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LINEAR ALGEBRA AND ITS APPLICATIONS 9, 209–222 (1974)

The Range of $A^{-1}A^*$ in $\mathbf{G}L(n,\mathbf{C})$

C. R. DePrima* and C. R. Johnson†

Mathematics Department, California Institute of Technology, Pasadena, California, 91109

Applied Mathematics Division National Bureau of Standards, Washington, D.C. 20234

Recommended by Olga Taussky Todd

ABSTRACT

Let A be an invertible linear operator on a finite dimensional complex Hilbert space. We carry out a detailed study of the map $A \to A^{-1}A^* \equiv \Phi(A)$. It is shown that the range of Φ is exactly the set of all invertible operators T for which T^{-1} is similar to T^*. In particular, unitaries and similarities of unitaries are in the range of Φ and we prove, among other things, the equivalence of the assertions: (i) T is similar to a unitary, (ii) every $A \in \Phi^{-1}(T)$ is congruent to a normal operator, (iii) there exists $B \in \Phi^{-1}(T)$ whose field of values omits the origin of the complex plane. For general T in the range of Φ, we determine *all* $A \in \Phi^{-1}(T)$ in terms of the self-adjoint invertible operators fixed by the map $X \to T^*XT$. Many of the results contained in this paper have known analogues for operators which are similar to their adjoints.

INTRODUCTION

On $\mathbf{G}L(n,\mathbf{C})$ the maps $A \to A^{-1}$ and $A \to A^*$ are involutory and the self-map Φ of $\mathbf{G}L(n,\mathbf{C})$, defined by $\Phi(A) = A^{-1}A^*$, intertwines these involutions, that is $\Phi(A^*) = \Phi(A)^{-1}$. This paper is primarily a study of the intertwining map Φ. In particular, we prove (Theorem 1) that the range of Φ is precisely the set of those $T \in \mathbf{G}L(n,\mathbf{C})$ for which T^{-1} and T^* are similar. The representation of T by $A^{-1}A^*$ is related to Hilbert's Theorem 90 for the

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particular group $\mathbf{G}L(n,\mathbf{C})$ and has occurred in various contexts in recent work, e.g., in [4, 7, 8, 10, 12, 13].

In view of Theorem 1, this paper may also be regarded as a study of those $T\in\mathbf{G}L(n,\mathbf{C})$ for which T^{-1} and T^* are similar. In this respect, our work is analogous to that of [1, 2, 10] where the set for which T and T^* are similar is studied. Actually these two sets may be mapped onto each other via appropriate Cayley transforms (a fact which we do not exploit in the present paper). Theorem 2 and 3 of §2, in which those T (and/or $\Phi(A)$) which are similar to a unitary are characterized, are analogous to corresponding characterizations in the above cited works of those T which are similar to a self adjoint.

In §3 we investigate $\Phi^{-1}(T)$ for particular T in the range of Φ. This work rests heavily on the construction of a specific square root of T (described in §4) and on the behavior of the hermitian congruence map: $S\in\mathbf{G}L(n,\mathbf{C})\to T^*ST$. Our results generally describe $\Phi^{-1}(T)$ in terms of the non-singular hermitian matrices fixed by this map.

1. PRINCIPAL RESULTS

$\mathbf{M}_n\equiv\mathbf{M}(n,\mathbf{C})$ denotes the algebra of all $n\times n$ matrices over the complex field. $\mathbf{G}_n\equiv\mathbf{G}L(n,\mathbf{C})$ is the general linear group of all non-singular elements of $\mathbf{M}_n$. If $T\in\mathbf{M}_n$, T^* denotes its adjoint (i.e., the transposed conjugate matrix) and $\sigma(T)$ denotes the spectrum of T. $\sigma(T)$ is a finite subset of the complex plane $\mathbf{C}$ consisting of at most n points.

