^{\mathrm{op}} \arrow[rrr, "\varphi"] \arrow[dr, "\cong", "\tau"'] \arrow[dd, "\partial_0=\widetilde{\epsilon}"'] & & & R^{\mathrm{op}} \arrow[dd, "\epsilon"] \\
& R^e \arrow[r, "\varphi"] \arrow[dl, "\partial_0=\widetilde{\epsilon}"] & R \arrow[dr, "\epsilon"] & \\
R^{\mathrm{op}} = R \arrow[rrr, "\epsilon"] & & & K
\end{tikzcd}
$$

Now consider an R-module, M (resp. an R^{op}-module, M), how does φ make M an R^e (resp. $(R^e)^{\mathrm{op}}$)-module? This way:

$$
\begin{aligned}
(r\otimes s^{\mathrm{op}})\cdot m &= \varphi(r\otimes s^{\mathrm{op}})\cdot m = r\epsilon(s)m \\
(\text{resp.}\quad (s^{\mathrm{op}}\otimes r)\cdot m &= \varphi(s^{\mathrm{op}}\otimes r)\cdot m = s^{\mathrm{op}}\epsilon(r)\cdot m = ms\epsilon(r)).
\end{aligned}
$$

That is, the R-module, M, goes over to the R^e-module $\epsilon_*^{\mathrm{op}}(M)$ and the R^{op}-module, M, goes over to the $(R^e)^{\mathrm{op}}$-module $\epsilon_*(M)$. Therefore, the map of pairs yields the *comparison maps*

$$
\begin{aligned}
&H_\bullet(M,\varphi)\colon H_\bullet(R,\epsilon_*(M)) = \mathrm{Tor}_\bullet^{R^e}(\epsilon_*(M),R) \longrightarrow \mathrm{Tor}_\bullet^{R}(M,K) = \overline{H}_\bullet(R,M)\\
&H^\bullet(M,\varphi)\colon \overline{H}^\bullet(R,M) = \mathrm{Ext}_R^\bullet(K,M) \longrightarrow \mathrm{Ext}_{R^e}^\bullet(R,\epsilon_*^{\mathrm{op}}(M)) = H^\bullet(R,\epsilon_*^{\mathrm{op}}(M)).
\end{aligned}
$$

Theorem 5.29 *If R is K-projective, then the comparison maps*

$$
H_\bullet(M,\varphi)\colon H_\bullet(R,\epsilon_*(M)) \longrightarrow \overline{H}_\bullet(R,M)
$$

and

$$
H^\bullet(M,\varphi)\colon \overline{H}^\bullet(R,M) \longrightarrow H^\bullet(R,\epsilon_*^{\mathrm{op}}(M))
$$

are isomorphisms of ∂ (resp. δ)-functors. Moreover, if $\widetilde{P}_\bullet \longrightarrow R \longrightarrow 0$ is an R^e-projective resolution of R, then $\widetilde{P}_\bullet \otimes_R K$ is an R-projective resolution of K.

Proof. Everything will follow from Theorem 5.27 once we verify conditions (1)a) and b) of that theorem. Here, there is the non-commutativity of R that might cause some confusion. Recall that R^e operates *on the right* on a module, N, *via*

$$n \cdot (r \otimes s^{\mathrm{op}}) = snr;$$

so, R^e operates on $\epsilon_* N$ *via*

$$n \cdot (r \otimes s^{\mathrm{op}}) = \epsilon(s)nr.$$

We apply this when N is R–itself and M is any two-sided R-module. For $\rho \otimes_{R^e} m \in \epsilon_* R \otimes_{R^e} M$, we observe that

$$\epsilon(s)\rho r \otimes_{R^e} m = \rho \cdot (r \otimes s^{\mathrm{op}}) \otimes_{R^e} m = \rho \otimes_{R^e} rms; \tag{$*$}$$

hence, the map

$$\alpha\colon \epsilon_* R \otimes_{R^e} M \longrightarrow M \otimes_R K$$

via

$$\alpha(\rho \otimes_{R^e} m) = \rho m \otimes_R 1$$

is well–defined. The only (mildly) tricky thing to check is that α preserves relation $(*)$. But, α of the left side of $(*)$ is $\epsilon(s)\rho r m \otimes_R 1$ and α of the right side of $(*)$ is $\rho rms \otimes_R 1$. Now,

$$zs \otimes_R 1 = z \otimes_R \epsilon(s) = z\epsilon(s) \otimes_R 1;$$

so, α agrees on the left and right sides of $(*)$. And now we see that α is an isomorphism of K-modules because the map

$$\beta\colon M \otimes_R K \longrightarrow \epsilon_* R \otimes_{R^e} M$$

via

$$\beta(m \otimes_R \kappa) = \kappa \otimes_{R^e} m$$

is its inverse. (Note that β is well-defined for:

$$m\rho \otimes_R \kappa = m \otimes_R \epsilon(\rho)\kappa$$

and

$$\epsilon(\rho)\kappa \otimes_{R^e} m = \kappa \cdot (1 \otimes \rho^{\mathrm{op}}) \otimes_{R^e} m = \kappa \otimes_{R^e} m\rho = \beta(m\rho \otimes_R \kappa),$$

while

$$\epsilon(\rho)\kappa \otimes_{R^e} m = \beta(m \otimes_R \epsilon(\rho)\kappa), \qquad \text{as required.)}$$

However,

$$\alpha\beta(m \otimes_R \kappa) = \alpha(\kappa \otimes_{R^e} m) = \kappa m \otimes_R 1 = m \otimes_R \kappa$$

$$\beta\alpha(\rho \otimes_{R^e} m) = \beta(\rho m \otimes_R 1) = 1 \otimes_{R^e} \rho m = \rho \otimes_{R^e} m.$$

We can now apply the K-module isomorphism α. First, take $M = R\,(= \widetilde{K})$. We find that

$$\alpha\colon \epsilon_* R \otimes_{R^e} R \xrightarrow{\sim} R \otimes_R K = K$$

and $\epsilon_* R$ is just R as $\widetilde{R}\,(= R^e\text{-module})$. This gives (1)a). To see (1)b), take $\widetilde{P}_\bullet \longrightarrow R \longrightarrow 0$ an $R^e\,(= \widetilde{R})$-projective resolution. We choose $M = \widetilde{P}_\bullet$ an R^e-module (i.e., a complex of same). Now apply α:

$$\mathrm{Tor}_\bullet^{\widetilde{R}}(R, \widetilde{K}) = \mathrm{Tor}_\bullet^{R^e}(\epsilon_* R, R) = H_\bullet(\epsilon_* R \otimes_{R^e} \widetilde{P}_\bullet) \underset{\alpha}{\xrightarrow{\sim}} H_\bullet(\widetilde{P}_\bullet \otimes_R K).$$

But, R is K-projective and so (by the usual arguments (DX)) $\widetilde{P}_\bullet$ is R^{op}-projective which means the last homology complex computes $\mathrm{Tor}_\bullet^R(R, K)$. We've shown

$$\mathrm{Tor}_n^{\widetilde{R}}(R, \widetilde{K}) \underset{\alpha}{\xrightarrow{\sim}} \mathrm{Tor}_n^R(R, K).$$

Yet, R is free (so flat) over R and so $\mathrm{Tor}_n^R(R, K) = (0)$ when $n > 0$; we are done. $\square$

Of course, we should apply all this to the standard resolution, $C_\bullet(R)$, when R is K-projective. Here,

$$C_n(R) \otimes_R K = R \otimes_K C_n[R] \otimes_K R \otimes_R K \xrightarrow{\sim} R \otimes_K C_n[R]$$

via the map

$$\Theta_n(r_0 \otimes \cdots \otimes r_{n+1} \otimes_R \kappa) = \epsilon(r_{n+1})\kappa(r_0 \otimes \cdots \otimes r_n).$$

As in the proof of Theorem 5.25, the standard boundary map induces a boundary map, $\overline{\partial}_n$, on $R \otimes_K C_n[R]$, by the formula $\overline{\partial}_n = \Theta_{n-1} \circ \partial_n \circ \Theta_n^{-1}$, and we find

$$\overline{\partial}_n(r_0 \otimes r_1 \otimes \cdots \otimes r_n) = \sum_{i=0}^{n-1} (-1)^i r_0 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n + (-1)^n \epsilon(r_n) r_0 \otimes \cdots \otimes r_{n-1}.$$

This gives us our R-projective resolution $R \otimes_K C_\bullet[R] \longrightarrow K \longrightarrow 0$ with which we can compute. The case when $r_0 = 1$ is most important:

$$\overline{\partial}_n(1 \otimes r_1 \otimes \cdots \otimes r_n) = r_1 \otimes \cdots \otimes r_n + \sum_{i=1}^{n-1} (-1)^i r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n + (-1)^n \epsilon(r_n)(1 \otimes r_1 \otimes \cdots \otimes r_{n-1}).$$

Now, for a right R-module, M, the groups $\mathrm{Tor}_\bullet^R(M, K)$ are the homology of

$$M \otimes_R R \otimes_K C_\bullet[R] = M \otimes_K C_\bullet[R]$$

under $1 \otimes_R \overline{\partial}$. We find

$$\begin{aligned}
\overline{\partial}_n(m \otimes_K r_1 \otimes_K \cdots \otimes_K r_n) &= m r_1 \otimes_K r_2 \otimes_K \cdots \otimes_K r_n \\
&\quad + \sum_{i=1}^{n-1} (-1)^i m \otimes_K r_1 \otimes_K \cdots \otimes_K r_i r_{i+1} \otimes_K \cdots \otimes_K r_n \\
&\quad + (-1)^n \epsilon(r_n) m \otimes_K r_1 \otimes_K \cdots \otimes_K r_{n-1}.
\end{aligned}$$

Therefore, we recover Hochschild's homology formula for $\epsilon_*(M)$, and when $R = \mathbb{Z}[G]$ and $K = \mathbb{Z}$ (with $\epsilon(\sigma) = 1$, all $\sigma \in G$) we also recover the explicit boundary formula for $H_\bullet(G, M)$.

For a left R-module, M, the groups $\mathrm{Ext}_R^\bullet(K, M)$ are the cohomology of

$$\mathrm{Hom}_R(R \otimes_R C_\bullet[R], M) = \mathrm{Hom}_K(C_\bullet[R], M).$$

If, as usual, we write $f(r_1, \ldots, r_n)$ for $f(r_1 \otimes_K r_2 \otimes_K \cdots \otimes_K r_n)$, then

$$\begin{aligned}
(\overline{\delta}_n f)(r_1, \ldots, r_{n+1}) &= r_1 f(r_2, \ldots, r_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1)^i f(r_1, \ldots, r_i r_{i+1}, \ldots, r_n) \\
&\quad + (-1)^{n+1} \epsilon(r_{n+1}) f(r_1, \ldots, r_n).
\end{aligned}$$

Here, $f \in \mathrm{Hom}_K(C_n[R], M)$. Once again, we recover Hochschild's cohomology formula for $\epsilon_*^{\mathrm{op}}(M)$, and when $R = \mathbb{Z}[G]$, *etc.*, we get our explicit coboundary formula for $H^\bullet(G, M)$.

But, we've done more; all this applies to any K-algebra, R, with a section (especially for K-projective algebras). In particular, we might apply it to $R = K[T_1, \ldots, T_n]$ or $R = K\langle T_1, \ldots, T_n \rangle$, with $\epsilon(T_j) = 0$ for $j = 1, 2, \ldots, n$. The standard resolution though is very inefficient for we must know $m \otimes r_1 \otimes \cdots \otimes r_l$

or $f(r_1, \ldots, r_l)$ on *all* monomials r_j of whatever degree. Instead we will find a better resolution, but we postpone this until Section 5.5 where it fits better.

Let us turn to the cohomology of sheaves and presheaves. These objects have been introduced already and we assume that Problem 69 has been mastered. Here, we'll be content to examine ordinary topological spaces (as in part (a) of that exercise) and (pre)sheaves on them. The most important fact is that the categories of presheaves and sheaves of R-modules on the space X have enough injectives. Let us denote by $\mathcal{P}(X, R\text{-mod})$ and $\mathcal{S}(X, R\text{-mod})$ these two abelian categories. Remember that

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

is exact in $\mathcal{P}(X, R\text{-mod})$ iff the sequence of R-modules

$$0 \longrightarrow F'(U) \longrightarrow F(U) \longrightarrow F''(U) \longrightarrow 0$$

is exact for *every* open U of X. But for sheaves, the situation is more complicated:

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

is exact in $\mathcal{S}(X, R\text{-mod})$ iff

(a) $0 \longrightarrow F'(U) \longrightarrow F(U) \longrightarrow F''(U)$ is exact for every open U or X *and*

(b) For each open U and each $\xi \in F''(U)$, there is an open cover $\{U_\alpha \longrightarrow U\}_\alpha$ so that each ξ_α $(= \rho_U^{U_\alpha}(\xi))$ is the image of some $\eta_\alpha \in F(U_\alpha)$ under the map $F(U_\alpha) \longrightarrow F''(U_\alpha)$.

A more perspicacious way of saying this is the following: Write $i\colon \mathcal{S}(X, R\text{-mod}) \rightsquigarrow \mathcal{P}(X, R\text{-mod})$ for the full embedding which regards a sheaf as a presheaf. There is a functor, $\#\colon \mathcal{P}(X, R\text{-mod}) \rightsquigarrow \mathcal{S}(X, R\text{-mod})$ which is left adjoint to i. That is, for $F \in \mathcal{S}$ and $G \in \mathcal{P}$, we have

$$\mathrm{Hom}_{\mathcal{S}}(G^{\#}, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{P}}(G, i(F)).$$ [6]

We can now say (b) this way: If $\mathrm{cok}(F \longrightarrow F'')$ is the presheaf cokernel

$$\mathrm{cok}(F \longrightarrow F'')(U) = \mathrm{cok}(F(U) \longrightarrow F''(U)),$$

then $\mathrm{cok}(F \longrightarrow F'')^{\#} = (0)$.

Given $x \in X$, and a (pre)sheaf, F, we define the *stalk of F at x*, denoted F_x, by

$$F_x = \varinjlim_{\{U \ni x\}} F(U).$$

It's easy to see that $(F^{\#})_x = F_x$ for any presheaf, F. Stalks are important because of the following simple fact:

Proposition 5.30 *If $F \xrightarrow{\varphi} G$ is a map of sheaves, then φ is injective (surjective, bijective) if and only if the induced map $\varphi_x\colon F_x \to G_x$ on stalks is injective (surjective, bijective) for every $x \in X$.*

We leave the proof as a (DX).

[6] One constructs $\#$ by two successive limits. Given U, open in X, write $G^{(+)}(U)$ for

$$G^{(+)}(U) = \varinjlim_{\{U_\alpha \longrightarrow U\}} \mathrm{Ker}\,\Big(\prod_\alpha G(U_\alpha) \rightrightarrows \prod_{\beta,\gamma} G(U_\beta \cap U_\gamma)\Big)$$

(the limit taken over all open covers of U) and set $G^{\#}(U) = G^{(+)(+)}(U)$.

This result is false for presheaves, they are not local enough.

Property (a) above shows that i is left-exact and the proposition shows $\#$ is exact. To get at the existence of enough injectives, we investigate what happens to $\mathcal{P}(X, R\text{-mod})$ and $\mathcal{S}(X, R\text{-mod})$ if we have a map of spaces $f\colon X \to Y$. In the first place, if F is a (pre)sheaf on X, we can define a (pre)sheaf f_*F, called the *direct image of F by f via*

$$(f_*F)(V) = F(f^{-1}(V)), \quad V \text{ open in } Y.$$

A simple check shows that the direct image of a sheaf is again a sheaf. Now, in the second place, we want a functor $f^*\colon \mathcal{P}(Y) \rightsquigarrow \mathcal{P}(X)$ (resp. $\mathcal{S}(Y) \rightsquigarrow \mathcal{S}(X)$) which will be left adjoint to f_*. If we knew the (classical) way to get a sheaf from its stalks, we could set $(f^*G)_x = G_{f(x)}$ for $G \in \mathcal{S}(Y)$ and $x \in X$ any point. But from our present point of view this can't be done. However, our aim is for an adjoint functor, so we can use the method of D. Kan [30].

We start we a presheaf, G, on Y and take an open set, U, of X. We set

$$(f^*G)(U) = \varinjlim_{\{f^{-1}(V) \supseteq U\}} G(V),$$

here, as noted, V ranges over all opens of Y with $f^{-1}(V) \supseteq U$. Then, f^*G is a presheaf (of R-modules) on X. If G is a sheaf on Y, we form f^*G, as above, and then take $(f^*G)^{\#}$. We'll continue to denote the latter sheaf by f^*G if no confusion results. Once the idea of defining f^*G by a direct limit is in hand, it is easy to prove (and the proof will be left as a (DX)):

Proposition 5.31 *If $f\colon X \to Y$ is a map of topological spaces, then the functors f^* from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$ (resp. from $\mathcal{S}(Y)$ to $\mathcal{S}(X)$) are left adjoint to the direct image functors. That is, for $G \in \mathcal{P}(Y)$ and $F \in \mathcal{P}(X)$ (resp. $G \in \mathcal{S}(Y)$ and $F \in \mathcal{S}(X)$), we have functorial isomorphisms*

$$\mathrm{Hom}_{\mathcal{P}(X)}(f^*G, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{P}(Y)}(G, f_*F)$$

(resp.

$$\mathrm{Hom}_{\mathcal{S}(X)}(f^*G, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{S}(Y)}(G, f_*F)).$$

Moreover, we have $(f^*G)_x = G_{f(x)}$, for all $x \in X$.

Since $\varinjlim$ is an exact functor on R-mod, our definition of the presheaf f^*G shows that f^* *is an exact functor* $\mathcal{P}(Y) \rightsquigarrow \mathcal{P}(X)$. The statement in the proposition about stalks shows (by Proposition 5.30) that f^* *is also an exact functor* $\mathcal{S}(Y) \rightsquigarrow \mathcal{S}(X)$. Of course, f_* *is a left-exact functor on sheaves and an exact functor on presheaves.*

There is a useful lemma that connects pairs of adjoint functors and injectives—it is what we'll use to get enough injectives in $\mathcal{P}$ and $\mathcal{S}$.

Lemma 5.32 *Say $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $\alpha\colon \mathcal{A} \rightsquigarrow \mathcal{B}$ and $\beta\colon \mathcal{B} \rightsquigarrow \mathcal{A}$ are functors with β left adjoint to α. If β is exact, then α carries injectives of $\mathcal{A}$ to injectives of $\mathcal{B}$.*

Proof. Take an injective, Q, of $\mathcal{A}$ and consider the co-functor (on $\mathcal{B}$)

$$T \rightsquigarrow \mathrm{Hom}_{\mathcal{B}}(T, \alpha(Q)).$$

By adjointness, this is exactly

$$T \rightsquigarrow \mathrm{Hom}_{\mathcal{A}}(\beta(T), Q).$$

Now, $\mathrm{Hom}_{\mathcal{A}}(\beta(-), Q)$ is the composition of the exact functor β with the exact functor $\mathrm{Hom}_{\mathcal{A}}(-, Q)$ (the latter being exact as Q is injective). But then, $\mathrm{Hom}_{\mathcal{B}}(-, \alpha(Q))$ is exact, i.e., $\alpha(Q)$ is injective in $\mathcal{B}$. $\square$

If we apply the lemma to the cases $\alpha = i$, $\beta = \#$; $\alpha = f_*$, $\beta = f^*$, we get

Corollary 5.33 *Let $f\colon X \to Y$ be a map of topological spaces and write $\mathcal{P}(X)$, etc. for the categories of R-module presheaves on X, etc. Further consider the functors $i\colon \mathcal{S}(X) \rightsquigarrow \mathcal{P}(X)$ and $\#\colon \mathcal{P}(X) \rightsquigarrow \mathcal{S}(X)$. Then,*

(1) If Q is an injective in $\mathcal{P}(X)$, the presheaf $f_*(Q)$ is injective on Y.

(2) If Q is an injective in $\mathcal{S}(X)$, the sheaf $f_*(Q)$ is injective on Y.

(3) If Q is an injective sheaf on X, then $i(Q)$ is an injective presheaf on X.

Theorem 5.34 *If X is a topological space, then the category $\mathcal{S}(X, R\text{-mod})$ possesses enough injectives.*

Proof. Pick any point, ξ, of X and consider the map of spaces $i_\xi\colon \{\xi\} \hookrightarrow X$. The categories $\mathcal{P}(\{\xi\})$ and $\mathcal{S}(\{\xi\})$ are each just R-mod, and for any module, M, we have

$$i_{\xi *}(M)(U) = \begin{cases} M & \text{if } \xi \in U \\ (0) & \text{if } \xi \notin U. \end{cases}$$

For any sheaf F on X, look at its stalk, F_ξ, at ξ and embed F_ξ into an injective R-module Q_ξ (say $j_\xi\colon F_\xi \hookrightarrow Q_\xi$ is the embedding). We form $i_{\xi *}(Q_\xi)$ which is an injective sheaf on X by Corollary 5.33 and then form $Q = \prod_{\xi \in X} i_{\xi *}(Q_\xi)$, again an injective sheaf on X. Note that

$$Q(U) = \prod_{\xi \in U} Q_\xi.$$

Now, I claim that the map $\theta\colon F \to Q$ *via*

$$\text{for } z \in F(U)\colon \theta(z) = (j_\xi(z_\xi))_{\xi \in U},$$

where z_ξ is the image of z in F_ξ, is the desired embedding. If $\theta(z) = 0$, then for each $\xi \in U$, the elements $j_\xi(z_\xi) = 0$; as j_ξ is an embedding, we get $z_\xi = 0$. By the definition of stalk, there is a neighborhood, U_ξ, of ξ in U where $\rho_U^{U_\xi}(z) = 0$. These neighborhoods give a covering of U, so we see that z goes to zero under the map

$$F(U) \longrightarrow \prod_{\xi \in U} F(U_\xi). \tag{+}$$

But, this map is injective by the sheaf axiom; so, $z = 0$. $\square$

Remark: The theorem is also true for presheaves and our proof above works for "good" presheaves; that is, those for which the maps (+) are indeed injective. (For general presheaves, G, the presheaf $G^{(+)}$ will satisfy (+) is injective). We can modify the argument to get the result for $\mathcal{P}(X)$ or use a different argument; this will be explored in the exercises.

To define cohomology with coefficients in a sheaf, F, on X, we consider the functor

$$\Gamma\colon F \rightsquigarrow F(X).$$

We already know this is left exact and we define *the cohomology of X with coefficients in F* by

$$H^\bullet(X, F) = (R^\bullet\Gamma)(F).$$

A little more generally, if U is open in X, we can set $\Gamma_U(F) = F(U)$ and then

$$H^\bullet(U, F) = (R^\bullet\Gamma_U)(F).$$

If we assume proved the existence of enough injectives in $\mathcal{P}(X)$, then for a presheaf, G, we set

$$\check{H}^0(X,G) = G^{(+)}(X) = \varinjlim_{\{U_\alpha \longrightarrow U\}} \mathrm{Ker}\,(\prod_\alpha G(U_\alpha) \rightrightarrows \prod_{\beta,\gamma} G(U_\beta \cap U_\gamma))$$

and define

$$\check{H}^\bullet(X,G) = (R^\bullet \check{H}^0)(G).$$

There is an explicit complex that computes $\check{H}^\bullet(X,G)$, see the exercises. The R-modules $\check{H}^\bullet(X,G)$ are called the *Čech cohomology groups of X with coefficients in the presheaf* G. Again, as above, we can generalize to cohomology over an open, U.

Pick open $U \subseteq X$, and write $\mathfrak{R}_U$ for the presheaf

$$\mathfrak{R}_U(V) = \begin{cases} R & \text{if } V \subseteq U \\ (0) & \text{if } V \not\subseteq U \end{cases}$$

(so, $\mathfrak{R}_X$ is the constant presheaf, R). Also write R_U for the sheaf $(\mathfrak{R}_U)^\#$. It turns our that the $\mathfrak{R}_U$ form a set of generators for $\mathcal{P}(X)$, while the same is true for the R_U in $\mathcal{S}(X)$. Moreover, we have

Proposition 5.35 *If X is a topological space and U is a given open set, then we have an isomorphism of δ-functors*

$$H^\bullet(U,-) \cong \mathrm{Ext}^\bullet_{\mathcal{S}(X)}(R_U,-)$$

*on the category $\mathcal{S}(X)$ to R-*mod.

Proof. All we have to check is that they agree in dimension 0. Now,

$$\mathrm{Hom}_{\mathcal{S}(X)}(R_U,F) \cong \mathrm{Hom}_{\mathcal{P}(X)}(\mathfrak{R}_U, i(F)).$$

Notice that $\rho^V_U\colon \mathfrak{R}(U) \to \mathfrak{R}(V)$ is just the identity if $V \subseteq U$ and is the zero map otherwise. It follows that

$$\mathrm{Hom}_{\mathcal{P}(X)}(\mathfrak{R}_U, i(F)) \cong \mathrm{Hom}_{R\text{-mod}}(R, F(U)) = F(U),$$

and we are done. $\square$

We don't compute $\mathrm{Ext}^\bullet_{\mathcal{S}(X)}(R_U,F)$ by projectively resolving R_U—such a resolution doesn't exist in $\mathcal{S}(X)$. Rather, we injectively resolve F.

Recall that we have the left exact functor $i\colon \mathcal{S}(X) \rightsquigarrow \mathcal{P}(X)$, so we can inquire as to its right derived functors, $R^\bullet i$. The usual notation for $(R^\bullet i)(F)$ is $\mathcal{H}^\bullet(F)$—these are presheaves. We compute them as follows:

Proposition 5.36 *The right derived functors $\mathcal{H}^\bullet(F)$ are given by*

$$\mathcal{H}^\bullet(F)(U) = H^\bullet(U,F).$$

Proof. It should be clear that for fixed F, each $H^p(U,F)$ is functorial in U; that is, $U \rightsquigarrow H^p(U,F)$ is a presheaf. Moreover, it is again clear that

$$F \rightsquigarrow H^\bullet(U,F)$$

is a δ-functor from $\mathcal{S}(X)$ to $\mathcal{P}(X)$. If Q is injective in $\mathcal{S}(X)$, we have $H^p(U,Q) = (0)$ when $p > 0$; so, our δ-functor is effaceable. But, for $p = 0$, the R-module $H^0(U,F)$ is just $F(U)$; i.e., it is just $\mathcal{H}^0(F)(U)$. By the uniqueness of universal δ-functors, we are done. $\square$

For the computation of the cohomology of sheaves, manageable injective resolutions turn out to be too hard to find. Sometimes one can prove cohomology can be computed by the Čech method applied to $i(F)$, and then the explicit complex of the exercises works quite well. This will depend on subtle properties of the space, X. More generally, Godement [18] showed that a weaker property than injectivity was all that was needed in a resolution of F to compute the R-module $H^\bullet(X,F)$. This is the notion of flasqueness.[7]

[7]The French word "flasque" can be loosely translated as "flabby".

