Chapter 3: Spectral Theory (HWK)

Note Title 5/28/2013

3.I.1. <u>Exercise</u>: Consider $E = \mathcal{P}_m$, the space of polynomials of degree $\leq m$ with coefficients in $\mathbb{R}$ and

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int_0^1 f(x)\, g(x)\, dx.$$

Show that the Gram Matrix of the std basis $\mathcal{E} = (1, x, \ldots, x^m)$ is

$$G_{\mathcal{E}} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{m+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{m+2} \\ \vdots & & \ddots & & \vdots \\ \frac{1}{m+1} & & \cdots & & \frac{1}{2m+1} \end{bmatrix}$$

Now, if we apply Gram-Schmidt to produce an ONB α of E, then $P = [\,{}^{\mathcal{E}}_{\alpha}\,]$ will produce a decomposition of $G_{\mathcal{E}}$ in the form P^*P. Check what the result would be for $m = 4$.

3.I.2. <u>Exercise</u>: Suppose E is a FDIPS and $f \in \mathcal{L}(E)$. If $W < E$ is both f- and f^*- invariant, verify that

$$(f|_W)^* = f^*|_W.$$

(<u>Note</u>: $f|_W$ denotes the restriction of f to a function $W \to W$, and similarly for f^*.)

simultaneous diagonalization theorem

(*) Theorem: Suppose $\mathcal{F} \subseteq \mathcal{L}(E)$ is a collection of commuting orthogonally/unitarily diagonalizable operators, that is:

① every $f \in \mathcal{F}$ has an ONB α_f of E s.t. $[f]_{\alpha_f} = \text{diag.}$;

② every $f, g \in \mathcal{F}$ satisfy $fg = gf$.

Then there exists an ONB α of E such that $[f]_\alpha$ is diagonal for all $f \in \mathcal{F}$.

3.II.1. Exercise: Prove this theorem using induction on $\dim E$, after proving a lemma: There exists a common eigenvector for $\mathcal{F}$ in E, that is there exists $0 \neq x \in E$ such that $f(x) \in \text{Span}(x)$ for every $f \in \mathcal{F}$.

Lemma: Let E be a FDIPS (Real or Complex), and $f \in \mathcal{L}(E)$. Then

① f symmetric or Hermitian $\Rightarrow \text{Spec}(f) \subseteq \mathbb{R}$

② f is skew-Symmetric/Hermitian $\Rightarrow \text{Spec}(f) \subseteq i\mathbb{R}$,

③ f is orthogonal/unitary $\Rightarrow \text{Spec}(f) \subseteq S^1$ unit circle in $\mathbb{C}$

Moreover, the converse implications hold whenever f is normal.

3.III.1 Exercise: Finish the proof of ② and ③.

3.IV.1. Exercise: find an Euclidean example where $\langle fx, x \rangle > 0$ for all $x \in E, x \neq 0$ but f is not self-adjoint.

Thm: The following are equivalent for a matrix $M \in M_n(\mathbb{K})$:

(1) M is positive definite,

(2) M is a Gram matrix of an inner product on an n-dim'l inner prod. space,

(3) M is a Gram matrix of an inner product on $\mathbb{K}^n$

(4) M is self-adjoint with strictly positive eigenvalues.

Criterion for inner products!

(*) (5) M is self-adjoint with all positive diagonal minors.

3.IV.2. **Exercise:** Find and read a proof of (5) being equivalent to M being positive definite, e.g. in Hoffman & Kunze's textbook on Linear Algebra.

Theorem: for any $f \in \mathcal{L}(E)$,

$$\|f\| \overset{\text{def}}{=} \max_{x \neq 0} \frac{\|f(x)\|}{\|x\|} = \max_{\|x\|=1} \|f(x)\| = \sigma_1(f)$$

In particular, $\mathrm{Spec}(f)$ is contained in the closed disc of radius $\sigma_1(f)$ about the origin in $\mathbb{C}$.

3.IV.3. **Exercise:** Prove that $\|\cdot\|$ as defined above is a norm on $\mathcal{L}(E)$ satisfying $\|fg\| \leq \|f\| \cdot \|g\|$ for all $f, g \in \mathcal{L}(E)$.

3.IV.4. **Exercise:** Suppose $f \in \mathcal{L}(E)$ has $\|f\| < 1$. Prove:

$$(I - f)^{-1} = \sum_{n=0}^{\infty} f^n$$

Some properties of $GL_n(\mathbb{K})$.

① Let $M_n(\mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, denote the algebra of $n \times n$ matrices with coefficients in $\mathbb{K}$, and let $GL_n(\mathbb{K})$ be the set of all invertible matrices in $M_n(\mathbb{K})$. We will consider $M_n(\mathbb{K})$ with the operator norm:

$$\|A\| \overset{\text{def}}{=} \max_{\|x\|=1} \|Ax\| \quad \text{where} \quad x \in \mathbb{K}^n \text{ and } \|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}.$$

Prove the following:

a) $GL_n(\mathbb{K})$ is an open subset of $M_n(\mathbb{K})$.

b) $GL_n(\mathbb{K})$ is a dense subset of $M_n(\mathbb{K})$

For a subset $X \subseteq \mathbb{R}^n$ we say that X is path-connected, if any two points $x, y \in X$ can be joined by a path in X, that is: there exists a continuous map $\alpha : [0,1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$.

c) Prove that the relation $x \sim y \Leftrightarrow$ there is a path from x to y in X is an equivalence relation on X.

d) Suppose X is path-connected. Show that it is impossible to find a pair of open disjoint subsets $U, V \subseteq \mathbb{R}^n$ such that

$$X = U \cup V, \quad U \cap X \neq \emptyset, \quad V \cap X \neq \emptyset.$$

e) Prove that $GL_n(\mathbb{R})$ is not path-connected.

f) Fill in the details of the following proof that $GL_n(\mathbb{C})$ is connected:

(i) It suffices to prove that any $M \in GL_n(\mathbb{C})$ is connected by a path to the identity matrix, I.

(ii) Consider the Euclidean plane

$$\Upsilon = \left\{ zI + (1-z)M \,\middle|\, z \in \mathbb{C} \right\} \subseteq M_n(\mathbb{C}).$$

Then Υ is contained in $GL_n(\mathbb{C})$ except, possibly, for finitely many points.

(iii) There is a path from I to M contained in $\Upsilon \cap GL_n(\mathbb{C})$.

Once again, for $X \subseteq \mathbb{R}^n$, we say that an equivalence class of the relation $(\sim)$ on X is a <u>path component</u> of X.

g) Use the standard block-diagonal form of a matrix $M \in SO(n)$ to prove that any such matrix is connected to the identity by a path in $SO(n)$.

h) Conclude that $O(n)$ has exactly two connected components, one of which is $SO(n)$.

i) Finally, use the polar decomposition to prove that $SL_n(\mathbb{R})$ and $GL_n(\mathbb{R})$ both have two components, and find those components.