Consider the map $\Phi:\mathbf{G}_n\to\mathbf{G}_n$ defined by

$$\Phi(A)=A^{-1}A^*,\qquad A\in\mathbf{G}_n \tag{1.1}$$

and let $\mathbf{F}_n$ designate the range of Φ, i.e. $\mathbf{F}_n=\Phi(\mathbf{G}_n)$. Hence $\mathbf{F}_n$ is the subset of $\mathbf{G}_n$ whose elements are representable in the factored form $A^{-1}A^*$. The map Φ is readily seen to satisfy the following properties:

$$\Phi(A^*)=\Phi(A)^{-1};\qquad \Phi(A)^*=\Phi(A^{-1}). \tag{1.2}$$

$$\alpha\Phi(A)=\Phi(\beta A),\qquad |\alpha|=1\ \text{ and }\ \beta^{-1}\bar{\beta}=\alpha. \tag{1.3}$$

$$R^{-1}\Phi(A)R=\Phi(R^*AR),\qquad R\in\mathbf{G}_n. \tag{1.4}$$

Consequently, in an obvious notation, $\mathbf{F}_n=\mathbf{F}_n^*=\mathbf{F}_n^{-1}=\alpha\mathbf{F}_n=R^{-1}\mathbf{F}_nR$ for all $\alpha\in\mathbf{C}$ with $|\alpha|=1$ and for all $R\in\mathbf{G}_n$. Moreover, a direct computation gives

PROPOSITION 1. [4]. *For $A\in\mathbf{G}_n$ the*

(i) $\Phi(A)$ *is*

(ii) *A is nor*

(iii) $A\Phi(A)=$

Let $\mathbf{D}_n$ denote the set of $T\in\mathbf{G}_n$ for w

$$\mathbf{D}_n=\{T\in\mathbf{G}_n|T^*ST=S$$

For $T\in\mathbf{G}_n$ set

$$\Delta(T)=\{S\in\mathbf{G}_n|$$
$$\Delta_*(T)=\{H\in\Delta($$

Then $T\in\mathbf{D}_n$ if and only if $\Delta(T)$ is non v

$$\Delta(T)=\Delta(T^{-1})=\Delta($$
$$\Delta_*(T)=\Delta_*(T^{-1})$$

Since $\Phi(A)^*A\Phi(A)=A$ for all $A\in\mathbf{G}_n$, so that $\mathbf{F}_n\subset\mathbf{D}_n$. In fact, we shall show constructing admissible factors for $T\in\mathbf{D}_n$ each of which is a natural generalization o [13] in the special instance when T is unit However, we first give another charact useful in what follows.

PROPOSITION 2. $\mathbf{D}_n=\{T\in\mathbf{G}_n|\Delta_*(T)\neq$

Proof. For $S\in\mathbf{G}_n$ and $z\in\mathbf{C}$ with $|z$ $S_z\in\mathbf{G}_n$ if and only if $-z^{-1}\bar{z}\notin\sigma(S^{-1}S^*)$ there certainly exists $z\in\mathbf{C}$, $|z|=1$, with th then $T^*S^*T=S^*$ and $T^*S_zT=S_z$.

THEOREM 1. $\mathbf{F}_n=\mathbf{D}_n$.

Proof. As noted above $\mathbf{F}_n\subset\mathbf{D}_n$. Henc

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$$|\alpha|=1 \quad\text{and}\quad \beta^{-1}\overline{\beta}=\alpha. \tag{1.3}$$

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$\mathbf{G}_n$. Moreover, a direct computation gives

Proposition 1. [4]. *For $A\in\mathbf{G}_n$ the following assertions are equivalent:*

(i) $\Phi(A)$ *is unitary*

(ii) A *is normal*

(iii) $A\Phi(A)=\Phi(A)A$.