Definition 5.8 A sheaf, F, on the space X is *flasque* if and only if for each pair of opens $V \subseteq U$ of X, the map
$$\rho_U^V \colon F(U) \to F(V)$$
is surjective. Of course, this is the same as $F(X) \longrightarrow F(U)$ being surjective for each open, U.

Here are two useful lemmas that begin to tell us how flasque sheaves intervene in cohomology.:

Lemma 5.37 *The following are equivalent statements about a sheaf, F', on the space X:*

(1) Every short exact sequence in $\mathcal{S}(X)$
$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$
is exact in $\mathcal{P}(X)$.

(2) For every open U of X, the R-module $H^1(U, F')$ is zero.

Proof. (1) $\Longrightarrow$ (2). Embed F' in an injective and pick open $U \subseteq X$. From $0 \longrightarrow F' \longrightarrow Q \longrightarrow \mathrm{cok} \longrightarrow 0$, we get
$$0 \longrightarrow F'(U) \longrightarrow Q(U) \longrightarrow \mathrm{cok}(U) \longrightarrow H^1(U, F') \longrightarrow (0);$$
By (1), $0 \longrightarrow F'(U) \longrightarrow Q(U) \longrightarrow \mathrm{cok}(U) \longrightarrow 0$ is exact; so, $H^1(U, F') = (0)$.

(2) $\Longrightarrow$ (1). Just apply cohomology over U to the short exact sequence $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$. We get
$$0 \longrightarrow F'(U) \longrightarrow F(U) \longrightarrow F''(U) \longrightarrow H^1(U, F').$$
By (2), we're done as U is an arbitrary open. $\square$

Lemma 5.38 *Say F' is a flasque sheaf and $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ is exact in $\mathcal{S}(X)$. Then it is exact in $\mathcal{P}(X)$. Moreover, if $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ is exact in $\mathcal{S}(X)$, then F is flasque if and only if F'' is flasque (of course, F' is always assumed to be flasque).*

Proof. Pick any open $U \subseteq X$; all we must prove is that $F(U) \longrightarrow F''(U)$ is surjective. Write Σ for the collection of pairs (V, σ) where V is open, $V \subseteq U$ and σ is a lifting to $F(V)$ of $\rho_U^V(s) \in F''(V)$ for some $s \in F''(U)$ fixed once and for all. As s admits liftings to F *locally on* U, our set Σ is non-empty. Now partially order Σ in the standard way: $(V, \sigma) \leq (\widetilde{V}, \widetilde{\sigma})$ iff $V \subseteq \widetilde{V}$ and $\rho_{\widetilde{V}}^V(\widetilde{\sigma}) = \sigma$. Of course, Σ is inductive and Zorn's Lemma yields a maximal lifting, σ_0, of s defined on $V_0 \subseteq U$. We must prove $V_0 = U$.

Were it not, there would exist $\xi \in U$ with $\xi \notin V_0$. Now the stalk map $F_\xi \longrightarrow F''_\xi$ is surjective, so the image of s in some small neighborhood, $U(\xi)$, of ξ in U lifts to an element $\tau \in F(U(\xi))$. We will get an immediate contradiction if $U(\xi) \cap V_0 = \emptyset$, for then $\widetilde{U} = U(\xi) \cup V_0$ has two opens as a *disjoint* cover and $F(\widetilde{U}) = F(U(\xi)) \prod F(V_0)$ by the sheaf axiom. The pair $\langle \tau, \sigma_0 \rangle$ is a lifting of s to a bigger open than V_0—a contradiction.

Therefore, we may assume $U(\xi) \cap V_0 \neq \emptyset$—it is here that the flasqueness of F' enters. For on the intersection both $\rho_{U(\xi)}^{U(\xi) \cap V_0}(\tau)$ and $\rho_{V_0}^{U(\xi) \cap V_0}(\sigma_0)$ lift $\rho_U^{U(\xi) \cap V_0}(s)$. Thus, there is an "error" $\epsilon \in F'(U(\xi) \cap V_0)$, so that
$$\rho_{V_0}^{U(\xi) \cap V_0}(\sigma_0) - \rho_{U(\xi)}^{U(\xi) \cap V_0}(\tau) = \epsilon.$$
As F' is flasque, ϵ lifts to $F'(U(\xi))$; call it ϵ again on this bigger open. Then $\tau + \epsilon$ also lifts $\rho_U^{U(\xi)}(s)$ and $\tau + \epsilon$ and σ_0 now agree on $U(\xi) \cap V_0$; so, the sheaf axiom shows we get a lifting to the bigger open $U(\xi) \cup V_0$—our last contradiction. Thus, $U = V_0$.

For the second statement, in which F' is given as a flasque sheaf, pick open $V \subseteq U$ in X. We have the commutative diagam

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F'(U) & \longrightarrow & F(U) & \longrightarrow & F''(U) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \rho'} & & \big\downarrow{\scriptstyle \rho} & & \big\downarrow{\scriptstyle \rho''} & & \\ 0 & \longrightarrow & F'(V) & \longrightarrow & F(V) & \longrightarrow & F''(V) & \longrightarrow & 0, \end{array}$$

in which $\mathrm{Coker}\,(\rho') = (0)$, By the snake lemma

$$\mathrm{Coker}\,(\rho) \xrightarrow{\sim} \mathrm{Coker}\,(\rho''),$$

and we are done. $\square$

Remark: There is an important addendum to Lemma 5.37. We mention this as the method of argument is fundamental in many applications. This addendum is: *The statement*

(3) $\check{H}^1(U, i(F')) = (0)$ for all U open in X, is equivalent to either properties (1) or (2) of Lemma 5.37.

Let us show (3) $\Longleftrightarrow$ (1). So say (3) holds. This means given any open cover of U, say $U = \bigcup_\alpha U_\alpha$, and any elements $z_{\alpha\beta} \in F'(U_\alpha \cap U_\beta)$ so that

$$z_{\alpha\beta} = -z_{\beta\alpha} \quad \text{and} \quad z_{\alpha\gamma} = z_{\alpha\beta} + z_{\beta\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \tag{$*$}$$

we can find elements $z_\alpha \in F'(U_\alpha)$ so that $z_{\alpha\beta} = z_\alpha - z_\beta$ on $U_\alpha \cap U_\beta$. Now suppose we have $s \in F''(U_\alpha)$, we can cover U by opens U_α so that the $s_\alpha = \rho_U^{U_\alpha}(s) \in F''(U_\alpha)$ lift to $\sigma_\alpha \in F(U_\alpha)$ for all α. The elements $\sigma_\alpha - \sigma_\beta \in F(U_\alpha \cap U_\beta)$ are not necessarily 0 but go to zero in $F''(U_\alpha \cap U_\beta)$. That is, if we set $z_{\alpha\beta} = \sigma_\alpha - \sigma_\beta$, the $z_{\alpha\beta}$ belong to $F'(U_\alpha \cap U_\beta)$. These $z_{\alpha\beta}$ satisfy $(*)$ and so by (3) we get $z_{\alpha\beta} = z_\alpha - z_\beta$ for various $z_\alpha \in F'(U_\alpha)$. Thus

$$z_\alpha - z_\beta = \sigma_\alpha - \sigma_\beta \quad \text{on } U_\alpha \cap U_\beta,$$

that is

$$\sigma_\alpha - z_\alpha = \sigma_\beta - z_\beta \quad \text{on } U_\alpha \cap U_\beta.$$

This equality and the sheaf axiom for F give us an element $\sigma \in F(U)$ with $\rho_U^{U_\alpha}(\sigma) = \sigma_\alpha - z_\alpha$. The z_α go to zero in F'', thus σ lifts s and this shows $F(U) \longrightarrow F''(U)$ is surjective.

To show (1) $\Longrightarrow$ (3), we simply embed F' in an injective again to get $0 \longrightarrow F' \longrightarrow Q \longrightarrow \mathrm{cok} \longrightarrow 0$ in $\mathcal{S}(X)$. By (1), the sequence

$$0 \longrightarrow i(F') \longrightarrow i(Q) \longrightarrow i(\mathrm{cok}) \longrightarrow 0$$

is exact in $\mathcal{P}(X)$ and $i(Q)$ is an injective of $\mathcal{P}(X)$. Apply Čech cohomology (a δ-functor on $\mathcal{P}(X)$):

$$0 \longrightarrow F'(U) \longrightarrow Q(U) \longrightarrow \mathrm{cok}(U) \longrightarrow \check{H}^1(U, i(F')) \longrightarrow 0$$

is exact. Since $Q(U) \longrightarrow \mathrm{cok}(U)$ is surjective, by (1), we get (3). $\square$

Proposition 5.39 *Every injective sheaf is a flasque sheaf. For every flasque sheaf, F, and every open U, we have $H^n(U, F) = (0)$ for $n > 0$.*

Proof. Pick open $V \subseteq U$, call our injective sheaf Q. Since $V \subseteq U$, we have the exact sequence

$$0 \longrightarrow R_V \longrightarrow R_U \longrightarrow \mathrm{cok} \longrightarrow 0$$

in $\mathcal{S}(X)$. Now $\mathrm{Hom}_{\mathcal{S}(X)}(-, Q)$ is an exact functor; so

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{S}(X)}(\mathrm{cok}, Q) \longrightarrow \mathrm{Hom}_{\mathcal{S}(X)}(R_U, Q) \longrightarrow \mathrm{Hom}_{\mathcal{S}(X)}(R_V, Q) \longrightarrow 0$$

is exact. The middle and right-hand groups are $Q(U)$ and $Q(V)$ respectively; so Q is flasque.

Let's prove the second statement by induction on n. Take a flasque sheaf, F. By Lemmas 5.37 and 5.38, we have $H^1(U, F) = (0)$ for all open U. Now embed F in an injective sheaf

$$0 \longrightarrow F \longrightarrow Q \longrightarrow \mathrm{cok} \longrightarrow 0.$$

By Lemma 5.38 and the above, the sheaf cok is also flasque. So by our induction hypothesis all the groups $H^p(U, \mathrm{cok}) = (0)$ for $1 \leq p \leq n-1$. Now apply cohomology to our exact sequence; this gives

$$H^p(U, \mathrm{cok}) \stackrel{\sim}{\longrightarrow} H^{p+1}(U, F), \quad \text{for } p \geq 1.$$

When $p = n-1$ this gives the induction step for F. $\square$

The important part of this proposition is that *flasque sheaves are acyclic objects over every open U of X*; that is, $H^n(U, F) = (0)$ for all $n > 0$ and every open U. And now we have the following general

Proposition 5.40 *Suppose $\mathcal{A}$ is an abelian category, T is a left exact functor from $\mathcal{A}$ to R-*mod *and*

$$0 \longrightarrow F \longrightarrow L_0 \longrightarrow L_1 \longrightarrow \cdots$$

is an acyclic resolution of F by $R^\bullet T$-acyclic objects $L_0, L_1, \ldots$ (That is, $(R^pT)(L_j) = (0)$, all $p > 0$, all $j \geq 0$.) Then the cohomology of the complex

$$T(L_0) \longrightarrow T(L_1) \longrightarrow T(L_2) \longrightarrow \cdots$$

computes the derived functors of T on the object F (i.e., the $(R^nT)(F)$, $n \geq 0$).

Proof. We use induction on n and décalage. (There are other methods of proof, *e.g.*, by double complexes.) Of course, the assertion for $n = 0$ is true and trivial, being independent of resolutions. So assume for an object G resolved by a sequence of the L's, we have $(R^pT)(G) = H^p(T(L_\bullet))$ for $0 \leq p \leq n-1$. Then the sequence,

$$0 \longrightarrow F \longrightarrow L_0 \longrightarrow \mathrm{cok} \longrightarrow 0$$

yields the resolution

$$0 \longrightarrow \mathrm{cok} \longrightarrow L_1 \longrightarrow L_2 \longrightarrow \cdots \tag{$\dagger$}$$

and the cohomology sequence

$$\begin{array}{l} 0 \longrightarrow T(F) \longrightarrow T(L_0) \longrightarrow T(\mathrm{cok}) \longrightarrow (R^1T)(F) \longrightarrow \cdots \\ \longrightarrow (R^pT)(L_0) \longrightarrow (R^pT)(\mathrm{cok}) \longrightarrow (R^{p+1}T)(F) \longrightarrow (R^{p+1}T)(L_0) \longrightarrow \cdots. \end{array}$$

If $p \geq 1$, we get

$$(R^pT)(\mathrm{cok}) \stackrel{\sim}{\longrightarrow} (R^{p+1}T)(F)$$

while for $p = 0$, we get

$$T(L_0) \longrightarrow T(\mathrm{cok}) \longrightarrow (R^1T)(F) \longrightarrow 0. \tag{$\dagger\dagger$}$$

Now by induction hypothesis on n, $(R^pT)(\mathrm{cok})$ is the pth cohomology of $T(L_1) \longrightarrow T(L_2) \longrightarrow \cdots$, for $p \leq n-1$. This is exactly the $p+1$ cohomology of $T(L_0) \longrightarrow T(L_1) \longrightarrow \cdots$, that is $H^{p+1}(T(L_\bullet))$. So,

$$H^{p+1}(T(L_\bullet)) = (R^pT)(\mathrm{cok}) \stackrel{\sim}{\longrightarrow} (R^{p+1}T)(F)$$

if $1 \leq p \leq n-1$. As long as $n \geq 2$, we can set $p = n-1$ and get the induction step.

So, all that remains is the step from $n = 0$ to $n = 1$. From (†), we see

$$T(\mathrm{cok}) \xrightarrow{\sim} \mathrm{Ker}\,(T(L_1) \longrightarrow T(L_2)).$$

By the short exact sequence for F, L_0, cok, we find that $\mathrm{Im}\,(T(L_0) \longrightarrow T(\mathrm{cok}))$ is exactly the image $(T(L_0) \longrightarrow T(L_1))$; that means

$$T(\mathrm{cok})/T(L_0) = H^1(T(L_\bullet)).$$

But, we know

$$T(\mathrm{cok})/T(L_0) \xrightarrow{\sim} (R^1T)(F)$$

by (††), and we are done. □

Of course, we apply this to resolving a sheaf, F, by flasque sheaves. If we do this, we get a complex (upon applying Γ_U) and so from its cohomology we compute the $H^p(U, F)$. It remains to give a canonical procedure for resolving each F by flasque sheaves. This is Godement's method of "discontinuous sections".

Definition 5.9 For a sheaf, F, write $\mathcal{G}(F)$ for the presheaf

$$\mathcal{G}(F)(U) = \prod_{x \in U} F_x,$$

and call $\mathcal{G}(F)$ the *sheaf of discontinuous sections of* F.

Remarks:

(1) $\mathcal{G}(F)$ is always a sheaf.

(2) $\mathcal{G}(F)$ *is flasque*. For, a section over V of $\mathcal{G}(F)$ is merely a function on V to $\bigcup_{x\in V} F_x$ so that its value at x lies in F_x. We merely extend by zero outside V and get our lifting to a section of U (with $U \supseteq V$).

(3) *There is a canonical embedding* $F \longrightarrow \mathcal{G}(F)$. To see this, if $s \in F(U)$, we have $s(x) \in F_x$, its image in $F_x = \varinjlim_{V \ni x} F(V)$. We send s to the function $x \mapsto s(x)$ which lies in $\mathcal{G}(F)(U)$. Now, if s and t go to the same element of $\mathcal{G}(F)(U)$, we know for each $x \in U$, there is a small open $U(x) \subseteq U$ where $s = t$ on $U(x)$ (i.e., $\rho_U^{U(x)}(s) = \rho_U^{U(x)}(t)$). But these $U(x)$ cover U, and the sheaf axiom says $s = t$ in $F(U)$.

Therefore, $\mathcal{S}(X, R\text{-mod})$ has enough flasques; *so every sheaf, F, possesses a canonical flasque resolution* (the Godement resolution) : Namely

$$\begin{array}{l} 0 \longrightarrow F \longrightarrow \mathcal{G}(F) \longrightarrow \mathrm{cok}_1 \longrightarrow 0 \\ 0 \longrightarrow \mathrm{cok}_1 \longrightarrow \mathcal{G}(\mathrm{cok}_1) \longrightarrow \mathrm{cok}_2 \longrightarrow 0 \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \longrightarrow \mathrm{cok}_n \longrightarrow \mathcal{G}(\mathrm{cok}_n) \longrightarrow \mathrm{cok}_{n+1} \longrightarrow 0 \cdots \end{array}$$

This gives

$$0 \longrightarrow F \longrightarrow \mathcal{G}_0(F) \longrightarrow \mathcal{G}_1(F) \longrightarrow \cdots \longrightarrow \mathcal{G}_n(F) \longrightarrow \cdots,$$

where we have set

$$\mathcal{G}_0(F) = \mathcal{G}(F) \quad \text{and} \quad \mathcal{G}_n(F) = \mathcal{G}(\mathrm{cok}_n) \quad \text{when } n \geq 1.$$

It's not hard to extend all our results on sheaves of R-modules to sheaves of $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is a *sheaf of rings on* X. To replace maps of spaces, we need the notion of a *map of ringed spaces* (i.e., of pairs $(X, \mathcal{O}_X)$ in which $\mathcal{O}_X$ is a sheaf of rings on X): By a map $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ of ringed spaces, we understand a pair (f, φ) in which f is a map $X \longrightarrow Y$ and φ is a map of sheaves $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ (over

Y). For intuition think of $\mathcal{O}_X$ as the sheaf of germs of continuous functions on X. If F is an $\mathcal{O}_X$-module, then f_*F will be an $\mathcal{O}_Y$-module thanks to the map $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$. But if G is an $\mathcal{O}_Y$-module, f^*G is *not* an $\mathcal{O}_X$-module. We must augment the notion of inverse image. Our map $\varphi\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ corresponds by adjunction to a map $\widetilde{\varphi}\colon f^*\mathcal{O}_Y \to \mathcal{O}_X$. Now f^*G *is* an $f^*\mathcal{O}_Y$-module, so we form

$$(f,\varphi)^*G = \mathcal{O}_X \otimes_{f^*\mathcal{O}_Y} f^*G$$

and get our improved notion of inverse image—an $\mathcal{O}_X$-module.

Finally, to end this long section we give some results (of a very elementary character) concerning $\mathrm{Tor}^R_\bullet(-,-)$ and properties of special rings, R. The first of these works for every ring:

Proposition 5.41 *Say M is an R-module (resp. R^{op}-module), then the following are equivalent:*

(1) M is R-flat.

(2) For all Z, we have $\mathrm{Tor}^R_n(Z,M) = (0)$, for all $n > 0$.

(3) For all Z, we have $\mathrm{Tor}^R_1(Z,M) = (0)$.

Proof. (1) $\Longrightarrow$ (2). Since the functor $Z \rightsquigarrow Z \otimes_R M$ is exact, its derived functors are zero for $n > 0$, i.e., (2) holds.

(2) $\Longrightarrow$ (3). This is a tautology.

(3) $\Longrightarrow$ (1). Given an exact sequence

$$0 \longrightarrow Z' \longrightarrow Z \longrightarrow Z'' \longrightarrow 0$$

tensor with M and take cohomology. We get the following piece of the long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}^R_1(Z'',M) \longrightarrow Z' \otimes_R M \longrightarrow Z \otimes_R M \longrightarrow Z'' \otimes_R M \longrightarrow 0.$$

By (3), we have $\mathrm{Tor}^R_1(Z'',M) = (0)$, so the tensored sequence is exact. $\square$

For the rest, we'll assume R is a domain. Now for a P.I.D. we know divisibility of a module is the same as injectivity. That's not true in general, but we have

Proposition 5.42 *If R is an integral domain, every injective R-module is divisible. Conversely, if a module is divisible and torsion free it is injective.*

Proof. We use the exact sequence

$$0 \longrightarrow R \xrightarrow{r} R$$

(R is a domain) for a given element ($\neq 0$) of R. The functor $\mathrm{Hom}_R(-,Q)$ is exact as Q is injective. Then we get

$$Q = \mathrm{Hom}_R(R,Q) \xrightarrow{r} \mathrm{Hom}_R(R,Q) = Q \longrightarrow 0$$

is exact. As r is arbitrary, Q is divisible.

Next, assume M is a torsion-free, divisible module. For an exact sequence

$$0 \longrightarrow \mathfrak{A} \longrightarrow R,$$

suppose we have an R-module map $\varphi\colon \mathfrak{A} \to M$. We need only prove φ extends to a map $R \longrightarrow M$. Of course, this means we need to find $m \in M$, the image of 1 under our extension of φ, so that

$$(\forall r \in \mathfrak{A})(\varphi(r) = rm).$$

Now for each fixed $r \in \mathfrak{A}$, the divisibility of M shows there is an element, $m(r) \in M$, so that

$$\varphi(r) = r \cdot m(r).$$

This element of M is *uniquely* determined by r because M is torsion-free. Now pick $s \in \mathfrak{A}$, $s \neq 0$, consider sr. We have

$$\varphi(sr) = s\varphi(r) = srm(r).$$

But, $sr = rs$; so

$$\varphi(sr) = \varphi(rs) = r\varphi(s) = rsm(s).$$

By torsion freeness, again, we find $m(r) = m(s)$. So, all the elements $m(r)$ are the same, m; and we're done. $\square$

Write $F = \mathrm{Frac}(R)$. The field F is a torsion-free divisible, R-module; it is therefore an injective R-module (in fact, it is the injective hull of R). The R-module, F/R, is an R-module of some importance. For example, $\mathrm{Hom}_R(F/R, M) = (0)$ provided M is torsion-free. In terms of F/R we have the

Corollary 5.43 *If M is a torsion-free module, then M is injective iff $\mathrm{Ext}^1_R(F/R, M) = (0)$. In particular, for torsion-free modules, M, the following are equivalent*

(1) M is injective

(2) $\mathrm{Ext}^n_R(F/R, M) = (0)$ all $n > 0$

(3) $\mathrm{Ext}^1_R(F/R, M) = (0)$.

Proof. Everything follows from the implication (3) $\Longrightarrow$ (1). For this, we have the exact sequence

$$0 \longrightarrow R \longrightarrow F \longrightarrow F/R \longrightarrow 0$$

and so (using $\mathrm{Hom}_R(F/R, M) = (0)$) we find

$$0 \longrightarrow \mathrm{Hom}_R(F, M) \xrightarrow{\theta} M \longrightarrow \mathrm{Ext}^1_R(F/R, M)$$

is exact. The map, θ, takes f to $f(1)$. By (3), θ is an isomorphism. Given $m \in M$ and $r \neq 0$ in R, there is some $f\colon F \to M$, with $f(1) = m$. Let $q = f(1/r)$, then

$$rq = rf(1/r) = f(1) = m;$$

so, M is divisible and Proposition 5.42 applies. $\square$

The field F is easily seen to be $\varinjlim_{\lambda} (\frac{1}{\lambda}R)$, where we use the Artin ordering on R: $\lambda \leq \mu$ iff $\lambda \mid \mu$. Consequently, F is a right limit of projective (indeed, free of rank one) modules. Now tensor commutes with right limits, therefore so does $\mathrm{Tor}^R_\bullet$ (DX). This gives us

$$\mathrm{Tor}^R_n(F, M) = \varinjlim_{\lambda} \mathrm{Tor}^R_n\left(\frac{1}{\lambda}R, M\right) = (0), \quad \text{if } n > 0.$$

That is, F is a flat R-module. Moreover, we have

Proposition 5.44 *If R is an integral domain and M is any R-module, then $\mathrm{Tor}^R_1(F/R, M) = t(M)$, the torsion submodule of M. The R-modules $\mathrm{Tor}^R_p(F/R, M)$ vanish if $p \geq 2$.*

Proof. Use the exact sequence

$$0 \longrightarrow R \longrightarrow F \longrightarrow F/R \longrightarrow 0$$

and tensor with M. We get

$$0 \longrightarrow \mathrm{Tor}_1^R(F/R, M) \longrightarrow R \otimes_R M\ (= M) \longrightarrow F \otimes_R M \longrightarrow F/R \otimes_R M \longrightarrow 0$$

and, further back along the homology sequence

$$(0) = \mathrm{Tor}_{p+1}^R(F, M) \longrightarrow \mathrm{Tor}_{p+1}^R(F/R, M) \longrightarrow \mathrm{Tor}_p^R(R, M) = (0)$$

for all $p \geq 1$. Thus, all will be proved when we show

$$t(M) = \mathrm{Ker}\,(M \longrightarrow F \otimes_R M).$$

Since $F \otimes_R M = \varinjlim_{\lambda} (\frac{1}{\lambda} R \otimes_R M)$, we see $\xi \in \mathrm{Ker}\,(M \longrightarrow F \otimes_R M)$ iff there is some $\lambda\ (\neq 0)$ with $\xi \in \mathrm{Ker}\,(M \longrightarrow \frac{1}{\lambda} R \otimes_R M)$. But, R is a domain, so multiplication by λ is an isomorphism of $\frac{1}{\lambda} R$ and R. This gives us the commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \left(\frac{1}{\lambda}\right) R \otimes_R M \arrow[d, "\text{mult. by } \lambda"] \\ M \arrow[r, equal] & R \otimes_R M \end{tikzcd}$$

and we see immediately that the left vectical arrow is also multiplication by λ. Hence $\xi \in \mathrm{Ker}\,(M \longrightarrow (\frac{1}{\lambda} R) \otimes_R M)$ when and only when $\lambda \xi = 0$, and we are done. $\square$

The name and symbol for $\mathrm{Tor}_\bullet^R$ arose from this proposition.

When R is a P.I.D., the module, F/R, being divisible is injective. Consequently,

Proposition 5.45 *If R is a P.I.D., the sequence*

$$0 \longrightarrow R \longrightarrow F \longrightarrow F/R \longrightarrow 0$$

is an injective resolution of R. Hence, $\mathrm{Ext}_R^p(M, R) = (0)$ if $p \geq 2$, while

$$\mathrm{Ext}_R^1(M, R) = \mathrm{Coker}\,(\mathrm{Hom}_R(M, F) \longrightarrow \mathrm{Hom}_R(M, F/R)).$$

When M is a finitely generated R-module, we find

$$\mathrm{Ext}_R^1(M, R) = \mathrm{Hom}_R(t(M), F/R).$$

Proof. We know the exact sequence is an injective resolution of R and we use it to compute the Ext's. This gives all but the last statement. For that, observe that

$$0 \longrightarrow t(M) \longrightarrow M \longrightarrow M/t(M) \longrightarrow 0$$

is split exact because $M/t(M)$ is free when R is a P.I.D. and M is f.g. Now F is torsion free, so

$$\mathrm{Hom}_R(M, F) = F^{\alpha}, \quad \alpha = \mathrm{rank}\, M/t(M)$$

and

$$\mathrm{Hom}_R(M, F/R) = \mathrm{Hom}_R(t(M), F/R) \amalg (F/R)^{\alpha}.$$

Therefore, $\mathrm{Ext}_R^1(M, R)$ computed as the cokernel has the value claimed above. $\square$

For torsion modules, M, the R-module $\mathrm{Hom}_R(M, F/R)$ is usually called the *dual of M* and its elements are *characters of M*. The notation for the dual of M is M^D. With this terminology, we obtain

Corollary 5.46 *Suppose R is a P.I.D. and M is a f.g. R-module. Then the equivalence classes of extensions of M by R are in 1–1 correspondence with the characters of the torsion submodule of M.*

5.4 Spectral Sequences; First Applications

The invariants provided by homological algebra are obtained from the computation of the (co)homology of a given complex. In general, this is not an easy task—we need all the help we can get. Experience shows that many complexes come with a natural filtration (for example, the complex of differential forms on a complex manifold with its Hodge filtration). In this case, if the filtration satisfies a few simple properties, we can go a long way toward computing (co)homology provided there is a suitable beginning provided for us.

So let $C^\bullet$ be a complex (say computing cohomology) and suppose $C^\bullet$ is filtered. This means there is a family of subobjects, $\{F^pC^\bullet\}_{p\in\mathbb{Z}}$, of $C^\bullet$ such that

$$\cdots \supseteq F^pC^\bullet \supseteq F^{p+1}C^\bullet \supseteq \cdots .$$

We also assume that $\bigcup_p F^pC^\bullet = C^\bullet$ and $\bigcap_p F^pC^\bullet = (0)$. Moreover, if d is the coboundary map of the complex $C^\bullet$ (also called *differentiation*), we assume that

(1) The filtration $\{F^pC^\bullet\}$ and d are compatible, which means that $d(F^pC^\bullet) \subseteq F^pC^\bullet$, for all p.

(2) The filtration $\{F^pC^\bullet\}$ is compatible with the grading on $C^\bullet$, i.e.,

$$F^pC^\bullet = \coprod_q F^pC^\bullet \cap C^{p+q} = \coprod_q C^{p,q},$$

where $C^{p,q} = F^pC^\bullet \cap C^{p+q}$. Then, each $F^pC^\bullet$ is itself a filtered graded complex as are the $F^pC^\bullet/F^{p+r}C^\bullet$, for all $r > 0$.

Remarks:

(1) We have $F^pC^\bullet = \coprod_q C^{p,q}$, the elements in $C^{p,q}$ have *degree* $p+q$.

(2) The $C^{p,q}$'s are subobjects of C^{p+q}.

(3) The $C^{p,q}$'s filter C^{p+q}, and p is the *index of filtration*.

Now, $C^\bullet$ possesses cohomology; $H^\bullet(C^\bullet)$. Also, $F^pC^\bullet$ possesses cohomology, $H^\bullet(F^pC^\bullet)$. There is a map of complexes $F^pC^\bullet \hookrightarrow C^\bullet$, so we have a map $H^\bullet(F^pC^\bullet) \longrightarrow H^\bullet(C^\bullet)$, the image is denoted $H^\bullet(C^\bullet)^p$ and the $H^\bullet(C^\bullet)^p$'s filter $H^\bullet(C^\bullet)$. So, $H^\bullet(C^\bullet)$ is graded and filtered. Thus, we can make

$$H(C)^{p,q} = H^\bullet(C^\bullet)^p \cap H^{p+q}(C^\bullet).$$

There is a graded complex, $\mathrm{gr}(C^\bullet)$, induced by F on $C^\bullet$, defined as

$$\mathrm{gr}(C^\bullet)^n = F^nC^\bullet/F^{n+1}C^\bullet.$$

So, we have $\mathrm{gr}(C^\bullet) = \coprod_n \mathrm{gr}(C^\bullet)^n$ and it follows that

$$\begin{aligned}
\mathrm{gr}(C^\bullet) &= \coprod_p (F^pC^\bullet/F^{p+1}C^\bullet) \\
&= \coprod_p \left[\left(\coprod_q F^pC^\bullet \cap C^{p+q}\right)\Big/\left(\coprod_q F^{p+1}C^\bullet \cap C^{p+q}\right)\right] \\
&= \coprod_p \coprod_q (F^pC^\bullet \cap C^{p+q})/(F^{p+1}C^\bullet \cap C^{p+q}) \\
&= \coprod_{p,q} C^{p,q}/C^{p+1,q-1}.
\end{aligned}$$

So, we get

$$\mathrm{gr}(C^\bullet) = \coprod_{p,q} C^{p,q}/C^{p+1,q-1} = \coprod_{p,q} \mathrm{gr}(C)^{p,q},$$

with $\mathrm{gr}(C)^{p,q} = C^{p,q}/C^{p+1,q-1}$. Similarly, $H^\bullet(\mathrm{gr}(C^\bullet))$ is also bigraded; we have

$$H^\bullet(\mathrm{gr}(C^\bullet)) = \coprod_{p,q} H(\mathrm{gr}(C))^{p,q},$$

where $H(\mathrm{gr}(C))^{p,q} = H^{p+q}(F^pC^\bullet/F^{p+1}C^\bullet)$.

Finally, we also have the graded pieces of $H^{p+q}(C^\bullet)$ in its filtration,

$$\mathrm{gr}(H(C))^{p,q} = H(C)^{p,q}/H(C)^{p+1,q-1} = H^{p+q}(C^\bullet) \cap H^\bullet(C^\bullet)^p/H^{p+q}(C^\bullet) \cap H^\bullet(C^\bullet)^{p+1}.$$

As a naive example of a filtration, we have $F^pC^\bullet = \coprod_{n\geq p} C^n$.

The rest of this section is replete with indices—a veritable orgy of indices. *The definitions to remember are four*: $C^{p,q}$, $\mathrm{gr}(C)^{p,q}$, $H(\mathrm{gr}(C))^{p,q}$ *and* $\mathrm{gr}(H(C))^{p,q}$. Now $C^\bullet$ is filtered and it leads to the graded object $\mathrm{gr}(C^\bullet)$. One always considers $\mathrm{gr}(C^\bullet)$ as a "simpler" object than $C^\bullet$. Here's an example to keep in mind which demonstrates this idea of "simpler". Let C be the ring of power series in one variable, x, over some field, k. Convergence is irrelevant here, just use formal power series. Let F^pC be the collection of power series beginning with terms involving x^{p+1} or higher. We feel that in F^pC the term of a series involving x^{p+1} is the "dominating term", but there are all the rest of the terms. How to get rid of them? Simply pass to $F^pC/F^{p+1}C$, in this object only the term involving x^{p+1} survives. So $\mathrm{gr}(C)$ is the coproduct of the simplest objects: the single terms $a_{p+1}x^{p+1}$. It is manifestly simpler than C. Ideally, we would like to compute the cohomology, $H^\bullet(C^\bullet)$, of $C^\bullet$. However, experience shows that this is usually not feasible, but instead we can begin by computing $H^\bullet(\mathrm{gr}(C^\bullet))$ because $\mathrm{gr}(C^\bullet)$ is simpler than $C^\bullet$. Then, a spectral sequence is just the passage from $H^\bullet(\mathrm{gr}(C^\bullet))$ to $\mathrm{gr}(H^\bullet(C^\bullet))$; this is not quite $H^\bullet(C^\bullet)$ but is usually good enough.

The following assumption makes life easier in dealing with the convergence of spectral sequences: A filtration is *regular* iff for every $n \geq 0$, there is some $\mu(n) \geq 0$, so that for all $p > \mu(n)$, we have $F^pC^\bullet \cap C^n = (0)$.

Definition 5.10 A *cohomological spectral sequence* is a quintuple,

$$\mathcal{E} = \langle E_r^{p,q}, d_r^{p,q}, \alpha_r^{p,q}, E, \beta^{p,q}\rangle,$$

where

(1) $E_r^{p,q}$ is some object in $\mathcal{O}\mathrm{b}(\mathcal{A})$, with $p,q \geq 0$ and $2 \leq r \leq \infty$ (the subscript r is called the *level*).

(2) $d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ is a morphism such that $d_r^{p,q} \circ d_r^{p-r,q+r-1} = 0$, for all $p,q \geq 0$ and all $r < \infty$.

(3) $\alpha_r^{p,q}\colon \mathrm{Ker}\, d_r^{p,q}/\mathrm{Im}\, d_r^{p-r,q+r-1} \to E_{r+1}^{p,q}$ is an isomorphism, for all p, q, all $r < \infty$.

(4) E is a graded, filtered object from $\mathcal{A}$, so that each F^pE is graded by the $E^{p,q} = F^pE \cap E^{p+q}$.

(5) $\beta^{p,q}\colon E_\infty^{p,q} \to \mathrm{gr}(E)^{p,q}$ is an isomorphism, for all p, q (where $\mathrm{gr}(E)^{p,q} = E^{p,q}/E^{p+1,q-1}$).

Remarks:

(1) The whole definition is written in the compact form

$$E_2^{p,q} \underset{p}{\Longrightarrow} E$$

and E is called the *end* of the spectral sequence. The index p is called the *filtration index*, $p+q$ is called the *total* or *grading index* and q the *complementary index*.

(2) If $r > q+1$, then Im $d_r^{p,q} = (0)$ and if $r > p$, then Im $d_r^{p-r,q+r-1} = (0)$. So, if $r > \max\{p, q+1\}$, then (3) implies that $E_r^{p,q} = E_{r+1}^{p,q}$, i.e., the sequence of $E_r^{p,q}$ stabilizes for $r >> 0$.

(3) In general, when $E_r^{p,q}$ stabilizes, $E_r^{p,q} \neq E_\infty^{p,q}$. Further assumptions must be made to get $E_r^{p,q} = E_\infty^{p,q}$ for $r >> 0$.

(4) One can instead make the definition of a *homological spectral sequence* by passing to the "third quadrant" ($p \leq 0$ and $q \leq 0$) and changing arrows around after lowering indices in the usual way, *viz*: H^{-n} becomes H_n. Further, one can make 2^{nd} or 4^{th} quadrant spectral sequences or those creeping beyond the quadrant boundaries. All this will be left to the reader—the cohomological case will be quite enough for us.

Spectral sequences can be introduced in many ways. The one chosen here leads immediately into applications involving double complexes but is weaker if one passes to triangulated and derived categories. No mastery is possible except by learning the various methods together with their strengths and weaknesses. In the existence proof given below there are many complicated diagrams and indices. I urge you to read as far as the definition of $Z_r^{p,q}$ and $B_r^{p,q}$ (one–half page) and skip the rest of the proof on a first reading.

Theorem 5.47 *Say $C^\bullet$ is a filtered right complex whose filtration is compatible with its grading and differentiation. Then, $H^\bullet(C^\bullet)$ possesses a filtration (and is graded) and there exists a spectral sequence*

$$E_2^{p,q} \underset{p}{\Longrightarrow} H^\bullet(C^\bullet),$$

in which $E_2^{p,q}$ is the cohomology of $H^\bullet(\mathrm{gr}(C^\bullet))$—so that $E_1^{p,q} = H(\mathrm{gr}(C))^{p,q} = H^{p+q}(F^pC^\bullet/F^{p+1}C^\bullet))$. If the filtration is regular, the objects $E_\infty^{p,q}$ $(= \mathrm{gr}(H^\bullet(C^\bullet))^{p,q} = H(C)^{p,q}/H(C)^{p+1,q-1} =$ composition factors in the filtration of $H^{p+q}(C^\bullet))$ are exactly the $E_r^{p,q}$ when $r >> 0$.

In the course of the proof of Theorem 5.47, we shall make heavy use of the following simple lemma whose proof will be left as an exercise:

Lemma 5.48 *(Lemma (L)) Let*

$$\begin{tikzcd} & B \arrow[d, "\varphi"] \arrow[dr, "\psi"] & \\ A' \arrow[ur] \arrow[r, "\varphi'"'] & A \arrow[r, "\eta"'] & A'' \end{tikzcd}$$

be a commutative diagram with exact bottom row. Then, η induces an isomorphism
Im $\varphi/$Im $\varphi' \xrightarrow{\sim}$ Im ψ.

Proof of Theorem 5.47. First, we need to make $Z_r^{p,q}$ and $B_r^{p,q}$ and set $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$.

Consider the exact sequence (we will *drop the notation $C^\bullet$ in favor of C for clarity*)

$$0 \longrightarrow F^pC \longrightarrow F^{p-r+1}C \longrightarrow F^{p-r+1}C/F^pC \longrightarrow 0.$$

Upon applying cohomology, we obtain

$$\cdots \longrightarrow H^{p+q-1}(F^{p-r+1}C) \longrightarrow H^{p+q-1}(F^{p-r+1}C/F^pC) \xrightarrow{\delta^*} H^{p+q}(F^pC) \longrightarrow \cdots$$

There is also the natural map $H^{p+q}(F^pC) \longrightarrow H^{p+q}(F^pC/F^{p+1}C)$ induced by the projection $F^pC \longrightarrow F^pC/F^{p+1}C$. Moreover, we have the projection $F^pC/F^{p+r}C \longrightarrow F^pC/F^{p+1}C$, which induces a map on cohomology

$$H^{p+q}(F^pC/F^{p+r}C) \longrightarrow H^{p+q}(F^pC/F^{p+1}C).$$

Set

$$\begin{aligned} Z_r^{p,q} &= \operatorname{Im}(H^{p+q}(F^pC/F^{p+r}C) \longrightarrow H^{p+q}(F^pC/F^{p+1}C)) \\ B_r^{p,q} &= \operatorname{Im}(H^{p+q-1}(F^{p-r+1}C/F^pC) \longrightarrow H^{p+q}(F^pC/F^{p+1}C)), \end{aligned}$$

the latter map being the composition of δ^* and the projection (where $r \geq 1$).

The inclusion $F^{p-r+1}C \subseteq F^{p-r}C$ yields a map $F^{p-r+1}C/F^pC \longrightarrow F^{p-r}C/F^pC$; hence we obtain the inclusion relation $B_r^{p,q} \subseteq B_{r+1}^{p,q}$. In a similar way, the projection $F^pC/F^{p+r+1}C \longrightarrow F^pC/F^{p+r}C$ yields the inclusion $Z_{r+1}^{p,q} \subseteq Z_r^{p,q}$. When $r = \infty$, the coboundary map yields the inclusion $B_\infty^{p,q} \subseteq Z_\infty^{p,q}$ (remember, $F^\infty C = (0)$). Consequently, we can write

$$\cdots \subseteq B_r^{p,q} \subseteq B_{r+1}^{p,q} \subseteq \cdots \subseteq B_\infty^{p,q} \subseteq Z_\infty^{p,q} \subseteq \cdots \subseteq Z_{r+1}^{p,q} \subseteq Z_r^{p,q} \subseteq \cdots.$$

Set

$$E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}, \quad \text{where } 1 \leq r \leq \infty, \text{ and} \quad E_n = H^n(C).$$

Then, $E = \coprod_n E_n = H(C)$, filtered by the $H(C)^p$'s, as explained earlier. When $r = 1$, $B_1^{p,q} = (0)$ and

$$Z_1^{p,q} = H^{p+q}(F^pC/F^{p+1}C);$$

We obtain $E_1^{p,q} = H^{p+q}(F^pC/F^{p+1}C) = H(\mathrm{gr}(C))^{p,q}$. On the other hand, when $r = \infty$ (remember, $F^{-\infty}C = C$), we get

$$\begin{aligned} Z_\infty^{p,q} &= \operatorname{Im}(H^{p+q}(F^pC) \longrightarrow H^{p+q}(F^pC/F^{p+1}C)) \\ B_\infty^{p,q} &= \operatorname{Im}(H^{p+q-1}(C/F^pC) \longrightarrow H^{p+q}(F^pC/F^{p+1}C)). \end{aligned}$$

Now the exact sequence $0 \longrightarrow F^pC/F^{p+1}C \longrightarrow C/F^{p+1}C \longrightarrow C/F^pC \longrightarrow 0$ yields the cohomology sequence

$$\cdots \longrightarrow H^{p+q-1}(C/F^pC) \xrightarrow{\delta^*} H^{p+q}(F^pC/F^{p+1}C) \longrightarrow H^{p+q}(C/F^{p+1}C) \longrightarrow \cdots$$

and the exact sequence $0 \longrightarrow F^pC \longrightarrow C \longrightarrow C/F^pC \longrightarrow 0$ gives rise to the connecting homomorphism $H^{p+q-1}(C/F^pC) \longrightarrow H^{p+q}(F^pC)$. Consequently, we obtain the commutative diagram (with exact bottom row)

$$\begin{tikzcd} & H^{p+q}(F^pC) \arrow[d] \arrow[dr] & \\ H^{p+q-1}(C/F^pC) \arrow[ur] \arrow[r] & H^{p+q}(F^pC/F^{p+1}C) \arrow[r] & H^{p+q}(C/F^{p+1}C) \end{tikzcd}$$

and Lemma (L) yields an isomorphism

$$\xi^{p,q}\colon E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q} \longrightarrow \operatorname{Im}(H^{p+q}(F^pC) \longrightarrow H^{p+q}(C/F^{p+1}C)).$$

But another application of Lemma (L) to the diagram

$$\begin{tikzcd} & H^{p+q}(F^pC) \arrow[d] \arrow[dr] & \\ H^{p+q}(F^{p+1}C) \arrow[ur] \arrow[r] & H^{p+q}(C) \arrow[r] & H^{p+q}(C/F^{p+1}C) \end{tikzcd}$$

gives us the isomorphism

$$\eta^{p,q}\colon \mathrm{gr}(H(C))^{p,q} \longrightarrow \operatorname{Im}(H^{p+q}(F^pC) \longrightarrow H^{p+q}(C/F^{p+1}C)).$$

Thus, $(\eta^{p,q})^{-1} \circ \xi^{p,q}$ is the isomorphism $\beta^{p,q}$ required by part (5) of Definition 5.10.

Only two things remain to be proven to complete the proof of Theorem 5.47. They are the verification of (2) and (3) of Definition 5.10, and the observation that $E_\infty^{p,q}$, as defined above, is the common value of the $E_r^{p,q}$ for $r >> 0$. The verification of (2) and (3) depends upon Lemma (L). Specifically, we have the two commutative diagrams (with obvious origins)

$$\begin{tikzcd}
& H^{p+q}(F^pC/F^{p+r}C) \arrow[d] \arrow[dr, "\theta"] & \\
H^{p+q}(F^pC/F^{p+r+1}C) \arrow[ur] \arrow[r] & H^{p+q}(F^pC/F^{p+1}C) \arrow[r, "\delta^*"'] & H^{p+q+1}(F^{p+1}C/F^{p+r+1}C)
\end{tikzcd}$$

and

$$\begin{tikzcd}
& H^{p+q}(F^pC/F^{p+r}C) \arrow[d] \arrow[dr, "\theta"] & \\
H^{p+q}(F^{p+1}C/F^{p+r}C) \arrow[ur] \arrow[r, "\delta^*"'] & H^{p+q+1}(F^{p+r}C/F^{p+r+1}C) \arrow[r] & H^{p+q+1}(F^{p+1}C/F^{p+r+1}C).
\end{tikzcd}$$

Here, the map θ is the composition

$$H^{p+q}(F^pC/F^{p+r}C) \longrightarrow H^{p+q+1}(F^{p+r}C) \longrightarrow H^{p+q+1}(F^{p+1}C/F^{p+r+1}C).$$

Now, Lemma (L) yields the following facts:

$$\begin{aligned}
Z_r^{p,q}/Z_{r+1}^{p,q} &\stackrel{\sim}{\longrightarrow} \text{Im } \theta, \\
B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1} &\stackrel{\sim}{\longrightarrow} \text{Im } \theta,
\end{aligned}$$

that is,

$$\delta_r^{p,q} \colon Z_r^{p,q}/Z_{r+1}^{p,q} \stackrel{\sim}{\longrightarrow} B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1}.$$

As $B_r^{p,q} \subseteq Z_s^{p,q}$ for every r and s, there is a surjection

$$\pi_r^{p,q} \colon E_r^{p,q} \longrightarrow Z_r^{p,q}/Z_{r+1}^{p,q}$$

with kernel $Z_{r+1}^{p,q}/B_r^{p,q}$; and there exists an injection

$$\sigma_{r+1}^{p+r,q-r+1} \colon B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1} \longrightarrow E_r^{p+r,q-r+1}.$$

The composition $\sigma_{r+1}^{p+r,q-r+1} \circ \delta_r^{p,q} \circ \pi_r^{p,q}$ is the map $d_r^{p,q}$ from $E_r^{p,q}$ to $E_r^{p+r,q-r+1}$ required by (2). Observe that,

$$\text{Im } d_r^{p-r,q+r-1} = B_{r+1}^{p,q}/B_r^{p,q} \subseteq Z_{r+1}^{p,q}/B_r^{p,q} = \text{Ker } d_r^{p,q};$$

hence

$$H(E_r^{p,q}) = \text{Ker } d_r^{p,q}/\text{Im } d_r^{p-r,q+r-1} \cong Z_{r+1}^{p,q}/B_{r+1}^{p,q} = E_{r+1}^{p,q},$$

as required by (3).

To prove that $E_\infty^{p,q}$ as defined above is the common value of $E_r^{p,q}$ for large enough r, we must make use of the regularity of our filtration. Consider then the commutative diagram

$$\begin{tikzcd}
& H^{p+q}(F^pC/F^{p+r}C) \arrow[d] \arrow[dr, "\lambda"] & \\
H^{p+q}(F^pC) \arrow[ur] \arrow[r] & H^{p+q}(F^pC/F^{p+1}C) \arrow[r] & H^{p+q+1}(F^{p+1}C)
\end{tikzcd}$$

where λ is the composition

$$H^{p+q}(F^pC/F^{p+r}C) \xrightarrow{\delta^*} H^{p+q+1}(F^{p+r}C) \longrightarrow H^{p+q+1}(F^{p+1}C).$$

By Lemma (L), we have $Z_r^{p,q}/Z_\infty^{p,q} \xrightarrow{\sim} \mathrm{Im}\ \lambda$. However, if $r > \mu(p+q+1) - p$, then δ^* is the zero map. This shows $\mathrm{Im}\ \lambda = (0)$; hence, we have proven

$$Z_r^{p,q} = Z_\infty^{p,q} \quad \text{for } r > \mu(p+q+1) - p.$$

By our assumptions, the filtration begins with $C = F^0C$, therefore if $r > p$ we find $B_r^{p,q} = B_\infty^{p,q}$. Hence, for

$$r > \max\{p, \mu(p+q+1) - p\}$$

the $E_r^{p,q}$ equal $E_\infty^{p,q}$. $\square$

Remark: Even if our filtration does not start at 0, we can still understand $E_\infty^{p,q}$ from the $E_r^{p,q}$ when the filtration is regular. To see this, note that since cohomology commutes with right limits, we have

$$\varinjlim_r B_r^{p,q} = B_\infty^{p,q},$$

and this implies $\bigcup_r B_r^{p,q} = B_\infty^{p,q}$. Hence, we obtain maps

$$E_r^{p,q} = Z_r^{p,q}/B_r^{p,q} \longrightarrow Z_s^{p,q}/B_s^{p,q} = E_s^{p,q}$$

for $s \geq r > \mu(p+q+1) - p$, and these maps are surjective. (The maps are in fact induced by the $d_r^{p-r,q+r-1}$'s because of the equality

$$E_r^{p,q}/\mathrm{Im}\ d_r^{p-r,q+r-1} = (Z_r^{p,q}/B_r^{p,q})/(B_{r+1}^{p,q}/B_r^{p,q}) = E_{r+1}^{p,q}$$

for $r > \mu(p+q+1) - p$.) Obviously, the right limit of the surjective mapping family

$$E_r^{p,q} \longrightarrow E_{r+1}^{p,q} \longrightarrow \cdots \longrightarrow E_s^{p,q} \longrightarrow \cdots$$

is the group $Z_\infty^{p,q}/(\bigcup B_r^{p,q}) = E_\infty^{p,q}$; so, each element of $E_\infty^{p,q}$ arises from $E_r^{p,q}$ if $r >> 0$ (for fixed p, q). Regularity is therefore still an important condition for spectral sequences that are first and second quadrant or first and fourth quadrant.

It is not true in general that $Z_\infty^{p,q} = \bigcap_r Z_r^{p,q}$ or that $\varprojlim_r Z_r^{p,q} = Z_\infty^{p,q}$. In the first case, we have a *weakly convergent* spectral sequence. In the second case, we have a *strongly convergent* spectral sequence.