Let $\mathbf{D}_n$ denote the set of $T\in\mathbf{G}_n$ for which T^{-1} is similar to T^*, i.e.,

$$\mathbf{D}_n=\{T\in\mathbf{G}_n\,|\,T^*ST=S\quad\text{for some}\quad S\in\mathbf{G}_n\}. \tag{1.5}$$

For $T\in\mathbf{G}_n$ set

$$\begin{cases}\Delta(T)=\{S\in\mathbf{G}_n\,|\,T^*ST=S\}\\ \Delta_*(T)=\{H\in\Delta(T)\,|\,H=H^*\}.\end{cases} \tag{1.6}$$

Then $T\in\mathbf{D}_n$ if and only if $\Delta(T)$ is non void. It is easy to verify that

$$\begin{cases}\Delta(T)=\Delta(T^{-1})=\Delta(T)^*=\Delta(T^*)^{-1}\\ \Delta_*(T)=\Delta_*(T^{-1})=\Delta_*(T^*)^{-1}.\end{cases} \tag{1.7}$$

Since $\Phi(A)^*A\Phi(A)=A$ for all $A\in\mathbf{G}_n$, we see that $\Delta(\Phi(A))$ is not empty, so that $\mathbf{F}_n\subset\mathbf{D}_n$. In fact, we shall show in Theorem 1 that $\mathbf{F}_n=\mathbf{D}_n$ by constructing admissible factors for $T\in\mathbf{D}_n$. We do this in two different ways, each of which is a natural generalization of the factors mentioned by Taussky [13] in the special instance when T is unitary (unitaries are obviously in $\mathbf{D}_n$). However, we first give another characterization of $\mathbf{D}_n$ which will prove useful in what follows.

Proposition 2. $\mathbf{D}_n=\{T\in\mathbf{G}_n\,|\,\Delta_*(T)\neq\Phi\}$.

Proof. For $S\in\mathbf{G}_n$ and $z\in\mathbf{C}$ with $|z|=1$, set $S_z=\bar{z}S+zS^*=S_z^*$. Then $S_z\in\mathbf{G}_n$ if and only if $-z^{-1}\bar{z}\notin\sigma(S^{-1}S^*)=\sigma(\Phi(S))$. Since $\sigma(\Phi(S))$ is finite, there certainly exists $z\in\mathbf{C}$, $|z|=1$, with the required property. If $T^*ST=S$, then $T^*S^*T=S^*$ and $T^*S_zT=S_z$. ■

Theorem 1. $\mathbf{F}_n=\mathbf{D}_n$.

Proof. As noted above $\mathbf{F}_n\subset\mathbf{D}_n$. Hence, for given $T\in\mathbf{D}_n$ we need only construct $A \in \mathbf{G}_n$ for which $T = \Phi(A)$. We proceed to give two such constructions.

(a) Assume $T^*HT = H = H^* \in \mathbf{G}_n$. As $\sigma(T)$ is finite, there exists $\alpha \in \mathbf{C}$, $|\alpha| = 1$, such that $\alpha \notin \sigma(T)$ and hence $\bar{\alpha} \notin \sigma(T^*)$. Let $\beta \in \mathbf{C}$ with $\beta^{-1}\bar{\beta} = \alpha$ and set

$$A = i\beta(1 - \alpha T^*)H.^1 \tag{1.8}$$

$A \in \mathbf{G}_n$ and $AT = i\beta(1-\alpha T^*)HT = i\beta H(T-\alpha) = -i\beta\alpha H(1-\bar{\alpha}T) = -i\bar{\beta}(1-\bar{\alpha}T) = A^*$. Hence $T = A^{-1}A^*$ so that $T \in \mathbf{F}_n$.

(b) The second construction is based on the existence of a particular square root of $T \in \mathbf{D}_n$. The relevant facts are contained in the following lemma whose proof is given in §4.

LEMMA 1. *Let $T \in \mathbf{G}_n$. There exists a unique $\tilde{T} \in \mathbf{G}_n$ satisfying*

$$\text{(i)}\quad \tilde{T}^2 = T, \qquad \text{(ii)}\quad -\frac{\pi}{2} < \arg\sigma(\tilde{T}) \leq \frac{\pi}{2},$$

$$\text{(iii)}\quad TC = CT \Rightarrow \tilde{T}C = C\tilde{T}.$$

Moreover $\Delta(T) = \Delta(\tilde{T})$.

Again, assume that $T^*HT = H = H^* \in \mathbf{G}_n$, then $\tilde{T}^*H\tilde{T} = H$. Set

$$B = \tilde{T}^*H, \tag{1.9}$$

then $B \in \mathbf{G}_n$ and $BT = \tilde{T}^*HT = \tilde{T}^*H\tilde{T}\tilde{T} = H\tilde{T} = B^*$. Hence $T = \Phi(B) = B^{-1}B^*$.

If U is unitary, U is called cramped if $\sigma(U)$ lies on an open arc of the unit circle of length π. We may then state

COROLLARY 1.1. [12]. *If V is unitary, then $V = \Phi(\tilde{V}^*)$ where $\tilde{V}^*$ is a cramped unitary.*

Proof. Since $V^*V = I$, $\tilde{V}^*\tilde{V} = I$, where $\tilde{V}$ is the square root of V given by Lemma 1. Thus $\tilde{V}$ is unitary, cramped, and $V = \Phi(\tilde{V}^*) = \tilde{V}^2$. ■

If $T \in \mathbf{D}_n$, T^{-1} is similar to T^*, so that $\sigma(T)$ is carried into itself by reflection in the unit circle. In other words, *if $T \in \mathbf{D}_n$, then necessarily $\lambda \in \sigma(T)$ implies $\bar{\lambda}^{-1} \in \sigma(T)$.* However, this condition is clearly not sufficient for membership in $\mathbf{D}_n$. A necessary and sufficient condition is given by

[1]Throughout we write $A - z$ for $A - zI$, where $A \in \mathbf{M}_n$, $z \in \mathbf{C}$, and I is the identity in $\mathbf{M}_n$.

PROPOSITION 3. *Let $T \in \mathbf{G}_n$, then $T \in$*

$\dim\ker(T-\lambda)^\mu = \dim\ker(T-\bar{\lambda}^{-1})^\mu,$

(ker *denotes kernel or null space and* dim

Proof. Since A, $B \in \mathbf{M}_n$ are similar i
canonical Jordan representations, it follow
if

$\dim\ker(A-z)^\mu = \dim\ker(B-z)$

Thus, if $T \in \mathbf{G}_n$, T is similar to T^*
$= \dim\ker(T^*-z)^\mu$, $z \in \mathbf{C}\backslash\{0\}$, $\mu = 1, 2,$
$= \ker(T-z^{-1})^\mu$ and $\dim\ker(T^*-z)^\mu = $
obtained by setting $\lambda = z^{-1}$.

In view of (1.10) of Proposition 3, any
unit circle belongs to $\mathbf{D}_n$. Hence by Theo

COROLLARY 1.2. *If $T \in \mathbf{M}_n$ and $\sigma(T)$*

2. UNITARIES AND THEIR SIMILAR

Proposition 1 and Corollary 1.1 assert
normal elements of $\mathbf{G}_n$ is precisely the su
we show among other things that the ra
non-singular hermitian congruences of th
of all similarities of the unitary group.
Theorem 3 and Corollary 3.1.

Recall that the numerical range (field

$W(A) = \{x^*Ax \mid x^*$

$W(A)$ is a compact convex subset of C w
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$\in \mathbf{G}_n$. As $\sigma(T)$ is finite, there exists $\alpha \in \mathbf{C}$,
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$C = CT \Rightarrow \tilde{T}C = C\tilde{T}$.

$H = H^* \in \mathbf{G}_n$, then $\tilde{T}^* H \tilde{T} = H$. Set

$B = \tilde{T}^* H,$ (1.9)

$^*H\tilde{T}\tilde{T} = H\tilde{T} = B^*$. Hence $T = \Phi(B) = B^{-1}B^*$.

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I, where $A \in \mathbf{M}_n$, $z \in \mathbf{C}$, and I is the identity in $\mathbf{M}_n$.