Remark: Let us keep up the convention of the above proof in which the complex C appears without the "dot". Then, by (5) of our theorem we find

$$E_\infty^{p,q} = (H^{p+q}(C) \cap H(C)^p)/(H^{p+q}(C) \cap H(C)^{p+1}),$$

so that, for $p + q = n$, *the* $E_\infty^{p,q} = E_\infty^{p,n-p}$ *are the composition factors in the filtration*

$$H^n(C) \supseteq H^n(C)^1 \supseteq H^n(C)^2 \supseteq \cdots \supseteq H^n(C)^\nu \supseteq \cdots.$$

To understand a spectral sequence, it is important to have in mind a pictorial representation of it in its entirety. We are to imagine an "apartment house"; on the r^{th} floor the apartments are labelled $E_r^{p,q}$ and a plan of the r^{th} floor is exactly the points of the pq-plane. The roof of the apartment building is the ∞-floor. In addition, there is the map $d_r^{p,q}$ on the r^{th} floor; it goes "over r and down $r - 1$". Hence, a picture of the r^{th} floor is

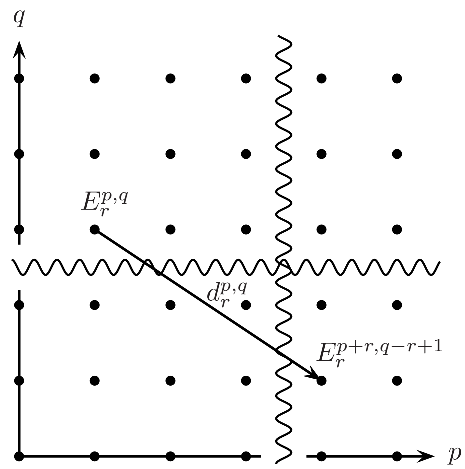

Figure 5.1: The $E_r^{p,q}$ terms of a spectral sequence ("r^{th} floor")

The entire edifice looks like

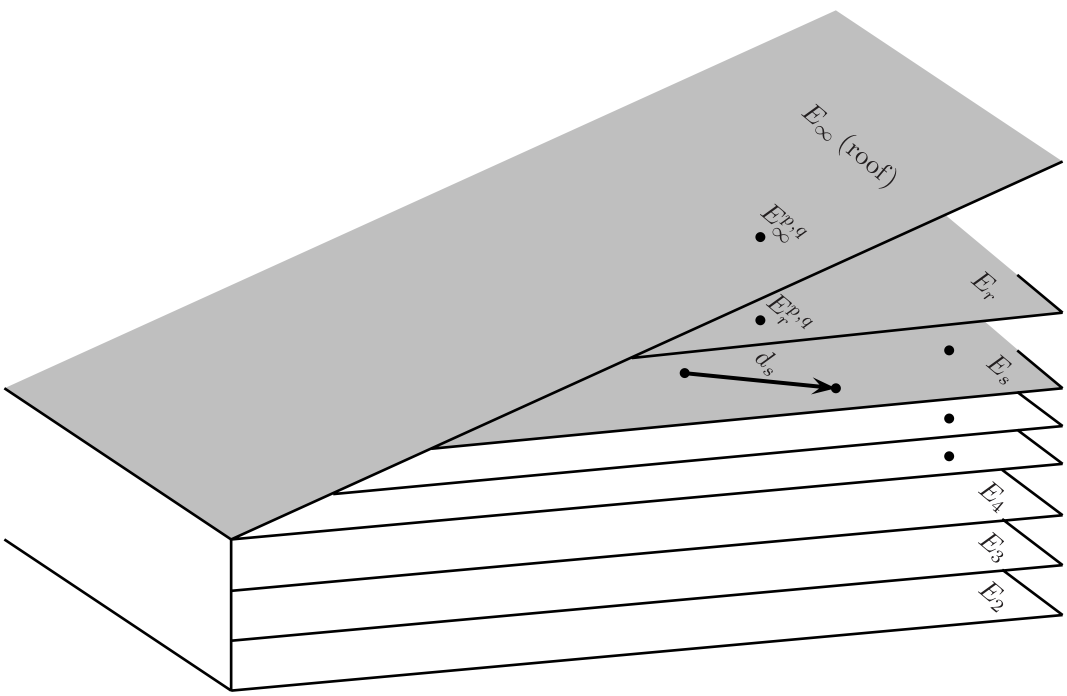

Figure 5.2: The entire spectral sequence (regular filtration)

One passes vertically directly to the apartment above by forming cohomology (with respect to d_r); so, one gets to the roof by repeated formings of cohomology at each higher level.

Once on the roof—at the ∞-level—the points on the line $p+q=n$, i.e., the groups $E_\infty^{0,n}, E_\infty^{1,n-1}, \ldots, E_\infty^{n,0}$, are the composition factors for the filtration of $H^n(C)$:

$$H^n(C) \supseteq H^n(C)^1 \supseteq H^n(C)^2 \supseteq \cdots \supseteq H^n(C)^n \supseteq (0).$$

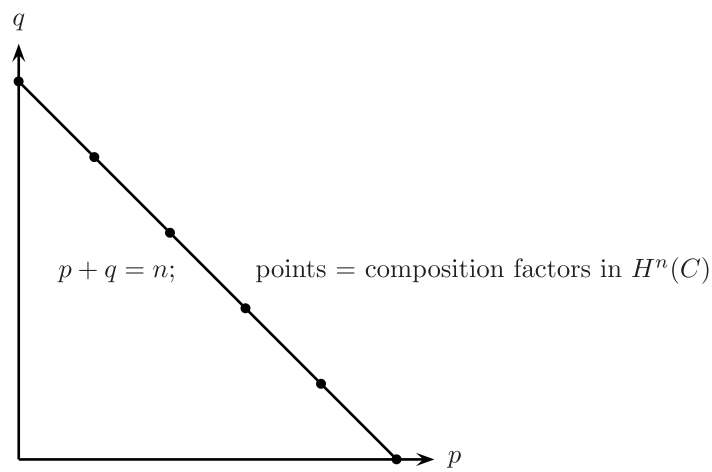

Figure 5.3: The $E_\infty^{p,q}$ terms of a spectral sequence ("roof level")

To draw further conclusions in situations that occur in practice, we need three technical lemmas. Their proofs should be skipped on a first reading and they are only used to isolate and formalize conditions frequently met in the spectral sequences of applications. We'll label them Lemmas A, B, C as their conclusions are only used to get useful theorems on the sequences.

First, observe that if for some r, there are integers n and $p_1 > p_0$ so that $E_r^{\nu,n-\nu} = (0)$ whenever $\nu \neq p_0, \nu \neq p_1$, then certainly $E_s^{\nu,n-\nu} = (0)$ for every s with $r \leq s \leq \infty$. If the filtration is regular, then $E_\infty^{p_0,n-p_0}$ and $E_\infty^{p_1,n-p_1}$ are the only possible non-zero composition factors for $H^n(C)$ and therefore we obtain the exact sequence

$$0 \longrightarrow E_\infty^{p_1,n-p_1} \longrightarrow H^n(C) \longrightarrow E_\infty^{p_0,n-p_0} \longrightarrow 0. \tag{$\dagger$}$$

Lemma 5.49 *(Lemma A) Let $E_2^{p,q} \Longrightarrow H^\bullet(C)$ be a spectral sequence with a regular filtration. Assume there are integers $r; p_1 > p_0; n$ so that*

$$E_r^{u,v} = (0) \quad for \quad \begin{cases} u+v = n, u \neq p_0, p_1 \\ u+v = n+1, u \geq p_1 + r \\ u+v = n-1, u \leq p_0 - r. \end{cases}$$

Then, there is an exact sequence

$$E_r^{p_1,n-p_1} \longrightarrow H^n(C) \longrightarrow E_r^{p_0,n-p_0}. \tag{A}$$

Proof. The remarks above and the first hypothesis yield sequence ($\dagger$). In the proof of Theorem 5.47, we saw that

$$\mathrm{Im}\, d_t^{p_0-t,n-p_0+t-1} = B_{t+1}^{p_0,n-p_0}/B_t^{p_0,n-p_0}.$$

We take $\infty > t \geq r$, let $u = p_0 - t$ and $v = n - p_0 + t - 1$. Using these u and v and the third hypothesis, we deduce $B_t^{p_0,n-p_0}$ is constant for $t \geq r$. Therefore, $B_\infty^{p_0,n-p_0} = B_r^{p_0,n-p_0}$. This gives an injection $E_\infty^{p_0,n-p_0} \hookrightarrow E_r^{p_0,n-p_0}$.

Next, with $u = p_1 + t; v = n - p_1 - t + 1$ and $\infty > t \geq r$, the second hypothesis shows that $\mathrm{Ker}\, d_t^{p_1+t,n-p_1-t+1} = (0)$ and the latter is $Z_{t+1}^{p_1+t,n-p_1-t+1}/B_t^{p_1+t,n-p_1-t+1}$. But,

$$B_r^{\bullet,\bullet} \subseteq B_t^{\bullet,\bullet} \subseteq B_\infty^{\bullet,\bullet} \subseteq Z_\infty^{\bullet,\bullet} \subseteq Z_{t+1}^{\bullet,\bullet},$$

and so we get

$$B_{t+1}^{p_1+t,n-p_1-t+1} = B_t^{p_1+t,n-p_1-t+1}, \quad \infty \geq t \geq r.$$

However, from the proof of Theorem 5.47, we find

$$Z_t^{p_1,n-p_1}/Z_{t+1}^{p_1,n-p_1} \simeq B_{t+1}^{p_1+t,n-p_1-t+1}/B_t^{p_1+t,n-p_1-t+1};$$

and therefore $Z_t^{p_1,n-p_1}$ is constant for $\infty > t \geq r$. By the regularity of the filtration, we find $Z_r^{p_1,n-p_1} = Z_\infty^{p_1,n-p_1}$. This gives a surjection $E_r^{p_1,n-p_1} \longrightarrow E_\infty^{p_1,n-p_1}$, and if we combine (†), our injection for $p_0, n-p_0$ and the surjection for $p_1, n-p_1$ we get sequence (A). $\square$

Lemma 5.50 *(Lemma B) Suppose that $E_2^{p,q} \Longrightarrow H^\bullet(C)$ is a spectral sequence with a regular filtration. Assume that there are integers $s \geq r; p, n$ so that*

$$E_r^{u,v} = (0) \quad \textit{for} \quad \begin{cases} u+v = n-1, u \leq p-r \\ u+v = n, u \neq p \textit{ and } u \leq p+s-r \\ u+v = n+1, p+r \leq u \textit{ and } u \neq p+s. \end{cases}$$

Then, there is an exact sequence

$$H^n(C) \longrightarrow E_r^{p,n-p} \longrightarrow E_r^{p+s,(n+1)-(p+s)}. \tag{B}$$

Proof. We apply $d_r^{p,n-p}$ to $E_r^{p,n-p}$ and land in $E_r^{p+r,n-p-r+1}$ which is (0) by hypothesis three. Also, $E_r^{p-r,n-p+r-1}$ is (0) by the first hypothesis, so the image of $d_r^{p-r,n-p+r-1}$ is (0). This shows $E_r^{p,n-p} = E_{r+1}^{p,n-p}$. Repeat, but with d_{r+1}; as long as $r+1 < s$ we can continue using hypotheses one and three. Thus we obtain $E_r^{p,n-p} = E_s^{p,n-p}$. Now apply $d_t^{p,n-p}$ to $E_t^{p,n-p}$ where $t \geq s+1$. Hypothesis three shows our map is zero and similarly the map $d_t^{p-t,n-p+t-1}$ is zero by hypothesis one. So, for all t, with $\infty > t \geq s+1$, we get $E_t^{p,n-p} = E_{t+1}^{p,n-t}$. As the filtration is regular, we obtain $E_{s+1}^{p,n-p} = E_\infty^{p,n-p}$.

Next, by hypothesis two with $u = p + (s-r)$ (provided $s > r$, otherwise there is nothing to prove), we see that $\mathrm{Im}\, d_r^{p+s-r,n-(p+s-r)}$ is (0). Thus,

$$B_{r+1}^{p+s,(n+1)-(p+s)} = B_r^{p+s,(n+1)-(p+s)}.$$

Should $s > r+1$, we continue because

$$(0) = \mathrm{Im}\, d_{r+1}^{p+s-(r+1),n-(p+s-(r+1))}.$$

This gives

$$B_{r+2}^{p+s,(n+1)-(p+s)} = B_{r+1}^{p+s,(n+1)-(p+s)}.$$

Hence, we get

$$B_s^{p+s,(n+1)-(p+s)} = B_r^{p+s,(n+1)-(p+s)}$$

by repetition. Of course, this gives the *inclusion*

$$E_s^{p+s,(n+1)-(p+s)} \subseteq E_r^{p+s,(n+1)-(p+s)}.$$

Lastly, by hypothesis one, $E_r^{p-s,(n-1)-(p-s)} = (0)$; so, $E_t^{p-s,(n-1)-(p-s)} = (0)$ for every $t \geq r$. Take $t = s$, then $d_s^{p-s,(n-1)-(p-s)}$ vanishes, and in the usual way we get

$$B_{s+1}^{p,n-p} = B_s^{p,n-p}.$$

But then, we obtain an inclusion

$$E_{s+1}^{p,n-p} \hookrightarrow E_s^{p,n-p}.$$

However, the kernel of $d_s^{p,n-p}$ is $Z_{s+1}^{p,n-p}/B_s^{p,n-p} = E_{s+1}^{p,n-p}$; therefore we get the exact sequence

$$0 \longrightarrow E_{s+1}^{p,n-p} \longrightarrow E_s^{p,n-p} \stackrel{d_s^{p,n-p}}{\longrightarrow} E_s^{p+s,(n+1)-(p+s)}.$$

And now we have a surjection $H^n(C) \longrightarrow E_\infty^{p,n-p}$ because $E_\infty^{u,n-u} = (0)$ when $u \leq p+s-r$ $(r \neq p)$ by hypothesis two. If we put all this together, we get sequence (B). $\square$

In a similar manner (see the exercises) one proves

Lemma 5.51 *(Lemma C) If $E_2^{p,q} \Longrightarrow H^\bullet(C)$ is a spectral sequence with a regular filtration and if there exist integers $s \geq r; p, n$ so that*

$$E_r^{u,v} = (0) \quad \textit{for} \quad \begin{cases} u+v = n+1, u \geq p+r \\ u+v = n, p+r-s \leq u \neq p \\ u+v = n-1, p-s \neq u \leq p-r, \end{cases}$$

then, there is an exact sequence

$$E_r^{p-s,(n-1)-(p-s)} \longrightarrow E_r^{p,n-p} \longrightarrow H^n(C). \tag{C}$$

Although Lemmas A, B, C are (dull and) technical, they do emphasize one important point: *For any level r, if $E_r^{p,q}$ lies on the line $p+q=n$, then d_r takes it to a group on the line $p+q=n+1$ and it receives a d_r from a group on the line $p+q=n-1$.* From this we obtain immediately

Corollary 5.52 *(Corollary D) Say $E_2^{p,q} \Longrightarrow H^\bullet(C)$ is a regularly filtered spectral sequence and there are integers r, n so that*

$$E_r^{p,q} = (0) \quad \textit{for} \quad \begin{cases} p+q = n-1 \\ p+q = n+1. \end{cases}$$

Then, $E_r^{p,n-p} = E_\infty^{p,n-p}$ and the $E_r^{p,n-p}$ are the composition factors for $H^n(C)$ in its filtration.

Now we wish to apply Lemmas A, B, C and we begin with the simplest case—a case for which we do not need these Lemmas. A spectral sequence $E_2^{p,q} \Longrightarrow H^\bullet(C)$ *degenerates at (level) r* when and only when for each n there is a $q(n)$ so that

$$E_r^{n-q,q} = (0) \quad \text{if } q \neq q(n).$$

Of course then $E_s^{n-q,q} = (0)$ when $q \neq q(n)$ for all $s \geq r$; so that, in the regular case, we have $E_\infty^{n-q,q} = (0)$ if $q \neq q(n)$. If we have $q(n+1) > q(n) - (r-1)$ for all n (e.g., if $q(n)$ is constant), then $E_r^{n-q,q} = E_\infty^{n-q,q}$ for every n and q and we deduce that

$$H^n(C) = E_\infty^{n-q(n),q(n)} = E_r^{n-q(n),q(n)}$$

for all n. This proves

Proposition 5.53 *When the filtration of C is regular and the spectral sequence*

$$E_2^{p,q} \Longrightarrow H^\bullet(C)$$

degenerates at r, then $H^n(C) = E_\infty^{n-q(n),q(n)}$. If in addition, $q(n+1) > q(n) - (r-1)$ for all n, then

$$H^n(C) \cong E_r^{n-q(n),q(n)}$$

for every n.

Theorem 5.54 *(Zipper Sequence) Suppose $E_2^{p,q} \Longrightarrow H^\bullet(C)$ is a regularly convergent spectral sequence and there exist integers p_0, p_1, r with $p_1 - p_0 \geq r \geq 1$ so that $E_r^{u,v} = (0)$ for all $u \neq p_0$ or p_1. Then we have the exact zipper sequence*

$$\cdots \longrightarrow E_r^{p_1, n-p_1} \longrightarrow H^n(C) \longrightarrow E_r^{p_0, n-p_0} \longrightarrow E_r^{p_1, n+1-p_1} \longrightarrow H^{n+1}(C) \longrightarrow \cdots$$

Dually, if there are integers q_0, q_1, r with $q_1 - q_0 \geq r - 1 \geq 1$ so that $E_r^{u,v} = (0)$ for $v \neq q_0$ or q_1, then the zipper sequence is

$$\cdots \longrightarrow E_r^{n-q_0, q_0} \longrightarrow H^n(C) \longrightarrow E_r^{n-q_1, q_1} \longrightarrow E_r^{n+1-q_0, q_0} \longrightarrow H^{n+1}(C) \longrightarrow \cdots$$

Proof. Write $s = p_1 - p_0 \geq r$ and apply Lemmas A, B and C (check the hypotheses using $u + v = n$). By splicing the exact sequences of those lemmas, we obtain the zipper sequence. Dually, write $s = 1 + q_1 - q_0 \geq r$, set $p_0 = n - q_1$ and $p_1 = n - q_0$. Then Lemmas A, B and C again apply and their exact sequences splice to give the zipper sequence. $\square$

The name "zipper sequence" comes from the following picture. In it, the dark arrows are the maps $E_r^{p_0, n-p_0} \longrightarrow E_r^{p_1, n+1-p_1}$ and the dotted arrows are the compositions $E_r^{p_1, n+1-p_1} \longrightarrow H^{n+1} \longrightarrow E_r^{p_0, n+1-p_0}$ (one is to imagine these arrows passing through the H^{n+1} somewhere behind the plane of the page). As you see, the arrows zip together the vertical lines $p = p_0$ and $p = p_1$.

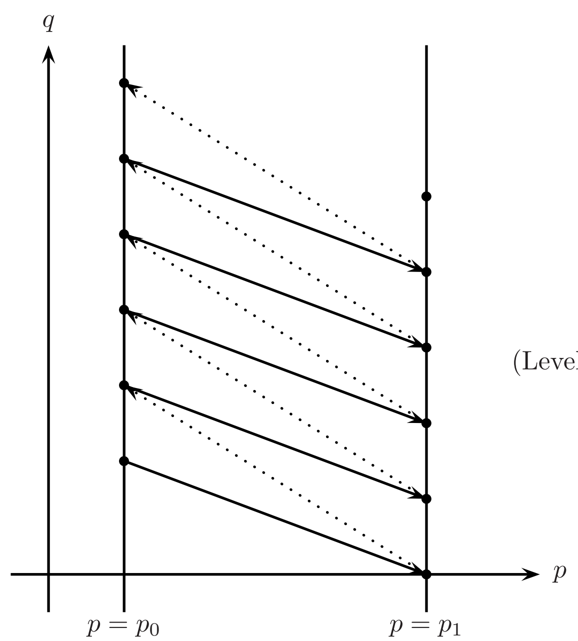

Figure 5.4: Zipper Sequence

Theorem 5.55 *(Edge Sequence) Suppose that $E_2^{p,q} \Longrightarrow H^\bullet(C)$ is a regularly convergent spectral sequence and assume there is an integer $n \geq 1$ so that $E_2^{p,q} = (0)$ for every q with $0 < q < n$ and all p (no hypothesis if $n = 1$). Then $E_2^{r,0} \cong H^r(C)$ for $r = 0, 1, 2, \ldots, n-1$ and*

$$0 \longrightarrow E_2^{n,0} \longrightarrow H^n(C) \longrightarrow E_2^{0,n} \longrightarrow E_2^{n+1,0} \longrightarrow H^{n+1}(C)$$

is exact (edge sequence). In particular, with no hypotheses on the vanishing of $E_2^{p,q}$, we have the exact sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(C) \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \longrightarrow H^2(C).$$

Proof. Since we have a cohomological (first quadrant) spectral sequence all the differentials $d_l^{r,0}$ vanish for all l and if $l \geq n$ no differential $d_l^{p,q}$ hits $E_l^{r,0}$ if $p \geq 0$ and $r \leq n-1$. All the differentials $d_l^{p,q}$ are 0 if $q < n$ and so we find $E_2^{r,0} \cong E_\infty^{r,0}$ for $0 \leq r \leq n-1$. But, only one non-zero term $E_\infty^{p,q}$ exists on the line $r = p+q$ for $r < n$ by our hypothesis on the vanishing; so, indeed $E_2^{r,0} \cong E_\infty^{r,0} = H^r(C)$ when $0 \leq r \leq n-1$.

For $E_l^{n,0}$, since $d_{n-p}^{p,n-p-1} \colon E_{n-p}^{p,n-p-1} \to E_{n-p}^{n,0}$, and since $p \geq 0$ implies $q \leq n-1$, we see that no non-zero differential hits $E_l^{n,0}$ for any l. Thus, $E_2^{n,0} \cong E_\infty^{n,0}$ and we get the injection $E_2^{n,0} \longrightarrow H^n(C)$. Apply Lemma A with $p_0 = 0, p_1 = n, r = 2$ to find the sequence

$$0 \longrightarrow E_2^{n,0} \longrightarrow H^n(C) \longrightarrow E_2^{0,n}. \tag{$*$}$$

Next, in Lemma B, take $r = 2, s = n+1 \geq 2$, and $p = 0$. Sequence (B) splices to $(*)$ to yield

$$0 \longrightarrow E_2^{n,0} \longrightarrow H^n(C) \longrightarrow E_2^{0,n} \longrightarrow E_2^{n+1,0}. \tag{$**$}$$

And, lastly, use Lemma C with $r = 2, s = n+1 \geq 2$, the n of Lemma C to be our $n+1 = s$ and $p = n+1$. Upon splicing Lemma C onto $(**)$ we find the edge sequence

$$0 \longrightarrow E_2^{n,0} \longrightarrow H^n(C) \longrightarrow E_2^{0,n} \longrightarrow E_2^{n+1,0} \longrightarrow H^{n+1}(C). \qquad \square$$

Obviously, the edge sequence gets its name from the fact that the $E_2^{p,q}$ which appear in it lie on the edge of the quadrant in the picture of E_2 as points (of the first quadrant) in the pq-plane. Equally obvious is the notion of a *morphism of spectral sequences*. Whenever C and $\widetilde{C}$ are graded, filtered complexes and $g \colon C \to \widetilde{C}$ is a morphism of such complexes, we find an induced morphism

$$\mathrm{ss}(g) \colon E_2^{p,q} \Longrightarrow H^\bullet(C) \mapsto \widetilde{E}_2^{p,q} \Longrightarrow H^\bullet(\widetilde{C})$$

of spectral sequences.

Theorem 5.56 *Suppose C and $\widetilde{C}$ are graded filtered complexes and write $E^{\bullet,\bullet}(C)$ and $E^{\bullet,\bullet}(\widetilde{C})$ for their associated spectral sequences. Assume both filtrations are regular and $g^\bullet \colon E^{\bullet,\bullet}(C) \to E^{\bullet,\bullet}(\widetilde{C})$ is a spectral sequence morphism. If, for some $r \geq 2$, the level r map $g_r^\bullet \colon E_r^{\bullet,\bullet} \to \widetilde{E}_r^{\bullet,\bullet}$ is an isomorphism, then for every $s \geq r$ the level s map, $g_s^\bullet$, is also an isomorphism (also for $s = \infty$) and we have an induced isomorphism on the graded cohomology*

$$\mathrm{gr}H(g^\bullet) \colon \mathrm{gr}H^\bullet(C) \xrightarrow{\sim} \mathrm{gr}H^\bullet(\widetilde{C}).$$

The proof of this is obvious because by regularity $E_\infty^{p,q} = E_s^{p,q}$ for $s >> 0$. But for H^n, its graded pieces are the $E_\infty^{p,n-p}$, and $p \geq 0$. Thus, $p \leq n$ and $q \leq n$; so, our choice $s = s(n) >> 0$ will do to get

$$E_s^{p,q} = E_\infty^{p,q} \qquad \text{(all } p, q \text{ with } p+q = n).$$

These groups are exactly the graded pieces of H^n as we've remarked and $\coprod_{p+q=n} g_s^{p,q}$ is our isomorphism. $\square$

Our technical results on spectral sequences are over, now we actually need some spectral sequences to use them on. Big sources of spectral sequences are double complexes. So, let $C = \coprod_{p,q} C^{p,q}$ be a doubly-graded complex (we assume that $p, q \geq 0$). We have two differentiations:

$$\begin{aligned} d_{\mathrm{I}}^{p,q} \colon C^{p,q} &\longrightarrow C^{p+1,q}, \qquad \text{(horizontal)} \\ d_{\mathrm{II}}^{p,q} \colon C^{p,q} &\longrightarrow C^{p,q+1} \qquad \text{(vertical)} \end{aligned}$$

such that

$$d_{\mathrm{I}} \circ d_{\mathrm{I}} = d_{\mathrm{II}} \circ d_{\mathrm{II}} = 0.$$

We will require

$$d_{\mathrm{II}}^{p+1,q} \circ d_{\mathrm{I}}^{p,q} + d_{\mathrm{I}}^{p,q+1} \circ d_{\mathrm{II}}^{p,q} = 0, \quad \textit{for all } p, q.$$

Then we get the (singly graded) *total complex*

$$C = \coprod_n \left(\coprod_{p+q=n} C^{p,q} \right)$$

with *total differential* $d_T = d_{\mathrm{I}} + d_{\mathrm{II}}$. We immediately check that $d_T \circ d_T = 0$. There are two filtrations

$$F_{\mathrm{I}}^p C = \coprod_{r \geq p, q} C^{r,q} \quad \text{and} \quad F_{\mathrm{II}}^q C = \coprod_{p, s \geq q} C^{p,s}.$$

Both have every compatibility necessary and give filtrations on the total complex and are regular. Therefore, we find two spectral sequences

$$\mathrm{I}_2^{p,q} \underset{p}{\Longrightarrow} H^\bullet(C) \quad \text{and} \quad \mathrm{II}_2^{p,q} \underset{q}{\Longrightarrow} H^\bullet(C).$$

Observe that

$$\begin{aligned} \mathrm{gr}_{\mathrm{I}}(C) &= \coprod \mathrm{gr}_{\mathrm{I}}^p(C) \\ &= \coprod (F_{\mathrm{I}}^p C / F_{\mathrm{I}}^{p+1} C) \\ &= \coprod_p \left(\coprod_q C^{p,q} \right) \end{aligned}$$

and $E_{\mathrm{I}}^{p,q} = H^{p,q}(\mathrm{gr}_{\mathrm{I}}^p(C))$, which is just $H_{\mathrm{II}}^{p,q}(C)$. Now, we need to compute $d_1^{p,q}$ in spectral sequence (I). It is induced by the connecting homomorphism arising from the short exact sequence

$$0 \longrightarrow F_{\mathrm{I}}^{p+1} C / F_{\mathrm{I}}^{p+2} C \longrightarrow F_{\mathrm{I}}^p C / F_{\mathrm{I}}^{p+2} C \longrightarrow F_{\mathrm{I}}^p C / F_{\mathrm{I}}^{p+1} C \longrightarrow 0.$$

Pick $\xi \in H_{\mathrm{II}}^{p,q}(C)$, represented by a cocycle with respect to d_{II} in C^{p+q}, call it x. The connecting homomorphism ($= d_1$) is given by "$d_T x$". But, $d_T x = d_{\mathrm{I}} x + d_{\mathrm{II}} x = d_{\mathrm{I}} x$, as $d_{\mathrm{II}} x = 0$. Therefore, d_1 is exactly the map induced on $H_{\mathrm{II}}^{p,q}(C)$ by d_{I}. It follows that

$$\mathrm{I}_2^{p,q} = Z_2^{p,q} / B_2^{p,q} = H_{\mathrm{I}}^p(H_{\mathrm{II}}^q(C)).$$

We have therefore proved

Theorem 5.57 *Given a double complex $C = \coprod_{p,q} C^{p,q}$, we have two regular spectral sequences converging to the cohomology of the associated total complex:*

$$H_{\mathrm{I}}^p(H_{\mathrm{II}}^q(C)) \underset{p}{\Longrightarrow} H^\bullet(C)$$

and

$$H_{\mathrm{II}}^q(H_{\mathrm{I}}^p(C)) \underset{q}{\Longrightarrow} H^\bullet(C).$$

It still is not apparent where we'll find an ample supply of double complexes so as to use the above theorem. A very common source appears as the answer to the following

Problem. Given two left-exact functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ between abelian categories (with enough injectives, etc.), we have $GF\colon \mathcal{A} \to \mathcal{C}$ (left-exact); how can we compute $R^n(GF)$ if we know $R^p F$ and $R^q G$?