PROPOSITION 3. *Let* $T \in \mathbf{G}_n$, *then* $T \in \mathbf{D}_n$ *if and only if*

$$\dim\ker(T-\lambda)^{\mu} = \dim\ker(T-\bar{\lambda}^{-1})^{\mu}, \qquad \lambda \in \mathbf{C}-\{0\}, \quad \mu = 1, 2, \dots \qquad (1.10)$$

(ker *denotes kernel or null space and* dim *is* (*complex*) *dimension*).

Proof. Since A, $B \in \mathbf{M}_n$ are similar if and only if they possess identical canonical Jordan representations, it follows that A is similar to B if and only if

$$\dim\ker(A-z)^{\mu} = \dim\ker(B-z)^{\mu}, \qquad z \in \mathbf{C}, \quad \mu = 1, 2, \dots.$$

Thus, if $T \in \mathbf{G}_n$, T is similar to T^* if and only if $\dim\ker(T^{-1}-z)^{\mu} = \dim\ker(T^*-z)^{\mu}$, $z \in \mathbf{C}\backslash\{0\}$, $\mu = 1, 2, \dots$. But for $z \neq 0$ $\ker(T^{-1}-z)^{\mu} = \ker(T-z^{-1})^{\mu}$ and $\dim\ker(T^*-z)^{\mu} = \dim\ker(T-\bar{z})^{\mu}$. The proposition is obtained by setting $\lambda = z^{-1}$. ■

In view of (1.10) of Proposition 3, any $T \in \mathbf{M}_n$ whose spectrum lies on the unit circle belongs to $\mathbf{D}_n$. Hence by Theorem 1 we have

COROLLARY 1.2. *If* $T \in \mathbf{M}_n$ *and* $\sigma(T) \subset \{z \in \mathbf{C} \mid |z| = 1\}$, *then* $T \in \mathbf{F}_n$.

2. UNITARIES AND THEIR SIMILARITIES

Proposition 1 and Corollary 1.1 assert that the range of Φ restricted to the normal elements of $\mathbf{G}_n$ is precisely the subgroup of unitaries. In this section, we show among other things that the range of Φ restricted to the set of all non-singular hermitian congruences of the normal elements of $\mathbf{G}_n$ is the set of all similarities of the unitary group. The relevant facts are contained in Theorem 3 and Corollary 3.1.

Recall that the numerical range (field of values) of $A \in \mathbf{M}_n$ is defined as

$$W(A) = \{x^*Ax \mid x^*x = 1, x \in \mathbf{C}^n\}. \qquad (2.1)$$

$W(A)$ is a compact convex subset of C which contains $\sigma(A)$. For α, β, $z \in \mathbf{C}$, $W(\alpha A + \beta B) \subset \alpha W(A) + \beta W(B)$ and $\mathrm{Re}\,\bar{z}W(A) = W(\mathrm{Re}\,\bar{z}A) = 1/2\, W(\bar{z}A + zA^*)$. $W(V^*AV) = W(A)$ if V is unitary. For $R \in \mathbf{G}_n$, $0 \notin W(R^*AR)$ if and only if $0 \notin W(A)$. $A \in \mathbf{M}_n$ is called convexoid if $W(A) = \mathrm{co}\,\sigma(A)$ (the convex hull of $\sigma(A)$). Normals are convexoid. $W(A)$ is real if and only if $A = A^*$ and $W(A) > 0$ if and only if $A > 0$ (A is positive definite).

PROPOSITION 4. *Let $T \in \mathbf{G}_n$, then $T^*ST = S$ with $0 \notin W(S)$ if and only if $T^*QT = Q$ for some $Q > 0$.*

Proof. Since $Q > 0$ implies $0 \notin W(Q)$ we need only consider $T^*ST = S$ with $0 \notin W(S)$. By virtue of the convexity of $W(S)$, there exists $z \in \mathbf{C}$, $|z| = 1$, such that $\operatorname{Re} \bar{z} W(S) = W(\operatorname{Re} \bar{z} S) > 0$. Hence $S_z = \bar{z}S + zS^* > 0$ for such z. But $T^*ST = S$ implies $T^*S_zT = S_z$. ■

THEOREM 2. *Let $T \in \mathbf{G}_n$, then the following assertions are equivalent:*

(a) *T is similar to a unitary*
(b) *$T^*QT = Q$ for some $Q > 0$*
(c) *$T^*ST = S$ for some S with $0 \notin W(S)$*
(d) *$T = X^{-1}Y$ with $X^*X = Y^*Y$.*

Proof. Suppose $V = RTR^{-1}$ with $R \in \mathbf{G}_n$ and V unitary, then $I = V^*V = R^{*-1}T^*R^*RTR^{-1}$. Hence $0 < R^*R \in \Delta(T)$ and (a) implies (b). If (b) is assumed to hold, set $X = Q^{1/2}$, the positive square root of $Q > 0$, then $T = X^{-1}(XT)$ and $X^*X = Q = T^*QT = (XT)^*(XT)$ so that (b) implies (d). On the other hand, if (d) is assumed then $X^*X = Y^*Y$ implies $YX^{-1} = YTY^{-1}$ is unitary and (a) follows. Since Proposition 4 gives the equivalence of (b) and (c), the theorem is proved. ■

THEOREM 3. *Let $A \in \mathbf{G}_n$, then the following assertions are equivalent*

(a′) *$\Phi(A)$ is similar to a unitary.*
(b′) *QAQ is normal for some $Q > 0$.*
(c′) *S^*AS is normal for some $S \in \mathbf{G}_n$.*
(d′) *$\Phi(A) = \Phi(B)$ with $0 \notin W(B)$.*

Proof. By (1.4) $S^{-1}\Phi(A)S = \Phi(S^*AS)$. Hence, by Proposition 1, (a′) holds if and only if (c′) holds. (b′) implies (c′) trivially. Assume (c′) and let $S^* = VQ$ be a polar decomposition of $S^* \in \mathbf{G}_n$ with V unitary and $Q = (SS^*)^{1/2} > 0$. Since $V(QAQ)V^*$ is normal and V is unitary, it follows that QAQ is normal. Thus (a′), (b′), and (c′) are equivalent. If (d′) holds, $B \in \Delta(\Phi(A))$ with $0 \notin W(B)$ so that by Theorem 2 $\Phi(A)$ is similar to a unitary and (a′) follows. On the other hand, if QAQ is normal, then $\Phi(QAQ) = Q^{-1}\Phi(A)Q = V$ is unitary by Proposition 1. By Corollary 1.1 $V = \Phi(\tilde{V}^*)$ with $\tilde{V}^*$ a cramped unitary so that $0 \notin W(\tilde{V}^*)$. Hence, $\Phi(A) = Q\Phi(\tilde{V}^*)Q^{-1} = \Phi(B)$ with $B = Q^{-1}\tilde{V}^*Q^{-1}$ and, by a remark following (2.1), $0 \notin W(B)$. ■

There is an analogue to Theorem 2, in case the set $\mathbf{D}_n$ is replaced by the set of all $T \in \mathbf{G}_n$ for which T is similar to T^*. This theorem appears in [2] and

[10] and may be stated as: For $T \in \mathbf{G}_n$, T i
if $T^*S = ST$ with $0 \notin W(S)$ if and only if T
proves a corollary, namely: If $T^*S = ST$
then $T = T^*$. The analogous corollary in o

COROLLARY 2.1. *Let $T \in \mathbf{G}_n$ and let o*
of Theorem 2 hold, then T normal implie

Proof. Since T is normal, it is unitaril
which must be unitary if (a) is assumed. I

It is interesting to note that a proof of t
modelled on that of Taussky [11], in whic
[9] of the Frobenius group commutator th
with $0 \notin W(S)$, then, since T is normal
$T^{-1}STS^{-1} = (T^*T)^{-1}$ and T^{-1} commute
the Marcus-Thompson theorem yields (T^*