In order to answer this question, we need to introduce special kinds of injective resolutions of complexes.

Definition 5.11 A *Cartan–Eilenberg injective resolution* of a complex, C, (with $C^k = (0)$ if $k < 0$) is a resolution

$$0 \longrightarrow C^\bullet \longrightarrow Q^{\bullet\, 0} \longrightarrow Q^{\bullet\, 1} \longrightarrow Q^{\bullet\, 2} \longrightarrow \cdots,$$

in which each $Q^{\bullet\, j} = \coprod_i Q^{i,j}$ is a complex (differential d^{ij}) and every $Q^{i,j}$ injective and so that if we write $Z^{i,j} = \mathrm{Ker}\ d^{i,j}$; $B^{i,j} = \mathrm{Im}\ d^{i-1,j}$ and $H^{i,j} = Z^{i,j}/B^{i,j}$, we have the injective resolutions

$$\begin{array}{ll}
(1) & 0 \longrightarrow C^i \longrightarrow Q^{i,0} \longrightarrow Q^{i,1} \longrightarrow \cdots \\
(2) & 0 \longrightarrow Z^i(C) \longrightarrow Z^{i,0} \longrightarrow Z^{i,1} \longrightarrow \cdots \\
(3) & 0 \longrightarrow B^i(C) \longrightarrow B^{i,0} \longrightarrow B^{i,1} \longrightarrow \cdots \\
(4) & 0 \longrightarrow H^i(C) \longrightarrow H^{i,0} \longrightarrow H^{i,1} \longrightarrow \cdots.
\end{array}$$

The way to remember this complicated definition is through the following diagram:

$$\begin{tikzcd}
 & {} & {} & {} & \\
0 \arrow[r] & C^{i+1} \arrow[u] \arrow[r] & Q^{i+1,0} \arrow[u] \arrow[r] & Q^{i+1,1} \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & C^{i} \arrow[u, "\delta^i"] \arrow[r] & Q^{i,0} \arrow[u, "d^{i,0}"] \arrow[r] & Q^{i,1} \arrow[u, "d^{i,1}"] \arrow[r] & \cdots \\
0 \arrow[r] & Z^{i} \arrow[u] \arrow[r] & Z^{i,0} \arrow[u] \arrow[r] & Z^{i,1} \arrow[u] \arrow[r] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Proposition 5.58 *Every complex, C, has a Cartan–Eilenberg resolution, $0 \longrightarrow C \longrightarrow Q^\bullet$, where the $\{Q^{i,j}\}$ form a double complex. Here, we have suppressed the grading indices of C and the Q^j.*

Proof. We begin with injective resolutions $0 \longrightarrow B^0(C) \longrightarrow B^{0,\bullet}$; $0 \longrightarrow B^1(C) \longrightarrow B^{1,\bullet}$ and $0 \longrightarrow H^0(C) \longrightarrow H^{0,\bullet}$ of $B^0(C)$; $B^1(C)$; $H^0(C)$. Now, we have exact sequences

$$0 \longrightarrow B^0(C) \longrightarrow Z^0(C) \longrightarrow H^0(C) \longrightarrow 0$$

and

$$0 \longrightarrow Z^0(C) \longrightarrow C^0 \xrightarrow{\delta^0} B^1(C) \longrightarrow 0;$$

so, by Proposition 5.1, we get injective resolutions $0 \longrightarrow Z^0(C) \longrightarrow Z^{0,\bullet}$ and $0 \longrightarrow C^0 \longrightarrow Q^{0,\bullet}$, so that

$$0 \longrightarrow B^{0,\bullet} \longrightarrow Z^{0,\bullet} \longrightarrow H^{0,\bullet} \longrightarrow 0$$

and

$$0 \longrightarrow Z^{0,\bullet} \longrightarrow Q^{0,\bullet} \longrightarrow B^{1,\bullet} \longrightarrow 0$$

are exact.

For the induction step, assume that the complexes $B^{i-1,\bullet}$, $Z^{i-1,\bullet}$, $H^{i-1,\bullet}$, $Q^{i-1,\bullet}$ and $B^{i,\bullet}$ are determined and satisfy the required exactness properties ($i \geq 1$). Pick any injective resolution $H^{i,\bullet}$ of $H^i(C)$, then using the exact sequence

$$0 \longrightarrow B^i(C) \longrightarrow Z^i(C) \longrightarrow H^i(C) \longrightarrow 0$$

and Proposition 5.1, we get an injective resolution $0 \longrightarrow Z^i(C) \longrightarrow Z^{i,\bullet}$ so that

$$0 \longrightarrow B^{i,\bullet} \longrightarrow Z^{i,\bullet} \longrightarrow H^{i,\bullet} \longrightarrow 0$$

is exact. Next, pick an injective resolution, $0 \longrightarrow B^{i+1}(C) \longrightarrow B^{i+1,\bullet}$, of $B^{i+1}(C)$ and use the exact sequence

$$0 \longrightarrow Z^i(C) \longrightarrow C^i \xrightarrow{\delta^i} B^{i+1}(C) \longrightarrow 0$$

and Proposition 5.1 to get an injective resolution $0 \longrightarrow C^i \longrightarrow Q^{i,\bullet}$ so that

$$0 \longrightarrow Z^{i,\bullet} \longrightarrow Q^{i,\bullet} \longrightarrow B^{i+1,\bullet} \longrightarrow 0$$

is exact. The differential $d_{\mathrm{II}}^{i,j}$ of the double complex $\{Q^{i,j}\}$ is the composition

$$Q^{i,j} \longrightarrow B^{i+1,j} \longrightarrow Z^{i+1,j} \longrightarrow Q^{i+1,j}$$

and the differential $d_{\mathrm{I}}^{i,j}$ is given by

$$d_{\mathrm{I}}^{i,j} = (-1)^i \epsilon^{i,j},$$

where, $\epsilon^{i,\bullet}$ is the differential of $Q^{i,\bullet}$. The reader should check that $\{Q^{i,j}\}$ is indeed a Cartan–Eilenberg resolution and a double complex (DX). $\square$

Note that, due to the exigencies of notation (we resolved our complex $C^\bullet$ horizontally) the usual conventions of horizontal and vertical were interchanged in the proof of Proposition 5.58 at least as far as Cartesian coordinate notation is concerned. This will be rectified during the proof of the next theorem, which is the result about spectral sequences having the greatest number of obvious applications and forms the solution to the problem posed before.

Theorem 5.59 *(Grothendieck) Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be two left-exact functors between abelian categories (with enough injectives, etc.) and suppose that $F(Q)$ is G-acyclic whenever Q is injective, which means that $R^pG(FQ) = (0)$, if $p > 0$. Then, we have the spectral sequence of composed functors*

$$R^qG((R^pF)(A)) \underset{q}{\Longrightarrow} (R^\bullet(GF))(A).$$

Proof. Pick some object $A \in \mathcal{A}$ and resolve it by injectives to obtain the resolution $0 \longrightarrow A \longrightarrow Q^\bullet(A)$:

$$0 \longrightarrow A \longrightarrow Q^0 \longrightarrow Q^1 \longrightarrow Q^2 \longrightarrow \cdots .$$

If we apply GF to $Q^\bullet(A)$ and compute cohomology, we get $R^n(GF)(A)$. If we just apply F to $Q^\bullet(A)$, we get the complex:

$$F(Q^0) \longrightarrow F(Q^1) \longrightarrow F(Q^2) \longrightarrow \cdots , \tag{$FQ^\bullet(A)$}$$

whose cohomology is $R^qF(A)$.

Now resolve the complex $FQ^\bullet(A)$ in the vertical direction by a Cartan-Eilenberg resolution. There results a double complex of injectives (with exact columns)

$$\begin{tikzcd}
\vdots & \vdots & & \vdots & \\
Q^{0,1} \arrow[u] \arrow[r] & Q^{1,1} \arrow[u] \arrow[r] & \cdots \arrow[r] & Q^{n,1} \arrow[u] \arrow[r] & \cdots \\
Q^{0,0} \arrow[u] \arrow[r] & Q^{1,0} \arrow[u] \arrow[r] & \cdots \arrow[r] & Q^{n,0} \arrow[u] \arrow[r] & \cdots \\
F(Q^0) \arrow[u] \arrow[r] & F(Q^1) \arrow[u] \arrow[r] & \cdots \arrow[r] & F(Q^n) \arrow[u] \arrow[r] & \cdots \\
0 \arrow[u] & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}$$

in the category $\mathcal{B}$. Apply the functor G to this double complex to obtain a new double complex we will label C:

$$\begin{tikzcd}
\vdots & \vdots & & \vdots & \\
G(Q^{0,1}) \arrow[u] \arrow[r] & G(Q^{1,1}) \arrow[u] \arrow[r] & \cdots \arrow[r] & G(Q^{n,1}) \arrow[u] \arrow[r] & \cdots \\
G(Q^{0,0}) \arrow[u] \arrow[r] & G(Q^{1,0}) \arrow[u] \arrow[r] & \cdots \arrow[r] & G(Q^{n,0}) \arrow[u] \arrow[r] & \cdots \\
GF(Q^0) \arrow[u] \arrow[r] & GF(Q^1) \arrow[u] \arrow[r] & \cdots \arrow[r] & GF(Q^n) \arrow[u] \arrow[r] & \cdots \\
0 \arrow[u] & 0 \arrow[u] & & 0 \arrow[u] & ,
\end{tikzcd} \tag{C}$$

in which, by hypothesis, *all the columns are still exact*. Therefore, using the notations for the two spectral sequences converging to $H^\bullet(C)$, we have $H^\bullet_{\mathrm{II}}(C) = (0)$ so that (by our first remarks)

$$H^\bullet(C) \cong R^\bullet(GF)(A).$$

From the second spectral sequence, we get

$$\mathrm{II}_2^{l,m} = H^l_{\mathrm{II}}(H^m_{\mathrm{I}}(C)) \underset{l}{\Longrightarrow} R^\bullet(GF)(A).$$

Since we used a Cartan-Eilenberg resolution of $FQ^\bullet(A)$, we have the following injective resolutions

$$\begin{aligned}
0 &\longrightarrow Z^p(FQ^\bullet(A)) \longrightarrow Z^{p,0} \longrightarrow Z^{p,1} \longrightarrow \cdots \\
0 &\longrightarrow B^p(FQ^\bullet(A)) \longrightarrow B^{p,0} \longrightarrow B^{p,1} \longrightarrow \cdots \\
0 &\longrightarrow H^p(FQ^\bullet(A)) \longrightarrow H^{p,0} \longrightarrow H^{p,1} \longrightarrow \cdots,
\end{aligned}$$

for all $p \geq 0$. Moreover, the exact sequences

$$0 \longrightarrow Z^{p,\bullet} \longrightarrow Q^{p,\bullet} \longrightarrow B^{p+1,\bullet} \longrightarrow 0$$

and
$$0 \longrightarrow B^{p,\bullet} \longrightarrow Z^{p,\bullet} \longrightarrow H^{p,\bullet} \longrightarrow 0$$
are split because the terms are injectives of $\mathcal{B}$. Therefore, the sequences
$$0 \longrightarrow G(Z^{p,\bullet}) \longrightarrow G(Q^{p,\bullet}) \longrightarrow G(B^{p+1,\bullet}) \longrightarrow 0$$
and
$$0 \longrightarrow G(B^{p,\bullet}) \longrightarrow G(Z^{p,\bullet}) \longrightarrow G(H^{p,\bullet}) \longrightarrow 0$$
are still exact and we find
$$H^p_{\mathrm{I}}(C^{\bullet,q}) = G(H^{p,q}).$$
But, the $H^{p,\bullet}$ form an injective resolution of $H^p(FQ^\bullet(A))$ and the latter is just $R^pF(A)$. So, $G(H^{p,\bullet})$ is the complex whose cohomology is exactly $R^qG(R^pF(A))$. Now, this cohomology is $H^q_{\mathrm{II}}(G(H^{p,\bullet}))$ and $H^p_{\mathrm{I}}(C^{\bullet,\bullet})$ is $G(H^{p,\bullet})$ by the above. We obtain
$$R^qG(R^pF(A)) = H^q_{\mathrm{II}}(H^p_{\mathrm{I}}(C^{\bullet,\bullet})) = \mathrm{II}_2^{q,p} \underset{q}{\Longrightarrow} H^\bullet(C).$$
Since $H^\bullet(C) \cong R^\bullet(GF)(A)$, we are done. $\square$

There are many applications of the Spectral Sequence of Composed Functors. We give just a few of these.

(I) The Hochschild-Serre Spectral Sequence for the Cohomology of Groups

Write G for a (topological) group, N for a (closed) normal subgroup and A for a (continuous) G-module. (Our main interest for non-finite or non-discrete groups is in the case of profinite groups because of their connection with Galois cohomology in the non-finite case. For a profinite group, the G-module is always given the discrete topology and the action $G \prod A \longrightarrow A$ is assumed continuous.)

We have three categories: G-mod, G/N-mod and $\mathcal{A}$b. And we have the two functors
$$A \rightsquigarrow H^0(N, A) = A^N \qquad (G\text{-mod} \rightsquigarrow G/N\text{-mod}),$$
and
$$B \rightsquigarrow H^0(G/N, B) = B^{G/N} \qquad (G/N\text{-mod} \rightsquigarrow \mathcal{A}\mathrm{b}).$$
Of course, their composition is exactly $A \rightsquigarrow A^G$. To apply Grothendieck's Theorem, we have to show that if Q is an injective G-module, then Q^N is G/N-cohomologically trivial. But, I claim Q^N is, in fact, G/N-injective. To see this, take $0 \longrightarrow M' \longrightarrow M$ exact in G/N-mod and look at the diagram (in G-mod)
$$\begin{tikzcd}
& Q & \\
& Q^N \arrow[u, hook] & \\
0 \arrow[r] & M' \arrow[u] \arrow[r] & M \arrow[uul, dotted]
\end{tikzcd}$$
Every G/N-module is a G-module (*via* the map $G \longrightarrow G/N$) and Q is G-injective; so, the dotted arrow exists as a G-homomorphism rendering the diagram commutative. Let θ be the dotted arrow; look at $\mathrm{Im}\,\theta$. If $q = \theta(m)$ and $\sigma \in N \subseteq G$, then $\sigma q = \theta(\sigma m) = \theta(m) = q$, because M is a G/N-module so N acts trivially on it. Therefore $q \in Q^N$ and so θ factors through Q^N, as required.

We obtain the Hochschild-Serre SS
$$H^p(G/N, H^q(N, A)) \underset{p}{\Longrightarrow} H^\bullet(G, A). \tag{HS}$$

Here is an application of importance for profinite groups (and Galois cohomology). If G is profinite, write $\mathrm{c.d.}(G) \le r$ (resp. $\mathrm{c.d.}_p(G) \le r$) provided $H^s(G, M) = (0)$ whenever M is a $\mathbb{Z}$-torsion G-module (resp. p-torsion G-module) and $s > r$. This notion is uninteresting for finite groups (see the exercises for the reason).

Theorem 5.60 *(Tower Theorem) If G is a profinite group and N is a closed normal subgroup, then*

$$\mathrm{c.d.}(G) \le \mathrm{c.d.}(N) + \mathrm{c.d.}(G/N).$$

(also true for $\mathrm{c.d.}_p$*).*

Proof. We may assume $\mathrm{c.d.}(N) \le a < \infty$ and $\mathrm{c.d.}(G/N) \le b < \infty$, otherwise the result is trivial. Let M be a torsion G-module and suppose $n > a + b$. All we need show is $H^n(G, M) = (0)$. Write $n = p + q$ with $p \ge 0$, $q \ge 0$. In the Hochschild-Serre SS, the terms

$$E_2^{p,q} = H^p(G/N, H^q(N, M))$$

must vanish. For if $p \le b$, then $q > a$ and $H^q(N, M)$ is zero by hypothesis. Now M is torsion therefore M^N is torsion and we saw in Chapter 4 that $H^q(N, M)$ is always torsion if $q > 0$ as it is a right limit of torsion groups. So, if $q \le a$, then $p > b$ and $E_2^{p,q} = (0)$ by hypothesis on G/N. Therefore, $E_s^{p,q} = (0)$ for all s with $2 \le s \le \infty$, when $p + q = n > a + b$. Hence, the terms in the composition series for $H^n(G, M)$ all vanish and we're done. $\square$

(II) The Leray Spectral Sequence

The set-up here is a morphism

$$\pi\colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

of ringed spaces (c.f. Section 5.3) and the three categories are: $\mathcal{S}(X)$, $\mathcal{S}(Y)$, $\mathcal{A}\mathrm{b}$. The functors are

$$\pi_*\colon \mathcal{S}(X) \rightsquigarrow \mathcal{S}(Y)$$

and

$$H^0(Y, -)\colon \mathcal{S}(Y) \rightsquigarrow \mathcal{A}\mathrm{b}.$$

Of course, $H^0(X, -)\colon \mathcal{S}(X) \rightsquigarrow \mathcal{A}\mathrm{b}$ is the composition $H^0(Y, -) \circ \pi_*$. We must show that if Q is an injective sheaf on X, then $\pi_* Q$ is cohomologically trivial on Y. Now every injective is flasque and flasque sheaves are cohomologically trivial; so, it will suffice to prove π_* takes flasque sheaves on X to flasque sheaves on Y.

But this is trivial, for if U and V are open on Y and $V \subseteq U$, then $\pi^{-1}(V) \subseteq \pi^{-1}(U)$ and

$$\begin{tikzcd}
\pi_* F(U) \arrow[r, equal] \arrow[d] & F(\pi^{-1}(U)) \arrow[d] \\
\pi_* F(V) \arrow[r, equal] & F(\pi^{-1}(V))
\end{tikzcd}$$

shows that surjectivity of the left vertical arrow follows from surjectivity on the right. We therefore obtain the Leray Spectral Sequence

$$H^p(Y, R^q\pi_* F) \underset{p}{\Longrightarrow} H^\bullet(X, F). \tag{LSS}$$

Unfortunately, full use of this spectral sequence demands considerable control of the sheaves $R^q\pi_* F$ and this is vitally affected by the map π; that is, by the "relative geometry and topology of X *vis a vis* Y". We must leave matters as they stand here.

(III) The Čech Cohomology Spectral Sequence

Once again, let $(X, \mathcal{O}_X)$ be a ringed space and write $\mathcal{S}(X)$ and $\mathcal{P}(X)$ for the categories of sheaves of $\mathcal{O}_X$-modules and presheaves of $\mathcal{O}_X$-modules. We also have two left exact functors from $\mathcal{P}(X)$ to $\mathcal{A}$b. Namely, if $\{U_\alpha \longrightarrow X\}_\alpha$ is an open cover of X and $G \in \mathcal{P}(X)$, then $H^0(\{U_\alpha \longrightarrow X\}_\alpha, G)$ is in $\mathcal{A}$b and we have $\check{H}^0(X, G)$, where the latter abelian group is what we called $G^{(+)}(X)$ in footnote 6 of Section 5.3. For the three abelian categories: $\mathcal{S}(X)$, $\mathcal{P}(X)$, $\mathcal{A}$b we now have the two composed functors

$$\mathcal{S}(X) \xhookrightarrow{i} \mathcal{P}(X) \xrightarrow{H^0(\{U_\alpha \to X\}, -)} \mathcal{A}\mathrm{b}$$

$$\mathcal{S}(X) \xhookrightarrow{i} \mathcal{P}(X) \xrightarrow{\check{H}^0(X, -)} \mathcal{A}\mathrm{b}.$$

Observe that both composed functors are the same functor:

$$F \in \mathcal{S}(X) \rightsquigarrow H^0(X, F) \in \mathcal{A}\mathrm{b}.$$

We need to show that if Q is an injective sheaf, then $i(Q)$ is acyclic for either $H^0(\{U_\alpha \to X\}, -)$ or $\check{H}^0(X, -)$. However, part (3) of Corollary 5.33 says that $i(Q)$ is injective as presheaf and is therefore acyclic. From Grothendieck's Theorem, we obtain the two *Čech Cohomology Spectral Sequences*:

$$H^p(\{U_\alpha \longrightarrow X\}_\alpha, \mathcal{H}^q(F)) \underset{p}{\Longrightarrow} H^\bullet(X, F) \tag{CCI}$$

$$\check{H}^p(X, \mathcal{H}^q(F)) \underset{p}{\Longrightarrow} H^\bullet(X, F) \tag{CCII}$$

Now it turns out that $\mathcal{H}^q(F)^\# = (0)$ for every $q > 0$ and every sheaf, F. (See the exercises.) Also, $\mathcal{H}^q(F)^{(+)} \subseteq \mathcal{H}^q(F)^\#$; so, we find

$$E_2^{0,q} = \check{H}^0(X, \mathcal{H}^q(F)) = \mathcal{H}^q(F)^{(+)} = (0), \quad \text{when} \quad q > 0.$$

If we apply the edge sequence to (CCII), we deduce

Proposition 5.61 *If $(X, \mathcal{O}_X)$ is a ringed space and F is a sheaf of $\mathcal{O}_X$-modules and if we continue to write F when F is considered as a presheaf (instead of $i(F)$), then*

(1) $\check{H}^1(X, F) \longrightarrow H^1(X, F)$ is an isomorphism and

(2) $\check{H}^2(X, F) \longrightarrow H^2(X, F)$ is injective.

(IV) The Local to Global Ext **Spectral Sequence**

Again, let $(X, \mathcal{O}_X)$ be a ringed space and fix a sheaf, A, of $\mathcal{O}_X$-modules on X. Write $\mathcal{S}(X)$ for the (abelian) category of $\mathcal{O}_X$-modules. We can make a functor from $\mathcal{S}(X)$ to itself, denoted $\mathcal{H}om_{\mathcal{O}_X}(A, -)$ *via*

$$\mathcal{H}om_{\mathcal{O}_X}(A, B)(U) = \mathrm{Hom}_{\mathcal{O}_X \restriction U}(A \restriction U, B \restriction U).$$

Here, U is open in X, the functor $\mathcal{H}om_{\mathcal{O}_X}(A, -)$ is usually called the *sheaf Hom*, it is (of course) left exact and its right derived functors (called *sheaf Ext*) are denoted $\mathcal{E}xt^\bullet_{\mathcal{O}_X}(A, -)$.

Therefore, we have the situation of three categories $\mathcal{S}(X)$, $\mathcal{S}(X)$, $\mathcal{A}$b and the two functors

$$\mathcal{H}om_{\mathcal{O}_X}(A, -) \colon \mathcal{S}(X) \rightsquigarrow \mathcal{S}(X)$$
$$H^0(X, -) = \Gamma(X, -) \colon \mathcal{S}(X) \rightsquigarrow \mathcal{A}\mathrm{b}$$

whose composition is the functor $\mathrm{Hom}_{\mathcal{O}_X}(A, -)$. In order to apply Grothendieck's Theorem, we must show that *if Q is injective in $\mathcal{S}(X)$, then $\mathcal{H}om_{\mathcal{O}_X}(A, Q)$ is an acyclic sheaf*. This, in turn, follows from

Proposition 5.62 *Suppose that Q is an injective sheaf of $\mathcal{O}_X$-modules. Then $\mathcal{H}om_{\mathcal{O}_X}(A, Q)$ is a flasque $\mathcal{O}_X$-module.*

Proof. If U is open in X, recall we have the presheaf A_U defined by

$$A_U(V) = \begin{cases} A(V) & \text{if } V \subseteq U \\ (0) & \text{if } V \nsubseteq U \end{cases}$$

and this gives rise to the associated sheaf $(A_U)^\sharp$. Now by adjointness,

$$\mathrm{Hom}_{\mathcal{O}_X}((A_U)^\sharp, B) \cong \mathrm{Hom}_{\mathcal{O}_X\text{-presheaves}}(A_U, i(B)).$$

On the right hand side, if V is open and $V \subseteq U$, then an element of $\mathrm{Hom}_{\mathcal{O}_X}(A_U, i(B))$ gives the map $A(V) \longrightarrow B(V)$ (consistent with restrictions). But, if $V \nsubseteq U$, we just get 0. However, this is exactly what we get from $\mathrm{Hom}_{\mathcal{O}_U}(A \restriction U, B \restriction U)$; therefore

$$\mathrm{Hom}_{\mathcal{O}_X}((A_U)^\sharp, B) = \mathrm{Hom}_{\mathcal{O}_X \restriction U}(A \restriction U, B \restriction U).$$

Now take Q to be an injective sheaf, we have to show that

$$\mathrm{Hom}_{\mathcal{O}_X}(A, Q) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X \restriction U}(A \restriction U, Q \restriction U)$$

is surjective for each open U of X. This means we must show that

$$\mathrm{Hom}_{\mathcal{O}_X}(A, Q) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}((A_U)^\sharp, Q)$$

is surjective. But, $0 \longrightarrow (A_U)^\sharp \longrightarrow A$ is exact and Q is injective; so, we are done. $\square$

We obtain the *local to global* Ext *spectral sequence*

$$H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(A, B)) \underset{p}{\Longrightarrow} \mathrm{Ext}^\bullet_{\mathcal{O}_X}(A, B). \tag{LGExt}$$

Remark: If $j \colon U \hookrightarrow X$ is the inclusion of the open set U in X, then the sheaf we have denoted $(A_U)^\sharp$ above is usually denoted $j_!A$. The functor, $j_!$, is left-exact and so we have a basic sequence of sheaf invariants $R^\bullet j_!$. Of course, we also have $R^\bullet \pi_*$ (for a morphism $\pi \colon Y \to X$) as well as π^*, $j^!$ (adjoint to $j_!$). The *six operations*

$$R^\bullet \pi_*,\ R^\bullet j_!,\ \pi^*,\ j^!,\ R^\bullet \mathcal{H}om,\ \otimes$$

were singled out by A. Grothendieck as the important test cases for the permanence of sheaf properties under morphisms.

(V) "Associativity" Spectral Sequences for Ext and Tor

In the proof of Grothendieck's Theorem on the spectral sequence for composed functors, there were two parts. In the first part, we used the essential hypothesis that $F(Q)$ was G-acyclic to compute the cohomology of the total complex (of our double complex) as $R^\bullet(GF(A))$—this is the ending of the spectral sequence. In the second part, which depends only on using a Cartan-Eilenberg resolution and did **not** use the G-acyclicity of $F(Q)$, we computed the spectral sequence $\mathrm{II}_2^{p,q} \Longrightarrow H^\bullet(C)$ and found $R^pG(R^qF(A)) \Longrightarrow H^\bullet(C)$. This second part is always available to us by Proposition 5.58 and we'll make use of it below.

We consider modules over various rings. In order that we have enough flexibility to specialize to varying cases of interest, we begin with three K-algebras, R, S, T and modules A, B, C as follows:

$$(\dagger) \begin{cases} A \text{ is a right } R \text{ and a right } S\text{-module} \\ B \text{ is a left } R\text{-module and a right } T\text{-module} \\ C \text{ is a right } S \text{ and a right } T\text{-module.} \end{cases}$$

Then

$A \otimes_R B$ is a right $S \otimes_K T$-module

and

$\mathrm{Hom}_T(B, C)$ is a right $R \otimes_K S$-module.

Observe that A is then a right $R \otimes_K S$-module *via*

$$a(r \otimes s) = (ar)s$$

because to say A is a right R and a right S-module is to imply

$$(ar)s = (as)r \qquad (\text{all } a \in A, r \in R, s \in S).$$

Also, C is a right $S \otimes_K T$-module. We know in this situation there is an "associativity" isomorphism

$$\mathrm{Hom}_{R\otimes_K S}(A, \mathrm{Hom}_T(B, C)) \cong \mathrm{Hom}_{S\otimes_K T}(A \otimes_R B, C). \tag{$*$}$$

If S is K-projective and $P_\bullet \longrightarrow A \longrightarrow 0$ is an $R \otimes_K S$-projective resolution of A, then $P_\bullet \longrightarrow A \longrightarrow 0$ is still an R-projective resolution of A and similarly if $0 \longrightarrow C \longrightarrow Q^\bullet$ is an $S \otimes_K T$-injective resolution, it still is a T-injective resolution of C. Our spectral sequences $\mathrm{II}_2^{p,q}$ then give us two spectral sequences with the same ending (by $(*)$):

$$\begin{aligned}
\mathrm{Ext}^p_{R\otimes_K S}(A, \mathrm{Ext}^q_T(B, C)) &\implies \mathrm{Ending}^\bullet \\
\mathrm{Ext}^p_{S\otimes_K T}(\mathrm{Tor}^R_q(A, B), C) &\implies \mathrm{Ending}^\bullet.
\end{aligned}$$

In a similar way, but this time if C is a (left) S and T-module, we get the "associativity" isomorphism

$$A \otimes_{R\otimes_K S} (B \otimes_T C) \cong (A \otimes_R B) \otimes_{S\otimes_K T} C. \tag{$**$}$$

Again, we assume S is K-projective and we get two spectral sequences with the same ending (by $(**)$):

$$\begin{aligned}
\mathrm{Tor}_p^{R\otimes_K S}(A, \mathrm{Tor}_q^T(B, C)) &\implies \widetilde{\mathrm{Ending}}^\bullet \\
\mathrm{Tor}_p^{S\otimes_K T}(\mathrm{Tor}_q^R(A, B), C) &\implies \widetilde{\mathrm{Ending}}^\bullet.
\end{aligned}$$

However, it is not clear how to compute the endings in these general cases. If we assume more, this can be done. For example, say $\mathrm{Tor}_q^R(A, B) = (0)$ if $q > 0$—this will be true when either A or B is flat over R—then the second Ext sequence and second Tor sequence collapse and we find

$$\begin{aligned}
\mathrm{Ext}^p_{R\otimes_K S}(A, \mathrm{Ext}^q_T(B, C)) &\implies \mathrm{Ext}^\bullet_{S\otimes_K T}(A \otimes_R B, C) \\
\mathrm{Tor}_p^{R\otimes_K S}(A, \mathrm{Tor}_q^T(B, C)) &\implies \mathrm{Tor}_\bullet^{S\otimes_K T}(A \otimes_R B, C).
\end{aligned}$$

We have proved all but the last statement of

Proposition 5.63 *Suppose R, S, T are K-algebras with S projective over K and say A is an R and S right module, C is an S and T right (resp. left) module and B is a left R and right T-module. Then there are spectral sequences with the same ending*

$$\begin{aligned}
\mathrm{Ext}^p_{R\otimes_K S}(A, \mathrm{Ext}^q_T(B, C)) &\implies \mathrm{Ending}^\bullet \\
\mathrm{Ext}^p_{S\otimes_K T}(\mathrm{Tor}^R_q(A, B), C) &\implies \mathrm{Ending}^\bullet
\end{aligned}$$

(resp.

$$\begin{aligned}\mathrm{Tor}_p^{R\otimes_K S}(A, \mathrm{Tor}_q^T(B,C)) &\Longrightarrow \widetilde{\mathrm{Ending}}^{\bullet}\\ \mathrm{Tor}_p^{S\otimes_K T}(\mathrm{Tor}_q^R(A,B), C) &\Longrightarrow \widetilde{\mathrm{Ending}}^{\bullet}\,)\end{aligned}$$

If $\mathrm{Tor}_q^R(A,B) = (0)$ *when* $q > 0$ *(e.g. if* A *or* B *is* R*-flat) then*

$$\mathrm{Ext}^p_{R\otimes_K S}(A, \mathrm{Ext}^q_T(B,C)) \Longrightarrow \mathrm{Ext}^{\bullet}_{S\otimes_K T}(A\otimes_R B, C) \tag{Ext}$$

and

$$\mathrm{Tor}_p^{R\otimes_K S}(A, \mathrm{Tor}_q^T(B,C)) \Longrightarrow \mathrm{Tor}_{\bullet}^{S\otimes_K T}(A\otimes_R B, C)). \tag{Tor}$$

Lastly, if B *is* T*-projective (more generally* $\mathrm{Ext}^q_T(B,C)$ *vanishes if* $q > 0$ *and* $\mathrm{Tor}_q^T(B,C)$ *vanishes if* $q > 0$*), then we have the* Ext *and* Tor *associativity formulae*

$$\mathrm{Ext}^p_{R\otimes_K S}(A, \mathrm{Hom}_T(B,C)) \cong \mathrm{Ext}^p_{S\otimes_K T}(A\otimes_R B, C)$$

and

$$\mathrm{Tor}_p^{R\otimes_K S}(A, B\otimes_T C) \cong \mathrm{Tor}_p^{S\otimes_K T}(A\otimes_R B, C).$$

Proof. The last statement is trivial as our spectral sequences (Ext), (Tor) collapse. □

Upon specializing the K-algebras R, S, T and the modules A, B, C, we can obtain several corollaries of interest. For example, let $S = R^{\mathrm{op}}$ and $A = R$. Then $\mathrm{Ext}^p_{R\otimes R^{\mathrm{op}}}(R, -) = H^p(R, -)$ in Hochschild's sense (by Section 5.3) and if R is K-projective the spectral sequences involving Ext yield

Corollary 5.64 *If* R *is* K*-projective then there is a spectral sequence*

$$H^p(R, \mathrm{Ext}^q_T(B,C)) \Longrightarrow \mathrm{Ext}^{\bullet}_{R^{\mathrm{op}}\otimes_K T}(B,C)$$

provided B *is a left* R *and right* T*-module and* C *is also a left* R *and right* T*-module.*

Note that this is reminiscent of the local-global Ext spectral sequence. Note further that if B is also *T-projective*, we deduce an isomorphism

$$H^p(R, \mathrm{Hom}_T(B,C)) \cong \mathrm{Ext}^p_{R^{\mathrm{op}}\otimes T}(B,C).$$

Next, let $A = B = R = K$ in the Ext-sequences. If S is K-projective the second Ext sequence collapses and gives $\mathrm{Ext}^p_{S\otimes_K T}(K,C) \cong \mathrm{Ending}^p$. The first spectral sequence then yields

Corollary 5.65 *Say* S *is* K*-projective and* S *and* T *possess augmentations to* K*, then we have the spectral sequence*

$$\mathrm{Ext}^p_S(K, \mathrm{Ext}^q_T(K,C)) \Longrightarrow \mathrm{Ext}^{\bullet}_{S\otimes_K T}(K,C),$$

where C *is a right* S *and right* T*-module.*

Here is another corollary:

Corollary 5.66 *Say* S *and* T *are* K*-algebras with* S *being* K*-projective. Assume* C *is a two-sided* $S\otimes_K T$*-module, then there is a spectral sequence*

$$H^p(S, H^q(T,C)) \Longrightarrow H^{\bullet}(S\otimes_K T, C).$$

Proof. For this use K, S^e, T^e in place of R, S, T. Now S^e is K-projective as S is so. Further replace A, B, C by (S, T, C)—this is O.K. because C is indeed both a right S^e and right T^e-module by hypothesis. The second Ext sequence collapses; so,

$$\mathrm{Ext}^p_{(S\otimes_K T)^e}(S \otimes_K T, C) \cong \mathrm{Ending}^p.$$

But, the left-side is just $H^p(S \otimes_K T, C)$ by definition. Now the $E_2^{p,q}$ term of our first Ext sequence is

$$\mathrm{Ext}^p_{S^e}(S, \mathrm{Ext}^q_{T^e}(T, C))$$

that is, it equals $H^p(S, H^q(T, C))$; so our proposition concludes the proof. $\square$

Clearly, there are analogous results for homology. Here are the conclusions, the exact hypotheses and the proofs will be left as (DX).

$$\begin{array}{rcl} H_p(T, \mathrm{Tor}^R_q(A, B)) & \Longrightarrow & \mathrm{Tor}^{R\otimes_K T^{\mathrm{op}}}_\bullet(A, B) \\ \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(A, K), K) & \Longrightarrow & \mathrm{Tor}^{R\otimes_K S}_\bullet(A, K) \\ H_p(S, H_q(R, A)) & \Longrightarrow & H_\bullet(R \otimes_K S, A). \end{array}$$

5.5 The Koszul Complex and Applications

In our previous work on the Hochschild cohomology of algebras, we studied the standard or bar complex, but we saw that it was inefficient in several cases of interest. As mentioned there, we have another, much better complex—the *Koszul complex*—which will serve for varied applications and which we turn to now.

Let A be a ring and M a module over this ring. For simplicity, we'll assume A is commutative as the main applications occur in this case. But, all can be done wih appropriate care in the general case. The Koszul complex is defined with respect to any given sequence $(f_1, \ldots, f_r)$ of elements of A. We write

$$\overrightarrow{f} = (f_1, \ldots, f_r).$$

Form the graded exterior power $\bigwedge^\bullet A^r$. We make $\bigwedge^\bullet A^r$ into a complex according to the following prescription: Since

$$\bigwedge^\bullet A^r = \coprod_{k=0}^{r} \bigwedge^k A^r,$$

it is a graded module, and we just have to define differentiation. Let $(e_1, \ldots, e_r)$ be the canonical basis of A^r, and set

$$de_j = f_j \in \bigwedge^0 A^r = A,$$

then extend d to be an antiderivation. That is, extend d *via*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta.$$

For example,

$$d(e_i \wedge e_j) = f_i e_j - f_j e_i,$$

and

$$\begin{aligned} d(e_i \wedge e_j \wedge e_k) &= d(e_i \wedge e_j) \wedge e_k + (e_i \wedge e_j) \wedge de_k \\ &= (f_i e_j - f_j e_i) \wedge e_k + f_k (e_i \wedge e_j) \\ &= f_i \widehat{e_i} \wedge e_j \wedge e_k - f_j e_i \wedge \widehat{e_j} \wedge e_k + f_k e_i \wedge e_j \wedge \widehat{e_k}, \end{aligned}$$

where, as usual, the hat above a symbol means that this symbol is omitted. By an easy induction, we get the formula:

$$d(e_{i_1} \wedge \cdots \wedge e_{i_t}) = \sum_{j=1}^{t} (-1)^{j-1} f_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_t}.$$

We denote this complex by $K_\bullet(\overrightarrow{f})$, i.e., it is the graded module $\bigwedge^\bullet A^r$ with the antiderivation d that we just defined. This is the *Koszul complex*.

Given an A-module M, we can make two Koszul complexes for the module M, namely:

$$\begin{aligned} K_\bullet(\overrightarrow{f}, M) &= K_\bullet(\overrightarrow{f}) \otimes_A M, \\ K^\bullet(\overrightarrow{f}, M) &= \mathrm{Hom}_A(K_\bullet(\overrightarrow{f}), M). \end{aligned}$$

We can take the homology and the cohomology respectively of these complexes, and we get the modules

$$H_\bullet(\overrightarrow{f}, M) \quad \text{and} \quad H^\bullet(\overrightarrow{f}, M).$$

For the cohomology complex, we need the explicit form of δ. Now,

$$K^t(\overrightarrow{f}, M) = \mathrm{Hom}_A(\bigwedge^t A^r, M),$$

and the family of elements of the form

$$e_{i_1} \wedge \cdots \wedge e_{i_t} \quad \text{with } 1 \leq i_1 < i_2 < \ldots < i_t \leq r,$$

is a basis of $\bigwedge^t A^r$; thus, $\mathrm{Hom}_A(\bigwedge^t A^r, M)$ is isomorphic to the set of alternating functions, g, from the set of sequences $(i_1, \ldots, i_t)$ of length t in $\{1, \ldots, r\}$ to M. Hence, the coboundary δ is given (on elements $g \in \mathrm{Hom}_A(\bigwedge^t A^r, M)$) by

$$(\delta g)(i_1, \ldots i_{t+1}) = \sum_{j=1}^{t+1} (-1)^{j-1} f_{i_j} g(i_1, \ldots, \widehat{i_j}, \ldots, i_{t+1}).$$

We have $H^0(\overrightarrow{f}, M) = Z^0(\overrightarrow{f}, M) = \mathrm{Ker}\, \delta$. (Note that $K^0(\overrightarrow{f}, M) = M$, *via* the map $g \mapsto g(1)$.) Then,

$$\delta g(e_i) = f_i g(1) = f_i m,$$

so $\delta f = 0$ implies that $f_i m = 0$ for all i. We find that

$$H^0(\overrightarrow{f}, M) = \{m \in M \mid \mathfrak{A} m = 0\}, \tag{5.1}$$

where $\mathfrak{A}$ is the ideal generated by $\{f_1, \ldots, f_r\}$. Also, it is clear that

$$H^t(\overrightarrow{f}, M) = 0 \quad \text{if } t < 0 \text{ or } t > r. \tag{5.2}$$

Let us compute the top cohomology group $H^r(\overrightarrow{f}, M)$. We have

$$Z^r(\overrightarrow{f}, M) = K^r(\overrightarrow{f}, M) = \mathrm{Hom}_A(\bigwedge^r A^r, M) = M,$$

via the map $g \mapsto g(e_1 \wedge \cdots \wedge e_r)$. Now, $\mathrm{Im}\, \delta_{r-1} = B^r(\overrightarrow{f}, M)$, but what is $B^r(\overrightarrow{f}, M)$? If $g \in K^{r-1}(\overrightarrow{f}, M)$ is an alternating function on $i_1, \ldots, i_{r-1}$, then

$$\delta_{r-1} g(1, \ldots, r) = (\delta_{r-1} g)(e_1 \wedge \cdots \wedge e_r) = \sum_{j=1}^{r} (-1)^{j-1} f_j g(1, \ldots, \widehat{j}, \ldots, r).$$

Therefore,

$$B^r = f_1 M + \cdots + f_r M,$$

and we find that

$$H^r(\overrightarrow{f}, M) = M/(f_1 M + \cdots + f_r M) = M/\mathfrak{A} M.$$

It is important to connect the Koszul homology (whose boundary map is

$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_t} \otimes m) = \sum_{j=1}^{t} (-1)^{j-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots e_{i_t} \otimes f_{i_j} m)$$

and cohomology *via* the notion of *Koszul duality*. This is the following: Consider $K_t(\overrightarrow{f}, M)$; an element of $K_t(\overrightarrow{f}, M)$ has the form

$$h = \sum e_{i_1} \wedge \cdots \wedge e_{i_t} \otimes z_{i_1 \ldots i_t}, \quad \text{where } 1 \leq i_1 < i_2 < \ldots < i_t \leq r.$$

We define a map (the duality map)

$$\Theta\colon K_t(\overrightarrow{f}, M) \longrightarrow K^{r-t}(\overrightarrow{f}, M)$$

as follows: Pick $j_1 < j_2 < \cdots < j_{r-t}$, and set

$$\Theta(h)(j_1, \ldots, j_{r-t}) = \epsilon z_{i_1 \ldots i_t},$$

where

(α) $i_1, \ldots, i_t$ is the set of complementary indices of $j_1, \ldots, j_{r-t}$ taken in ascending order,

(β) ϵ is the sign of the permutation

$$(1, 2, \ldots, r) \mapsto (i_1, \ldots, i_t, j_1, \ldots, j_{r-t}),$$

where both $i_1, \ldots, i_t$ and $j_1, \ldots, j_{r-t}$ are in ascending order.

We find (DX) that

$$\Theta(\partial h) = \delta\Theta(h),$$

where ∂ is the homology boundary map described above. So, the isomorphism, Θ, induces an isomorphism

$$H_t(\overrightarrow{f}, M) \cong H^{r-t}(\overrightarrow{f}, M) \quad \text{for all } t \geq 0,$$

which is called *Koszul duality*. This notion of Koszul duality does not look like a duality, but we can make it look so. For this, write $Q(A)$ for "the" injective hull of A as A-module and set $M^D = \mathrm{Hom}_A(M, Q(A))$. The cofunctor $M \rightsquigarrow M^D$ is exact; we'll refer to M^D as the dual of M. Now the associativity isomorphism

$$\mathrm{Hom}_A(M \otimes_A N, Z) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, Z))$$

shows that $(K_t(\overrightarrow{f}, M))^D$ is isomorphic to $K^t(\overrightarrow{f}, M^D)$. Moreover, it is easy to see that

$$\begin{tikzcd}
(K_t(\overrightarrow{f}, M))^D \arrow[r, "\cong"] & K^t(\overrightarrow{f}, M^D) \\
(K_{t-1}(\overrightarrow{f}, M))^D \arrow[r, "\cong"] \arrow[u, "\partial_t^D"] & K^{t-1}(\overrightarrow{f}, M^D) \arrow[u, "\delta^{t-1}"]
\end{tikzcd}$$

is a commutative diagram. So, it follows (by the exactness of $M \rightsquigarrow M^D$) (DX) that our isomorphisms yield isomorphisms

$$H_t(\overrightarrow{f}, M)^D \cong H^t(\overrightarrow{f}, M^D), \quad \text{for all } t \geq 0. \tag{5.3}$$

Put these together with the above notion of Koszul duality and obtain the *duality isomorphisms*

$$\begin{aligned}
H^t(\overrightarrow{f}, M^D) &\cong H^{r-t}(\overrightarrow{f}, M)^D \\
H_t(\overrightarrow{f}, M^D) &\cong H_{r-t}(\overrightarrow{f}, M)^D, \quad \text{for all } t \geq 0.
\end{aligned}$$

Gathering together what we have proved above, we find the following

Proposition 5.67 *If A is a (commutative) ring, M is an A-module, and $\overrightarrow{f} = (f_1, \ldots, f_r)$ an ordered set of r elements from A, then for the Koszul homology and cohomology of M we have*

(0) $H_t(\overrightarrow{f}, M) = H^t(\overrightarrow{f}, M) = (0)$ *if* $t < 0$ *or* $t > r$,

(1) (Koszul duality) There is an isomorphism

$$H_t(\overrightarrow{f}, M) \cong H^{r-t}(\overrightarrow{f}, M), \quad \textit{all } t \geq 0,$$

$$\begin{aligned} H_0(\overrightarrow{f}, M) = H^r(\overrightarrow{f}, M) &= M/\mathfrak{A}M, \\ H^0(\overrightarrow{f}, M) = H_r(\overrightarrow{f}, M) &= \{m \mid \mathfrak{A}m = 0\}, \end{aligned} \tag{2}$$

where $\mathfrak{A}$ *is the ideal generated by* $f_1, \ldots, f_r$.

Write $M^D = \mathrm{Hom}_A(M, Q(A))$ *with* $Q(A)$ *the injective hull of* A*, then*

(3) $H_t(\overrightarrow{f}, M)^D \cong H^t(\overrightarrow{f}, M^D)$

and Koszul duality becomes

$$\begin{aligned} H^t(\overrightarrow{f}, M^D) &\cong H^{r-t}(\overrightarrow{f}, M)^D, \\ H_t(\overrightarrow{f}, M^D) &\cong H_{r-t}(\overrightarrow{f}, M)^D, \quad \textit{for all } t \geq 0. \end{aligned}$$

We need one more definition to exhibit the main algebraic property of the Koszul complex.

Definition 5.12 The sequence $\overrightarrow{f} = (f_1, \ldots, f_r)$ is *regular* for M or M*-regular* if for every i, with $1 \leq i \leq r$, the map

$$z \mapsto f_i z$$

is an injection of $M/(f_1M + \cdots + f_{i-1}M)$ to itself.

By its very definition, the notion of M-regularity appears to depend on the order of the elements $f_1, \ldots, f_r$. This is indeed the case as the following classical example [39] shows: Let A be $\mathbb{C}[X, Y, Z]$ and $f_1 = X(Y-1)$; $f_2 = Y$; $f_3 = Z(Y-1)$. Then unique factorization in A shows that f_1, f_2, f_3 is A-regular, but f_1, f_3, f_2 is certainly not A-regular as f_3X is zero in A/f_1A but X is not zero there. In the special case that A is graded, M is a graded module and the f_j are homogeneous elements of A, the order of an M-sequence does not matter.

If $\mathfrak{A}$ is a given ideal of A and $f_1, \ldots, f_r \in \mathfrak{A}$ (the f_j are not necessarily generators of $\mathfrak{A}$), and if $f_1, \ldots, f_r$ is an M-regular sequence but no for other element $g \in \mathfrak{A}$ is $f_1, \ldots, f_r, g$ an M-regular sequence, then $f_1, \ldots, f_r$ is a *maximal* M*-regular sequence from* $\mathfrak{A}$. It turns out that *the number of elements in a maximal* M*-regular sequence from* $\mathfrak{A}$ *is independent of the choice of such a sequence*; this number is called the $\mathfrak{A}$*-depth* of M and denoted $\mathrm{depth}_{\mathfrak{A}} M$. (When A is a local ring and $\mathfrak{A} = \mathfrak{M}$ is its maximal ideal, one writes $\mathrm{depth}\, M$ and omits any reference to $\mathfrak{M}$.)

Here is the main property of the Koszul complex *vis a vis* M-regularity (and, hence, depth):

Proposition 5.68 *(Koszul) Suppose* M *is an* A*-module and* $\overrightarrow{f}$ *is an* M*-regular sequence of length* r*. Then the Koszul complexes* $K_\bullet(\overrightarrow{f}, M)$ *and* $K^\bullet(\overrightarrow{f}, M)$ *are acyclic and consequently*

$$H_i(\overrightarrow{f}, M) = (0) \quad \textit{if} \quad i \neq 0 \quad \textit{and} \quad H^i(\overrightarrow{f}, M) = (0) \quad \textit{if} \quad i \neq r.$$

Proof. The two Koszul complexes

$$K_\bullet(\overrightarrow{f}, M)\colon M \xrightarrow{\partial_r} \bigwedge^{r-1} A^r \otimes_A M \xrightarrow{\partial} \cdots \xrightarrow{\partial} A^r \otimes_A M \xrightarrow{\partial_1} M$$

$$K^\bullet(\overrightarrow{f}, M)\colon M \xrightarrow{\delta^0} \mathrm{Hom}_A(A^r, M) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathrm{Hom}_A(\bigwedge^{r-1} A^r, M) \xrightarrow{\delta^{r-1}} M$$

will be exact sequences when $H_1(\overrightarrow{f}, M) = \cdots = H_{r-1}(\overrightarrow{f}, M) = (0)$ and when $H^1(\overrightarrow{f}, M) = \cdots = H^{r-1}(\overrightarrow{f}, M) = (0)$; so the vanishing statement of the conclusion appears stronger than acyclicity. But, under our hypothesis the modules

$$H_r(\overrightarrow{f}, M) = H^0(\overrightarrow{f}, M) = \{m \mid \mathfrak{A}m = (0)\}$$

automatically vanish since f_1 is a non-zero divisor on M.

We will prove the vanishing statements and, of course, by Koszul duality all we need prove is that $H_t(\overrightarrow{f}, M) = (0)$ for all $t > 0$. There are several ways of proving this; all use induction on r, the length of the M-sequence. We choose a method involving the tensor product of complexes.

If $C_\bullet$ and $D_\bullet$ are left complexes, we make their tensor product $C_\bullet \otimes D_\bullet$ by setting

$$(C_\bullet \otimes D_\bullet)_t = \coprod_{i+j=t} C_i \otimes D_j$$

and defining differentiation by

$$d(\alpha \otimes \beta) = d_C(\alpha) \otimes \beta + (-1)^{\deg \alpha} \alpha \otimes d_D(\beta).$$

Then, $(C_\bullet \otimes D_\bullet)_\bullet$ is a complex. Consider for example the Koszul complex for the single element $f \in A$. Namely,

$$K_\bullet(f)_t = \begin{cases} A & \text{if } t = 0 \text{ or } 1 \\ (0) & \text{if } t > 1 \end{cases}$$

a two term complex. Its differentiation is given by $d(e) = f$, where $e\ (= 1)$ is a base for A as A-module; in other words, d is just multiplication by f. With this notation, we have

$$K_\bullet(\overrightarrow{f}) = K_\bullet(f_1) \otimes \cdots \otimes K_\bullet(f_r).$$

Now the vanishing statements are true and trivial for $r = 0$ or 1. So, write $\overrightarrow{f'} = (f_1, \ldots, f_{r-1})$ and set $L_\bullet = K_\bullet(\overrightarrow{f'}, M)$. Since $\overrightarrow{f'}$ is M-regular we see that

$$H_t(\overrightarrow{f'}, M) = H_t(L_\bullet) = (0), \quad \text{for all } t > 0,$$

by the induction hypothesis. Further, set $M_\bullet = K_\bullet(f_r, M)$. Then $K_\bullet(\overrightarrow{f}, M) = (L_\bullet \otimes M_\bullet)_\bullet$, and this will enable our induction.

I claim that we have the exact sequence

$$\cdots \longrightarrow H_0(H_t(L_\bullet) \otimes M_\bullet) \longrightarrow H_t(L_\bullet \otimes M_\bullet) \longrightarrow H_1(H_{t-1}(L_\bullet) \otimes M_\bullet) \longrightarrow \cdots \tag{5.4}$$

for every $t \geq 0$. Suppose this claim is proved, take $t \geq 2$ (so that $t - 1 \geq 1$) and get

$$H_t(L_\bullet) = H_{t-1}(L_\bullet) = (0)$$

by the induction hypothesis. The exact sequence (5.4) tells us that $H_t(\overrightarrow{f}, M) = H_t(L_\bullet \otimes M_\bullet) = (0)$ when $t \geq 2$. If $t = 1$, we know that $H_1(L_\bullet)$ vanishes, so (5.4) gives us the exact sequence

$$0 \longrightarrow H_1(\overrightarrow{f}, M) \longrightarrow H_1(H_0(L_\bullet) \otimes M_\bullet).$$

But $H_1(-) = H^0(-)$ by Koszul duality for $M_\bullet$ and the latter is the kernel of multiplication by f_r on $(-)$. However, in this case $(-)$ is $H_0(L_\bullet) = M/(f_1 M + \cdots + f_{r-1} M)$; the kernel of multiplication by f_r on this last module is zero because $f_1, \ldots, f_r$ is M-regular. We conclude $H_1(H_0(L_\bullet) \otimes M_\bullet)$ is zero, finishing our induction.

There remains only the proof of exact sequence (5.4). It, in turn, follows from a general homological lemma:

Lemma 5.69 *Suppose M is a two-term complex of A-modules, zero in degree $\neq 0, 1$ and for which M_0 and M_1 are free A-modules. If $L_\bullet$ is any complex of A-modules, we have the exact sequence*

$$\cdots \longrightarrow H_0(H_t(L_\bullet) \otimes M_\bullet) \longrightarrow H_t(L_\bullet \otimes M_\bullet) \longrightarrow H_1(H_{t-1}(L_\bullet) \otimes M_\bullet) \longrightarrow \cdots \tag{5.4}$$

for all $t \geq 0$.

Proof. Once again, we have more than one proof available. We'll sketch the first and give the second in detail. The modules comprising $M_\bullet$ are A-free, so there is a "Künneth Formula" spectral sequence

$$E^2_{p,q} = H_p(H_q(L_\bullet) \otimes_A M_\bullet) \Longrightarrow H_\bullet(L_\bullet \otimes_A M_\bullet).$$

(For example, see Corollary 5.66 and its homology analog.) But, as $M_\bullet$ is a two-term complex, $E^2_{p,q} = (0)$ if $p \neq 0, 1$ and we obtain the zipper sequence (5.4) of our lemma.

More explicitly (our second proof), we make two one-term complexes, M_i, in which M_i has its one term in degree i $(i = 0, 1)$. Each differentiation in these complexes is to be the trivial map. We form the tensor product complexes $L_\bullet \otimes M_i$ and recall that

$$\begin{cases} (L_\bullet \otimes M_0)_p = L_p \otimes M_0 \\ d(\alpha \otimes \beta) = d_L(\alpha) \otimes \beta \\ H_p(L_\bullet \otimes M_0) = H_p(L_\bullet) \otimes M_0 \end{cases}$$

and

$$\begin{cases} (L_\bullet \otimes M_1)_p = L_{p-1} \otimes M_1 \\ d(\alpha \otimes \beta) = d_L(\alpha) \otimes \beta \\ H_p(L_\bullet \otimes M_1) = H_{p-1}(L_\bullet) \otimes M_1. \end{cases}$$

Then, we obtain an exact sequence of complexes

$$0 \longrightarrow L_\bullet \otimes M_0 \longrightarrow L_\bullet \otimes M_\bullet \longrightarrow L_\bullet \otimes M_1 \longrightarrow 0$$

and its corresponding long exact homology sequence

$$\cdots H_{p+1}(L_\bullet \otimes M_1) \xrightarrow{\partial} H_p(L_\bullet \otimes M_0) \longrightarrow H_p(L_\bullet \otimes M_\bullet) \longrightarrow H_p(L_\bullet \otimes M_1) \longrightarrow \cdots.$$

But, ∂ is just $1 \otimes d_{M_\bullet}$ and so the above homology sequence is exactly (5.4). $\square$

The main applications we shall make of the Koszul complex concern the notion of "dimension"—and even here most applications will be to commutative rings. We begin by defining the various notions of dimension. Suppose R is a ring (not necessarily commutative) and M is an R-module.

Definition 5.13 The module M has *projective dimension* (resp. *injective dimension*) $\leq n$ if and only if it possesses a projective (resp. injective) resolution $P_\bullet \longrightarrow M \longrightarrow 0$ (resp. $0 \longrightarrow M \longrightarrow Q^\bullet$) for which $P_t = 0$ (resp. $Q^t = 0$) when $t > n$. The infimum of the integers n for which M has projective (resp. injective) dimension $\leq n$ is called the *projective dimension* (resp. *injective dimension*) *of* M.

Remark: Of course, if no n exists so that proj dim $M \leq n$, then we write proj dim $M = \infty$ and similarly for injective dimension. A module is projective (resp. injective) iff it has proj (resp. inj) dim $= 0$. It is convenient to set proj (or inj) dim (0) equal to $-\infty$. If M is a right R-module, it is an R^{op}-module and so it has proj (and inj) dimension as R^{op}-module. Therefore, it makes sense to include R in the notation and we'll write $\dim_R M$ for the projective or injective dimension of M (as R-module) when no confusion can arise.

By this time, the following propositions, characterizing the various dimensions, are all routine to prove. So, we'll omit all the proofs leaving them as (DX's).

Proposition 5.70 *If R is a ring and M is an R-module, then the following are equivalent conditions:*

(1) M has projective dimension $\leq n$ (here, $n \geq 0$)

(2) $\mathrm{Ext}_R^{n+1}(M, -) = (0)$

(3) $\mathrm{Ext}_R^{n}(M, -)$ *is a right exact functor*

(4) If $0 \longrightarrow X_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ is an acyclic resolution of M and if $P_0, \ldots, P_{n-1}$ are R-projective, then X_n is also R-projective

Also, the following four conditions are mutually equivalent:

(1′) M has injective dimension $\leq n$ (here, $n \geq 0$)

(2′) $\mathrm{Ext}_R^{n+1}(-, M) = (0)$

(3′) $\mathrm{Ext}_R^{n}(-, M)$ *is a right exact functor*

(4′) If $0 \longrightarrow M \longrightarrow Q^0 \longrightarrow \cdots \longrightarrow Q^{n-1} \longrightarrow X^n \longrightarrow 0$ is an acyclic resolution of M and if $Q^0, \ldots, Q^{n-1}$ are R-injective, then X^n is also R-injective.

If we use the long exact sequence of (co)homology, we get a corollary of the above:

Corollary 5.71 *Say $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of R-modules.*

(1) If $\dim_R M'$ *and* $\dim_R M'' \leq n$ *(either both projective or both injective dimension), then* $\dim_R M \leq n$

(2) Suppose M is projective, then either

(a) $\dim_R M'' = 0$ *(i.e., M'' is projective), in which case M' is also projective; or*

(b) $\dim_R M'' \geq 1$, *in which case* $\dim_R M' = \dim_R M'' - 1$.

To get an invariant of the underlying ring, R, we ask for those n for which $\mathrm{projdim}_R\, M \leq n$ (resp. $\mathrm{injdim}_R\, M \leq n$) for *all* R-modules M. For such an n, we write $\mathrm{gldim}\, R \leq n$ and say the *global dimension of R is less than or equal to n*. (It will turn out that we can check this using either projdim for all M or injdim for all M; so, no confusion can arise.) Of course, the infimum of all n so that $\mathrm{gldim}\, R \leq n$ is called the *global dimension of R*. When we use right R-modules, we are using R^{op}-modules and so are computing $\mathrm{gldim}\, R^{\mathrm{op}}$.

Notice that $\mathrm{gldim}\, R$ is an invariant computed from the category of R-modules. So, *if R and S are rings and if there is an equivalence of categories R-mod $\approx$ S-mod, then* $\mathrm{gldim}\, R = \mathrm{gldim}\, S$. Rings for which R-mod $\approx$ S-mod are called *Morita equivalent rings*. For commutative rings, it turns out that Morita equivalence is just isomorphism; this is not true for non-commutative rings. Indeed, if R is a ring and $M_n(R)$ denotes, as usual, the ring of $n \times n$ matrices over R, then $R \approx M_n(R)$. Moreover, this is almost the full story. Also, if $R \approx S$, then $R^{\mathrm{op}} \approx S^{\mathrm{op}}$. Now for a field, K, we clearly have $\mathrm{gldim}\, K = 0$; so, we find $\mathrm{gldim}\, M_n(K) = 0$, as well. If A is a commutative ring and G is a group, then the map $\sigma \mapsto \sigma^{-1}$ gives an isomorphism of $A[G]$ onto $A[G]^{\mathrm{op}}$. Hence, $\mathrm{gldim}\, A[G] = \mathrm{gldim}\, A[G]^{\mathrm{op}}$.

Proposition 5.72 *Let R be a ring and let n be a non-negative integer. Then the following statements are equivalent:*

(1) Every R-module, M, has $\mathrm{projdim}_R \leq n$.

(2) Every R-module, M, has $\mathrm{injdim}_R \leq n$.

(3) $\mathrm{gldim}\, R \leq n$.

(4) $\mathrm{Ext}^t_R(-,-) = (0)$ *for all $t > n$.*

(5) $\mathrm{Ext}^{n+1}_R(-,-) = (0)$.

(6) $\mathrm{Ext}^n_R(-,-)$ *is right-exact.*

A ring R is called *semi-simple* if and only if every submodule, N, of each R-module, M, possesses an R-complement. (We say M is *completely reducible.*) That is, iff given $N \subseteq M$, there is a submodule $\widetilde{N} \subseteq M$ so that the natural map $N \coprod \widetilde{N} \longrightarrow M$ is an isomorphism (of R-modules). Of course each field, K, or division ring, D, is semi-simple. But, again, semi-simplicity is a property of the category R-mod; so $M_n(K)$ and $M_n(D)$ are also semi-simple. It is a theorem of Maschke that if K is a field, G is a finite group, and $(\mathrm{ch}(K), \#(G)) = 1$, then the group algebra, $K[G]$, is semi-simple. See Problem 134 for this result. Again, there are many equivalent ways to characterize semi-simplicity:

Proposition 5.73 *For any ring, R, the following statements are equivalent:*

(1) R is semi-simple.

(2) R^{op} is semi-simple.

(3) R, as R-module, is a coproduct of simple R-modules.

(4) R, as R-module, is completely reducible.

(5) Each (left) ideal of R is an injective module.

(6) Every R-module is completely reducible.

(7) In R-mod, every exact sequence splits.

(8) Every R-module is projective.

(9) Every R-module is injective.

(10) $\mathrm{gldim}\, R = 0$.

The proofs of these equivalences will be left as the material of Problem 145. Note that dimension is defined using $\mathrm{Ext}^{\bullet}_R(-,-)$ and $\mathrm{Tor}^R_{\bullet}$ is not mentioned. There are two reasons for this. First, while Hom, Ext, projective and injective are properties of abelian categories, tensor and Tor are generally not. Second, the vanishing of Tor characterizes flatness which is a weaker property than projectivity. However, for commutative rings, the notions of dimension and global dimension are frequently reduced by localization to the case of local rings. For noetherian local rings, we already know flatness and freeness are equivalent for f.g. modules; so over noetherian local rings the vanishing of Tor is connected with dimension (at least on the category of f.g. modules). In the general case, when we use Tor, we call the resulting invariant the Tor-*dimension*. It's easy to see that when R is a PID we have $\mathrm{gldim}\, R \leq 1$.

For our main applications of the Koszul complex, we return to the situation of a pair (R, Q) in which R is a ring and $\epsilon\colon R \to Q$ is a *surjective* R-module map. Such a pair is an *augmented ring*, the map ϵ is the augmentation (as discussed in Section 5.3) and Q is the *augmentation module*. As usual, write I for the augmentation ideal (just a left ideal, in general): $I = \mathrm{Ker}\ \epsilon$. Then the exact sequence

$$0 \longrightarrow I \longrightarrow R \xrightarrow{\epsilon} Q \longrightarrow 0$$

and Corollary 5.71 above show:

Either Q is projective (so that I is projective) or $1 + \dim_R I = \dim_R Q$.

Note that if R is commutative then I is a 2-sided ideal and Q becomes a ring if we set $\epsilon(r)\cdot\epsilon(p) = r\cdot\epsilon(p)$; i.e., if we make ϵ a ring homomorphism. The map ϵ is then a section in case R is a Q-algebra. Here is the main result on which our computations will be based.

Theorem 5.74 *Assume (R, Q) is an augmented ring and suppose I is finitely generated (as R-ideal) by elements $f_1, \ldots, f_r$* **which commute with each other***. If $f_1, \ldots, f_r$ form an R-regular sequence, then $\dim_R Q = r$ (if $Q \neq (0)$). In particular, $\mathrm{gldim}\, R \geq r$.*

Proof. Write A for the commutative ring $\mathbb{Z}[T_1, \ldots, T_r]$, then as the $f_1, \ldots, f_r$ commute with each other, R becomes an A-module if we make T_j operate *via* $\rho T_j = \rho f_j$ for all $\rho \in R$. We form the Koszul complex $K_{\bullet}(\overrightarrow{T})$ for A and then form $R \otimes_A K_{\bullet}(\overrightarrow{T})$. The latter is clearly the Koszul complex $K_{\bullet}(\overrightarrow{f}, R)$ and as $(f_1, \ldots, f_r)$ is an R-regular sequence, $K_{\bullet}(\overrightarrow{f}, R)$ is acyclic. Thus, we obtain the exact sequence

$$0 \longrightarrow R \xrightarrow{\partial_r} \bigwedge^{r-1}(R^r) \xrightarrow{\partial_{r-1}} \cdots \xrightarrow{\partial_2} \bigwedge^{1}(R^r) \xrightarrow{\partial_1} R \longrightarrow Q \longrightarrow 0 \qquad (*)$$

because we know the image of ∂_1 is the (left) ideal generated by $f_1, \ldots, f_r$; that is, $\mathrm{Im}\ \partial_1 = I$. Now $(*)$ is visibly an R-projective resolution of Q and so $\dim_R Q \leq r$.

Since $(*)$ is an R-projective resolution of Q, we can use it to compute $\mathrm{Ext}^{\bullet}_R(Q, -)$. In particular, we can compute $\mathrm{Ext}^{\bullet}_R(Q, Q)$—this is the cohomology of the complex $\mathrm{Hom}_R((*), Q)$. But, the latter complex is just $K^{\bullet}(\overrightarrow{f}, Q)$. We find

$$\mathrm{Ext}^r_R(Q, Q) = H^r(\overrightarrow{f}, Q) = Q$$

and so $\dim_R Q = r$ provided $Q \neq (0)$. $\square$

Corollary 5.75 *If K is a ring (not necessarily commutative) and R is the graded ring $K[T_1, \ldots, T_r]$, then $\dim_R K = r$. If K is a field or division ring and R is the local ring of formal power series $K[[T_1, \ldots, T_r]]$, then $\dim_R K = r$. (This is also true if K is any ring though R may not be local.) Lastly, if K is a field complete with respect to a valuation and R is the local ring of converging power series $K\{T_1, \ldots, T_r\}$, then $\dim_R K = r$. In all these cases, $\mathrm{gldim}\, R \geq r$.*

Proof. In each case, the variables $T_1, \ldots, T_r$ play the role of the $f_1, \ldots, f_r$ of our theorem; all hypotheses are satisfied. $\square$

Notice that for $A\ (= \mathbb{Z}[T_1, \ldots, T_r])$, the Koszul resolution

$$0 \longrightarrow A \longrightarrow \bigwedge^{r-1}(A^r) \longrightarrow \cdots \bigwedge^{1}(A^r) \longrightarrow A \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{$**$}$$

can be used to compute $\mathrm{Tor}_{\bullet}^{A}(-, \mathbb{Z})$ as well as $\mathrm{Ext}_{A}^{\bullet}(\mathbb{Z}, -)$. So for M, any A-module,

$$\mathrm{Tor}_p^A(M, \mathbb{Z}) = H_p(\overrightarrow{T}, M) \quad \text{and} \quad \mathrm{Ext}_A^p(\mathbb{Z}, M) = H^p(\overrightarrow{T}, M).$$

By Koszul duality,

$$\mathrm{Tor}_p^A(M, \mathbb{Z}) \cong \mathrm{Ext}_A^{r-p}(\mathbb{Z}, M).$$

Further, the acyclicity of $M \otimes_A K_{\bullet}(\overrightarrow{T})$ is equivalent with $\mathrm{Tor}_p^A(M, \mathbb{Z}) = (0)$ when $p > 0$.

Now, recall that, for a ring R possessing a section $R \xrightarrow{\epsilon} K$ (here, R is a K-algebra), we defined the homology and cohomology "bar" groups by

$$\begin{aligned} \overline{H}_n(R, M) &= \mathrm{Tor}_n^R(M, K) \qquad (M \text{ an } R^{\mathrm{op}}\text{-module}) \\ \overline{H}^n(R, M) &= \mathrm{Ext}_R^n(K, M) \qquad (M \text{ an } R\text{-module}). \end{aligned}$$

In the cases

(1) $R = K[T_1, \ldots, T_r]$

(2) $R = K[[T_1, \ldots, T_r]]$

(3) $R = K\{T_1, \ldots, T_r\}$ $\quad$ (K has a topology),

our discussion above shows that

$$\overline{H}_n(R, M) = H_n(\overrightarrow{T}, M) \quad \text{and} \quad \overline{H}^n(R, M) = H^n(\overrightarrow{T}, M).$$

So, by the Hochschild (co)homology comparison theorem (Theorem 5.29), we see that *the Hochschild groups* $H_n(R, \epsilon_*(M))$ *and* $H^n(R, \epsilon_*^{\mathrm{op}}(M))$ *can be computed by the Koszul complexes* $K_{\bullet}(\overrightarrow{T}, M)$ *and* $K^{\bullet}(\overrightarrow{T}, M)$ *in cases (1)–(3) above*. This is what we alluded to at the end of the discussion following Theorem 5.29.

We now face the problem of the global dimension of a ring R. We assume R is not only an augmented ring (with augmentation module, Q, and ideal, I) but in fact *that I is a two-sided ideal so that Q is a ring and $\epsilon\colon R \to Q$ is a ring homomorphism*. Experience shows that for certain types of rings some subclasses of modules have more importance than others. For example, if R is a graded ring, the graded modules are the important ones for these are the ones giving rise to sheaves over the geometric object corresponding to R (a generalized projective algebraic variety) and the cohomology groups of these sheaves are geometric invariants of the object in question. Again, if R is a (noetherian) local ring, the finitely generated modules are the important ones as we saw in Chapter 3. It makes sense therefore to compute the global dimension of R with respect to the class of "important" R-modules, that is to define

$$\mathcal{I}\text{-gldim}\, R = \inf\{m \mid \mathcal{I}\text{-gldim}_R \leq m\},$$

where $\mathcal{I}\text{-gldim}_R \leq m$ iff for every *important* module, M, we have $\dim_R M \leq m$; (here, $\mathcal{I}$–stands for "important").

Eilenberg ([9]) abstracted the essential properties of the graded and finitely generated modules to give an axiomatic treatment of the notion of the class of "important" modules. As may be expected, the factor ring, Q, plays a decisive role. Here is the abstract treatment together with the verification that for graded (resp. local) rings, the graded (resp. finitely generated) modules satisfy the axioms.

(A) Call an R^{op}-module, M, *pertinent* provided $M \otimes_R Q \neq (0)$ when $M \neq (0)$; also (0) is to be pertinent.

If R is a graded ring, say $R = \coprod_{j \geq 0} R_j$, then we set $I = R^{(+)} = \coprod_{j>0} R_j$ and $Q = R_0$. *When M is a graded R^{op}-module with* **grading bounded below** *then M is pertinent*. For we have $M = \coprod_{n \geq B} M_n$; so, $MI = \coprod_{n \geq B+1} M_n \neq M$. But $M \otimes_R Q = M/MI$. When R is a local ring, we set $I = \mathfrak{M}_R$ (its maximal ideal) and then $Q = \kappa(R)$–the residue field. Of course, *all f.g. R^{op}-modules are pertinent* by Nakayama's Lemma.

If S is a subset of a module, M, write $F(S)$ for the free R (or R^{op})-module generated by S. Of course, there's a natural map $F(S) \longrightarrow M$ and we get an exact sequence

$$0 \longrightarrow \mathrm{Ker}\,(S) \longrightarrow F(S) \longrightarrow M \longrightarrow \mathrm{cok}(S) \longrightarrow 0.$$

(B) The subset, S, of M is *good* provided $0 \in S$ and for each $T \subseteq S$, in the exact sequence

$$0 \longrightarrow \mathrm{Ker}\,(T) \longrightarrow F(T) \longrightarrow M \longrightarrow \mathrm{cok}(T) \longrightarrow 0,$$

the terms $\mathrm{Ker}\,(T)$ and $\mathrm{cok}(T)$ are pertinent.

Notice right away that free modules are pertinent; so, if S is good and we take $\{0\} = T$, then, as the map $F(\{0\}) \longrightarrow M$ is the zero map, we find $M = \mathrm{cok}(\{0\})$ and therefore M is pertinent. That is, any module possessing a good subset is automatically pertinent. Conversely, if M is pertinent, then clearly $S = \{0\}$ is a good subset; so, we've proved

Proposition 5.76 *If R is an augmented ring and I is two-sided, the following are equivalent conditions on an R^{op}-module, M:*

(a) M possesses a good subset

(b) M is pertinent

(c) $\{0\}$ is a good subset of M.

In the case that R is a graded ring, *we shall restrict all attention to modules whose homogeneous elements (if any) have degrees bounded below. In this case, any set $S \subseteq M$ consisting of 0 and homogeneous elements is good.* For suppose $T \subseteq S$, then we grade $F(T)$ by the requirement that $F(T) \longrightarrow M$ be a map of degree zero (remember; M is graded and, further, observe if $0 \in T$ it goes to 0 in M and causes no trouble). But then, $\mathrm{Ker}\,(T)$ and $\mathrm{cok}(T)$ are automatically graded (with grading bounded below) and so are pertinent. If R^{op} is a noetherian local ring and M is a finitely generated R^{op}-module, then *any finite set containing 0 is good*. For if S is finite, then any $T \subseteq S$ is also finite and so all of $F(T)$, M, $\mathrm{cok}(T)$ are f.g. But since R^{op} is noetherian, $\mathrm{Ker}\,(T)$ is also f.g.

(C) A family, $\mathcal{F}$, of R^{op}-modules is *a class of important modules* provided

(1) If $M \in \mathcal{F}$ it has a good set S which generates M, and

(2) In the exact sequence
$$0 \longrightarrow \mathrm{Ker}\,(S) \longrightarrow F(S) \longrightarrow M \longrightarrow 0$$
resulting from (1), we have $\mathrm{Ker}\,(S) \in \mathcal{F}$.

For graded rings, R, *the graded modules (whose degrees are bounded below) form an important family*. This is easy since such modules are always generated by their homogeneous (= good) elements and $\mathrm{Ker}\,(S)$ is clearly graded and has degrees bounded below. In the case that R is noetherian local, *the family of all f.g. modules is important*. Again, this is easy as such modules are generated by finite (= good) sets and $\mathrm{Ker}\,(S)$ is again f.g. because R is noetherian.

In what follows, one should keep in mind the two motivating examples and the specific translations of the abstract concepts: pertinent modules, good sets of elements, the class of important modules.

Abstract R	**R graded**	**R noetherian local**
pertinent module	graded module with degrees bounded below	finitely generated module
good subset of a module	subset of homogeneous elements of a module	finite subset of module
class of important modules	class of graded modules with degrees bounded below	class of finitely generated modules

Since we have abstracted the local case, it is no surprise that we have a "generalized Nakayama's Lemma":

Proposition 5.77 *(Generalized Nakayama's Lemma) If (R, Q) is an augmented ring with I a two-sided ideal, then for any good subset, S, of an R^{op}-module, M, whenever the image of $S - \{0\}$ in $M \otimes_R Q$ generates $M \otimes_R Q$ as Q^{op}-module the set $S - \{0\}$ generates M. Moreover, if $\mathrm{Tor}_1^R(M, Q) = (0)$, and the image of $S - \{0\}$ freely generates $M \otimes_R Q$ as Q^{op}-module, then $S - \{0\}$ freely generates M as R^{op}-module.*

Proof. The proof is practically identical to the usual case (write S instead of $S - \{0\}$): We have the exact sequence

$$0 \longrightarrow \mathrm{Ker}\,(S) \longrightarrow F(S) \xrightarrow{\varphi} M \longrightarrow \mathrm{cok}(S) \longrightarrow 0,$$

and we tensor with Q. We obtain the exact sequence

$$F(S) \otimes_R Q \xrightarrow{\overline{\varphi}} M \otimes_R Q \longrightarrow \mathrm{cok}(S) \otimes_R Q \longrightarrow 0$$

and we've assumed $\overline{\varphi}$ is surjective. Thus $\mathrm{cok}(S) \otimes_R Q = (0)$, yet $\mathrm{cok}(S)$ is pertinent; so $\mathrm{cok}(S) = (0)$. Next, our original sequence has become

$$0 \longrightarrow \mathrm{Ker}\,(S) \longrightarrow F(S) \longrightarrow M \longrightarrow 0,$$

so we can tensor with Q again to obtain

$$\mathrm{Tor}_1^R(M, Q) \longrightarrow \mathrm{Ker}\,(S) \otimes_R Q \longrightarrow F(S) \otimes_R Q \xrightarrow{\overline{\varphi}} M \otimes_R Q \longrightarrow 0.$$

Since, in the second part, we've assume $\overline{\varphi}$ is an isomorphism and $\mathrm{Tor}_1^R(M, Q) = (0)$, we get $\mathrm{Ker}\,(S) \otimes_R Q = (0)$. But, $\mathrm{Ker}\,(S)$ is also pertinent and so $\mathrm{Ker}\,(S) = (0)$. $\square$

If we specialize Q, we get the following:

Corollary 5.78 *With (R, Q) as in the generalized Nakayama's Lemma and assuming Q is a (skew) field,*[8] *we have the following equivalent conditions for an R^{op}-module, M, which is generated by a good set:*

(1) M is free over R^{op}

(2) M is R^{op}-flat

(3) $\mathrm{Tor}_n^R(M, Q) = (0)$ if $n > 0$

(4) $\mathrm{Tor}_1^R(M, Q) = (0)$.

Moreover, under these equivalent conditions, every good generating set for M contains an R^{op}-basis for M.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are trivial or are tautologies.

(4) $\Longrightarrow$ (1). The image, $\overline{S}$, of our good generating set generates $M \otimes_R Q$. But, Q is a (skew) field; so $\overline{S}$ contains a basis and this has the form $\overline{T}$ for some $T \subseteq S$. Then $T \cup \{0\}$ is good and (4) with generalized Nakayama shows T is an R^{op}-basis for M. This gives (1) and even proves the last asssertion. $\square$

Finally, we have the abstract $\mathcal{I}$-gldim theorem, in which $\mathcal{I}$ is a class of important modules.

[8] A skew field is a division ring.

Theorem 5.79 *($\mathcal{I}$-Global Dimension Theorem) If (R, Q) is an augmented ring in which $\mathrm{Ker}\,(R \xrightarrow{\epsilon} Q)$ is a two-sided ideal, if Q is a (skew) field and if $\mathcal{I}$ is an important class of R^{op}-modules, then*

$$\mathcal{I}\text{-gldim}\, R^{\mathrm{op}} \leq \dim_R Q \qquad (\textit{proj. dim}).$$

Proof. Of course, w.m.a. $\dim_R Q < \infty$ else there is nothing to prove; so, write $n = \dim_R Q$. The proof is now practically forced, namely pick an important R^{op}-module, M, then by assumption there is a good generating set, S_0, in M and an exact sequence

$$0 \longrightarrow \mathrm{Ker}\,(S_0) \longrightarrow F(S_0) \longrightarrow M \longrightarrow 0.$$

Also, $\mathrm{Ker}\,(S_0)$ is important so, there's a good generating set, S_1, in $\mathrm{Ker}\,(S_0)$ and an exact sequence

$$0 \longrightarrow \mathrm{Ker}\,(S_1) \longrightarrow F(S_1) \longrightarrow \mathrm{Ker}\,(S_0) \longrightarrow 0$$

in which $\mathrm{Ker}\,(S_1)$ is again important. We repeat and obtain the chain of exact sequences

$$\left.\begin{array}{c} 0 \longrightarrow \mathrm{Ker}\,(S_0) \longrightarrow F(S_0) \longrightarrow M \longrightarrow 0 \\ 0 \longrightarrow \mathrm{Ker}\,(S_1) \longrightarrow F(S_1) \longrightarrow \mathrm{Ker}\,(S_0) \longrightarrow 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \longrightarrow \mathrm{Ker}\,(S_t) \longrightarrow F(S_t) \longrightarrow \mathrm{Ker}\,(S_{t-1}) \longrightarrow 0 \end{array}\right\} \qquad (\dagger)$$

for all t. Upon splicing these sequences, we get the exact sequence

$$0 \longrightarrow \mathrm{Ker}\,(S_t) \longrightarrow F(S_t) \longrightarrow F(S_{t-1}) \longrightarrow \cdots \longrightarrow F(S_1) \longrightarrow F(S_0) \longrightarrow M \longrightarrow 0. \qquad (\dagger\dagger)$$

Now in the sequence

$$0 \longrightarrow \mathrm{Ker}\,(S_t) \longrightarrow F(S_t) \longrightarrow \mathrm{Ker}\,(S_{t-1}) \longrightarrow 0$$

($t \geq 0$ and $\mathrm{Ker}\,(S_{-1}) = M$), we compute Tor and find

$$\cdots \longrightarrow \mathrm{Tor}_{r+1}^R(F(S_t), Q) \longrightarrow \mathrm{Tor}_{r+1}^R(\mathrm{Ker}\,(S_{t-1}), Q) \longrightarrow \mathrm{Tor}_r^R(\mathrm{Ker}\,(S_t), Q) \longrightarrow \mathrm{Tor}_r^R(F(S_t), Q) \longrightarrow \cdots$$

for all $r \geq 1$ and $t \geq 0$; hence the isomorphisms

$$\mathrm{Tor}_{r+1}^R(\mathrm{Ker}\,(S_{t-1}), Q) \cong \mathrm{Tor}_r^R(\mathrm{Ker}\,(S_t), Q).$$

Take $r = 1$ and $t = n - 1$, then

$$\mathrm{Tor}_2^R(\mathrm{Ker}\,(S_{n-2}), Q) \cong \mathrm{Tor}_1^R(\mathrm{Ker}\,(S_{n-1}), Q)$$

and similarly

$$\mathrm{Tor}_3^R(\mathrm{Ker}\,(S_{n-3}), Q) \cong \mathrm{Tor}_2^R(\mathrm{Ker}\,(S_{n-2}), Q),$$

etc. We find

$$\mathrm{Tor}_{n+1}^R(\mathrm{Ker}\,(S_{-1}), Q) \cong \mathrm{Tor}_1^R(\mathrm{Ker}\,(S_{n-1}), Q). \qquad (\dagger\dagger\dagger)$$

But, $\dim_R Q = n$ and so $\mathrm{Tor}_{n+1}^R(-, Q) = (0)$; thus,

$$\mathrm{Tor}_1^R(\mathrm{Ker}\,(S_{n-1}), Q) = (0).$$

Now $\mathrm{Ker}\,(S_{n-1})$ is important and the corollary to Generalized Nakayama shows that $\mathrm{Ker}\,(S_{n-1})$ is R^{op}-free. Thus,

$$0 \longrightarrow \mathrm{Ker}\,(S_{n-1}) \longrightarrow F(S_{n-1}) \longrightarrow \cdots \longrightarrow \mathrm{Ker}\,(S_1) \longrightarrow F(S_0) \longrightarrow M \longrightarrow 0$$

is a free resolution (length n) of M and this proves $\dim_{R^{\mathrm{op}}}(M) \leq n$. But, M is arbitrary and we're done. $\square$

Notice that the above argument is completely formal except at the very last stage where we used the vanishing of $\mathrm{Tor}_{n+1}^R(-, Q)$. But, this vanishing holds if Tor^R-$\dim(Q) \leq n$ and therefore we've actually proved the following stronger version of the $\mathcal{I}$-global dimension theorem:

Corollary 5.80 *Under the hypotheses of the $\mathcal{I}$-global dimension theorem, we have the strong $\mathcal{I}$-global dimension inequality*
$$\mathcal{I}\text{-gldim}(R^{\mathrm{op}}) \le \mathrm{Tor}^R\text{-dim}(Q).$$

To recapitulate and set these ideas firmly in mind, here are the two special, motivating cases:

Theorem 5.81 *(Syzygy[9] Theorem) Assume of the ring R that either*

(I) R is graded; $R = Q \amalg R_1 \amalg R_2 \amalg \cdots$, and Q is a (skew) field,

or

(II) R is local with R^{op} noetherian and $Q = \kappa(R)$ is a (skew) field.

Then, when $\mathcal{I}$ is, in case (I), the class of graded R^{op}-modules with degrees bounded below, or in case (II), the class of finitely generated R^{op}-modules, we have

$$\mathcal{I}\text{-gldim}(R^{\mathrm{op}}) \le \mathrm{Tor}^R\text{-dim}(Q).$$

Moreover, either Q is projective or else if $\mathfrak{A}$ is any R^{op}-ideal (which in case (I) is homogeneous), then we have
$$1 + \dim_{R^{\mathrm{op}}}(\mathfrak{A}) \le \dim_R(Q).$$

In case (I) of the Syzygy Theorem, note that Q is an R^{op}-module, too and that $\mathcal{I}$ can be taken to be the class of graded (with degrees bounded below) R-modules. Therefore, $\dim_{R^{\mathrm{op}}}(Q) \le \mathrm{Tor}^R\text{-dim}(Q) \le \dim_R(Q)$. Interchanging R and R^{op} as we may, we deduce

Corollary 5.82 *In case (I) of the Syzygy Theorem, we have*

(a) $\dim_R(Q) = \dim_{R^{\mathrm{op}}}(Q) = \mathrm{Tor}^R\text{-dim}(Q) = \mathrm{Tor}^{R^{\mathrm{op}}}\text{-dim}(Q)$,

(b) $\mathcal{I}\text{-gldim}(R) \le \mathrm{Tor}^R\text{-dim}(Q)$,

(c) $1 + \dim_R(\mathfrak{A}) \le \dim_R(Q)$ *if Q is not projective.*

Similarly, in case (II) of the Syzygy Theorem, provided we assume R noetherian, $\mathcal{I}$ is again the family of f.g. R-modules when we interchange R and R^{op}. There results

Corollary 5.83 *If R is local with both R and R^{op} noetherian, then*

(a) $\dim_R(\kappa(R)) = \dim_{R^{\mathrm{op}}}(\kappa(R)) = \mathrm{Tor}^R\text{-dim}(\kappa(R)) = \mathrm{Tor}^{R^{\mathrm{op}}}\text{-dim}(\kappa(R))$,

(b) $\mathcal{I}\text{-gldim}(R) \le \mathrm{Tor}^R\text{-dim}(\kappa(R))$,

(c) $1 + \dim_R(\mathfrak{A}) \le \mathrm{Tor}^R\text{-dim}(\kappa(R))$.

Finally, there are the cases that appeared first in the literature:

Corollary 5.84 *If K is a (skew) field and*

(I) (Hilbert Syzygy Theorem) $R = K[T_1, \ldots, T_n]$ and M is a graded R-module with degrees bounded from below or $\mathfrak{A}$ is a homogeneous R-ideal then

$$\dim_R M \le n \quad \textit{and} \quad \dim_R \mathfrak{A} \le n - 1.$$

[9] The Greek (later Latin) derived word "syzygy" means a coupling, pairing, relationship. Thus, for the exact sequence $0 \longrightarrow \mathrm{Ker}\,(S) \longrightarrow F(S) \longrightarrow M \longrightarrow 0$, the generators of $\mathrm{Ker}\,(S)$ are relations among the generators of M and generate all such relations. For $0 \longrightarrow \mathrm{Ker}\,(S_1) \longrightarrow F(S_1) \longrightarrow \mathrm{Ker}\,(S) \longrightarrow 0$, generators of $\mathrm{Ker}\,(S_1)$ are relations among the relations and so on. Each of $\mathrm{Ker}\,(S_j)$ is a *syzygy module*.

or

(II) $R = K[[T_1, \ldots, T_n]]$ *(resp.* $R = K\{T_1, \ldots, T_n\}$ *when* K *has a non-discrete (valuation) topology) and* M *is a f.g.* R*-module while* $\mathfrak{A}$ *is an* R*-ideal, then*

$$\dim_R M \le n \quad \textit{and} \quad \dim_R \mathfrak{A} \le n-1.$$

To prove these, we use the above and Theorem 5.74.

Remarks:

(1) Note that Q appears as the "worst" module, i.e., the one with the largest homological dimension. In the case of a commutative local ring, R, if $\mathfrak{M}$ is generated by an R-regular sequence (R is then called a *regular local ring*) of length n, we see that $\mathcal{I}$-gldim$(R) = n$ and $Q = \kappa(R)$ achieves this maximum dimension. The finiteness of global dimension turns out to be characteristic of regular local rings (Serre [46]).

(2) One might think of interchanging R and R^{op} in the general global dimension theorem. But this is not generally possible because the class of important modules usually does not behave well under this interchange. The trouble comes from the self-referential nature of $\mathcal{I}$. The R-module Q is an R^{op}-module, pertinence will cause no difficulty, nor will good subsets cause difficulty (in general). But, we need $\mathrm{Ker}\,(S)$ to be important in the sequence

$$0 \longrightarrow \mathrm{Ker}\,(S) \longrightarrow F(S) \longrightarrow M \longrightarrow 0$$

if M is to be important, so we cannot get our hands on how to characterize importance "externally" in the general case.

For the global dimension of R (that is, when $\mathcal{I} = R$-mod itself) we must restrict attention to more special rings than arbitrary augmented rings. Fix a *commutative* ring K and assume R is a K-algebra as in the Hochschild Theory of Section 5.3. An obvious kind of cohomological dimension is then the smallest n so that $H^{n+1}(R, M) = (0)$ for all R^e $(= R \otimes_K R^{\mathrm{op}})$-modules, M; where the cohomology is Hochschild cohomology. But, this is not a new notion because, by definition,

$$H^r(R, M) = \mathrm{Ext}^r_{R^e}(R, M).$$

Hence, the Hochschild cohomological dimension is exactly $\mathrm{projdim}_{R^e}(R)$. Let us agree to write $\dim_{R^e}(R)$ instead of $\mathrm{projdim}_{R^e}(R)$. It's important to know the behavior of $\dim_{R^e}(R)$ under base extension of K as well as under various natural operations on the K-algebra R. Here are the relevant results.

Proposition 5.85 *Suppose* R *is projective over* K *and let* L *be a commutative base extension of* K*. Then*

$$\dim_{(L\otimes_K R)^e}(L \otimes_K R) \le \dim_{R^e}(R).$$

If L *is faithfully flat over* K*, equality holds.*

Remark: To explain the (perhaps) puzzling inequality of our proposition, notice that the dimension of R is as a K-algebra while that of $L \otimes_K R$ is as an L-algebra as befits base extension. So we might have written $\dim_{R^e}(R; K)$, *etc.* and then the inequality might not have been so puzzling–but, one must try to rein in excess notation.

Proof. A proof can be based on the method of maps of pairs as given in Section 5.3, but it is just as simple and somewhat instructive to use the associativity spectral sequence and associativity formula for Ext (cf.

Proposition 5.63). To this end, we make the following substitutions for the objects, K, R, S, T, A, B, C of that Proposition:

$$\begin{aligned}&K \longrightarrow K,\ R \longrightarrow K,\ S \longrightarrow R^e,\ T \longrightarrow L,\\ &A \longrightarrow R,\ B \longrightarrow L,\ C \longrightarrow M \quad \text{(an arbitrary } (L \otimes_K R)^e\text{-module).}\end{aligned}$$

Since R is projective over K, the abstract hypothesis: A (our R) is R (our K)-flat is valid and moreover B (our L) is T (again our L)-projective. Hence, the Ext associativity gives

$$\mathrm{Ext}^p_{R^e}(R, M) \cong \mathrm{Ext}^p_{(L\otimes_K R)^e}(L \otimes_K R, M)$$

because $S \otimes_K T$ is equal to $(L \otimes_K R)^e$. Now M is an L and an R^e-module; so, if $p > \dim_{R^e}(R)$ the left side vanishes and therefore so does the right side. But, M is arbitrary and the inequality follows. (One could also use Corollary 5.66).

We have an inequality simply because we cannot say that an arbitrary R^e-module is also an L-module. Now suppose L is faithfully flat as K-algebra, then L splits as K-*module* into $K\,(= K \cdot 1) \amalg V$ so that we have a K-morphism $\pi\colon L \to K$. The composition $K \overset{i}{\hookrightarrow} L \xrightarrow{\pi} K$ is the identity. If M is any R^e-module, then $L \otimes_K M$ is an R^e and an L-module and we may apply our above Ext associativity to $L \otimes_K M$. We find the isomorphism

$$\mathrm{Ext}^p_{R^e}(R, L \otimes_K M) \cong \mathrm{Ext}^p_{(L\otimes_K R)^e}(L \otimes_K R, L \otimes_K M). \qquad (*)$$

However, the composition

$$M = K \otimes_K M \overset{i\otimes 1}{\hookrightarrow} L \otimes_K M \xrightarrow{\pi\otimes 1} K \otimes_K M = M$$

is the identity; so, applied to $(*)$ it gives

$$\mathrm{Ext}^p_{R^e}(R, M) \longrightarrow \mathrm{Ext}^p_{R^e}(R, L \otimes_K M) \longrightarrow \mathrm{Ext}^p_{R^e}(R, M) \qquad (**)$$

whose composition is again the identity. If $p > \dim_{(L\otimes_K R)^e}(L \otimes_K R)$ the middle group is (0) and so $(**)$ shows $\mathrm{Ext}^p_{R^e}(R, M) = (0)$. M is arbitrary, therefore $\dim_{R^e}(R) \le \dim_{(L\otimes_K R)^e}(L \otimes_K R)$. $\square$

Of course, faithful flatness is always true if K is a field; so, we find

Corollary 5.86 *If K is a field and R is a K-algebra, then for any commutative K-algebra, L, we have*

$$\dim_{R^e}(R) = \dim_{(L\otimes_K R)^e}(L \otimes_K R).$$

In particular, the notion of dimension is "geometric", i.e., it is independent of the field extension.

If we're given a pair of K-algebras, say R and S, then we get two new K-algebras $R \prod S$ and $R \otimes_K S$. Now, consider $R \prod S$. It has the two projections pr_1 and pr_2 to R and S and so we get the two functors pr_1^* and pr_2^* from R-mod (resp. S-mod) to $R \prod S-\mathrm{mod}$. If M is an $R \prod S$-module, then we get two further functors

$$\begin{aligned}{pr_1}_*\colon R \prod S - \mathrm{mod} &\rightsquigarrow R\text{-mod}\\ {pr_2}_*\colon R \prod S - \mathrm{mod} &\rightsquigarrow S\text{-mod}\end{aligned}$$

via $M \rightsquigarrow (1,0)M$ (resp. $(0,1)M$). Observe that $pr_i^*({pr_i}_* M)$ is naturally an $R \prod S$-submodule of M, therefore we have two functors

$$F\colon R\text{-mod} \prod S\text{-mod} \rightsquigarrow R \prod S - \mathrm{mod}$$

via

$$F(M, \widetilde{M}) = pr_1^* M \amalg pr_2^* \widetilde{M} \quad (\text{in } R \prod S - \text{mod})$$

and

$$G\colon R \prod S - \text{mod} \rightsquigarrow R\text{-mod} \prod S\text{-mod}$$

via

$$G(M) = (pr_{1*} M, pr_{2*} M);$$

the above shows that F *and* G *establish an equivalence of categories*

$$R \prod S - \text{mod} \approx R\text{-mod} \prod S\text{-mod}.$$

If $T = R \prod S$, then $T^e = R^e \prod S^e$; so, applying the above, we get the category equivalence

$$T^e\text{-mod} = R^e \prod S^e - \text{mod} \approx R^e\text{-mod} \prod S^e\text{-mod}.$$

Then, obvious arguments show that

$$H^p(R \prod S, M) \cong H^p(R, pr_{1*} M) \amalg H^p(S, pr_{2*} M)$$

(where, M is an $(R \prod S)^e$-module). This proves the first statement of

Proposition 5.87 *Suppose R and S are K-algebras and R is K-projective then*

$$\dim_{(R \prod S)^e}(R \prod S) = \max(\dim_{R^e}(R), \dim_{S^e}(S))$$

and

$$\dim_{(R\otimes S)^e}(R \otimes S) \leq \dim_{R^e}(R) + \dim_{S^e}(S).$$

Proof. For the second statement, we have the spectral sequence (of Corollary 5.66)

$$H^p(R, H^q(S, M)) \Longrightarrow H^\bullet(R \otimes_K S, M).$$

Exactly the same arguments as used in the Tower Theorem (Theorem 5.60) for the Hochschild-Serre spectral sequence finish the proof. $\square$

Remark: The K-projectivity of R is only used to prove the inequality for $R \otimes_K S$.

We can go further using our spectral sequences.

Theorem 5.88 *Suppose R is a K-projective K-algebra, then*

$$\text{gldim}\, R^e \leq \dim_{R^e}(R) + \text{gldim}(R).$$

Further,

$$\text{gldim}(R) \leq \dim_{R^e}(R) + \text{gldim}\, K$$

and

$$\text{gldim}(R^{\text{op}}) \leq \dim_{R^e}(R) + \text{gldim}\, K.$$

Proof. We apply the spectral sequence

$$E_2^{p,q} = H^p(R, \mathrm{Ext}_T^q(B,C)) \Longrightarrow \mathrm{Ext}^{\bullet}_{R^{\mathrm{op}}\otimes_K T}(B,C)$$

of Corollary 5.64. Here, B and C are left R and right T-modules (and, in the $\mathrm{Ext}^{\bullet}$ of the ending, they are viewed as right $R^{\mathrm{op}} \otimes_K T$-modules, or left $R \otimes_K T^{\mathrm{op}}$-modules). We set $T = R$ and see that

$$\mathrm{Ext}_R^q(B,C) = (0) \quad \text{when} \quad q > \mathrm{gldim}(R).$$

But,

$$H^p(R,-) = \mathrm{Ext}^p_{R^e}(R,-)$$

and so

$$H^p(R,-) = (0) \quad \text{when} \quad p > \dim_{R^e}(R).$$

Therefore, $E_2^{p,q} = (0)$ when $p + q > \dim_{R^e}(R) + \mathrm{gldim}(R)$. Once again, exactly as in the Tower Theorem, we conclude $\mathrm{Ext}^n_{R\otimes_K R^{\mathrm{op}}}(B,C)$ vanishes for $n > \dim_{R^e}(R) + \mathrm{gldim}(R)$. This proves the first inequality.

For the second and third inequalities, we merely set $T = K$. Then, $\mathrm{Ext}^q_K(B,C)$ vanishes for all $q > \mathrm{gldim}(K)$ and $H^p(R,-)$ vanishes for all $p > \dim_{R^e}(R) = \dim_{R^e}(R^{\mathrm{op}})$. Our spectral sequence argument now yields the two desired inequalities. $\square$

Corollary 5.89 *If R is a projective K-algebra and R is semi-simple as K-algebra, then*

$$\mathrm{gldim}(R^e) = \dim_{R^e}(R).$$

If R is arbitrary but K is a semi-simple ring, then

$$\mathrm{gldim}(R) \leq \dim_{R^e}(R)$$

and

$$\mathrm{gldim}(R^{\mathrm{op}}) \leq \dim_{R^e}(R).$$

Proof. In the first inequality, $\mathrm{gldim}(R) = 0$, so

$$\mathrm{gldim}(R^e) \leq \dim_{R^e}(R).$$

The opposite inequality is always true by definition.

If now K is semi-simple, R is automatically K-projective; so, our other inequalities (of the theorem) finish the proof as $\mathrm{gldim}(K) = 0$. $\square$

Corollary 5.90 *If K is semi-simple and R is a K-algebra, then R^e is semi-simple if and only if R is a projective R^e-module (i.e., $\dim_{R^e}(R) = 0$).*

Proof. Suppose $\dim_{R^e}(R) = (0)$. Then, by Corollary 5.89 above, $\mathrm{gldim}(R) = 0$, i.e., R is itself a semi-simple ring. But then we apply the corollary one more time and deduce $\mathrm{gldim}(R^e) = 0$. Conversely, if $\mathrm{gldim}(R^e) = 0$, then $\dim_{R^e}(R) = 0$. $\square$

Corollary 5.91 *If K is semi-simple and if R^e is semi-simple, then R is semi-simple (as K-algebra).*

Proof. As $\mathrm{gldim}(R^e) = 0$ and K is semi-simple, we get $\dim_{R^e}(R) = 0$. But, $\mathrm{gldim}(R) \leq \dim_{R^e}(R)$; so, we are done. $\square$

We can now put together the Koszul complex and the material above to prove

Theorem 5.92 *(Global Dimension Theorem) Suppose K is a commutative ring and write $R = K[T_1, \ldots, T_n]$. Then,*

$$\dim_{R^e}(R) = \dim_R(K) = n.$$

We have the inequality

$$n \leq \operatorname{gldim} R \leq n + \operatorname{gldim} K,$$

and so if K is a semi-simple ring (e.g., a field), then

$$\operatorname{gldim} R = n.$$

Proof. By the main application of the Koszul complex to dimension (Theorem 5.74) we have $\dim_R(K) = n$ and so $\operatorname{gldim} R \geq n$. Here, K is an R-module *via* sending all T_j to 0. But if $\widetilde{\epsilon}$ is *any* K-algebra map $R \longrightarrow K$, we can perform the automorphism $T_j \mapsto T_j - \widetilde{\epsilon}(T_j)$ and this takes $\widetilde{\epsilon}$ to the usual augmentation in which all $T_j \longrightarrow 0$. Therefore, we still have $\dim_R K = n$ (and $\operatorname{gldim} R \geq n$) when viewing K as R-module *via* $\widetilde{\epsilon}$.

Now $R^{\mathrm{op}} = R$; so, $R^e = R \otimes_K R$, and thus

$$R^e = K[T_1, \ldots, T_n, Z_1, \ldots Z_n] = R[Z_1, \ldots, Z_n].$$

(Remember that T_j stands for $T_j \otimes 1$ and Z_j for $1 \otimes T_j$). The standard augmentation $\eta \colon R^e \to R$ is given by $\rho \otimes \widetilde{\rho} \mapsto \rho\widetilde{\rho}$ and it gives a map

$$R^e = R[Z_1, \ldots, Z_n] \longrightarrow R,$$

in which Z_j goes to $T_j \in R$. The Z_j's commute and we can apply Theorem 5.74 again to get

$$\dim_{R^e}(R) = n.$$

Finally, Theorem 5.88 shows that

$$\operatorname{gldim}(R) \leq n + \operatorname{gldim} K. \qquad \square$$

Remark: The global dimension theorem is a substantial improvement of Hilbert's Syzygy Theorem. For one thing we need not have K a field (but, in the semi-simple case this is inessential) and, more importantly, we need not restrict to graded modules. Also, the role of the global dimension of K becomes clear.

5.6 Concluding Remarks

The apparatus of (co)homological methods and constructions and, more importantly, their manifold applications to questions and situations in algebra and geometry has been the constant theme of this chapter. Indeed, upon looking back to all earlier chapters from the first appearance of group cohomology as a computational tool to help with group extensions through the use of sequences of modules and Galois cohomology in field theory to the theory of derived functors and spectral sequences to obtain new, subtle invariants in algebra and geometry, we see a unified ever deepening pattern in this theme. The theme and pattern are a major development of the last sixty years of the twentieth century—a century in which mathematics flowered as never before. Neither theme nor pattern gives a hint of stopping and we have penetrated just to middling ground. So read on and work on.

5.7 Supplementary Readings

The classic reference on homological algebra is Cartan and Eilenberg [9]. One may also consult Mac Lane [36], Rotman [44], Weibel [48], Hilton and Stammback [24], Bourbaki [5], Godement [18] and Grothendieck [20]. For recent developments and many more references, see Gelfand and Manin's excellent books [16, 17]. For a global perspective on the role of homological algebra in mathematics, see Dieudonné [10].