Actually better results are possible ir
hypothesis T normal by the weaker hypot
situation, T is similar to a self adjoint and t
if T is convexoid, $W(T)$ is real, and $T=$
slightly more complicated. We have

COROLLARY 2.2. *If $T \in \mathbf{G}_n$ and one o*
Theorem 2 holds, then T convexoid implie

Proof. Assume (a), i.e., T is similar to
able and $\sigma(T) = \{\alpha_1, \ldots, \alpha_k\}$ lies on the u
$W(T)$ is contained in the unit disk, so t
$W(T)$. Consequently $\ker(T - \alpha_j) = \ker(T^*$
232). Hence, these subspaces are mutual
diagonalizable, it follows that they reduc
unitarily equivalent to a diagonal unitary

We now consider a result which bears
which the hypothesis on T is dropped, bu
on $\Delta(T)$, namely

COROLLARY 2.3. *Let $T \in \mathbf{D}_n$ and st*
cramped unitary, then T is unitary.

Proof. If $V \in \Delta(T)$ and is unitary, the en $T^*ST=S$ *with* $0\notin W(S)$ *if and only if*

$\notin W(Q)$ we need only consider $T^*ST=S$
nvexity of $W(S)$, there exists $z\in\mathbf{C}$, $|z|=1$,
0. Hence $S_z=\bar{z}S+zS^*>0$ for such z. But ■

the following assertions are equivalent:

>0
th $0\notin W(S)$
$^*Y$.

vith $R\in\mathbf{G}_n$ and V unitary, then $I=V^*V$
$R^*R\in\Delta(T)$ and (a) implies (b). If (b) is
the positive square root of $Q>0$, then
$T=(XT)^*(XT)$ so that (b) implies (d). On
hen $X^*X=Y^*Y$ implies $YX^{-1}=YTY^{-1}$ is
position 4 gives the equivalence of (b) and ■

the following assertions are equivalent

itary.
ne $Q>0$.
ne $S\in\mathbf{G}_n$.
$V(B)$.

(S^*AS). Hence, by Proposition 1, (a′) holds
s (c′) trivially. Assume (c′) and let $S^*=VQ$
$\mathbf{G}_n$ with V unitary and $Q=(SS^*)^{1/2}>0$.
is unitary, it follows that QAQ is normal.
uivalent. If (d′) holds, $B\in\Delta(\Phi(A))$ with
(A) is similar to a unitary and (a′) follows.
ormal, then $\Phi(Q\tilde{A}Q)=Q^{-1}\Phi(A)Q=V$ is
ollary 1.1 $V=\Phi(\tilde{V}^*)$ with $\tilde{V}^*$ a cramped
nce, $\Phi(A)=Q\Phi(\tilde{V}^*)Q^{-1}=\Phi(B)$ with B
ollowing (2.1), $0\notin W(B)$. ■

m 2, in case the set $\mathbf{D}_n$ is replaced by the
ilar to T^*. This theorem appears in [2] and

[10] and may be stated as: For $T\in\mathbf{G}_n$, T is similar to a selfadjoint if and only if $T^*S=ST$ with $0\notin W(S)$ if and only if $T^*Q=QT$ with $Q>0$. Taussky [11] proves a corollary, namely: If $T^*S=ST$ with $0\notin W(S)$ and if T is normal, then $T=T^*$. The analogous corollary in our situation is

COROLLARY 2.1. *Let* $T\in\mathbf{G}_n$ *and let one of the assertions* (a) *through* (d) *of Theorem* 2 *hold, then T normal implies T unitary.*

Proof. Since T is normal, it is unitarily equivalent to a diagonal matrix D which must be unitary if (a) is assumed. Hence T is unitary. ■

It is interesting to note that a proof of the preceding corollary may also be modelled on that of Taussky [11], in which the Marcus-Thompson extension [9] of the Frobenius group commutator theorem is applied. Assume $T^*ST=S$ with $0\notin W(S)$, then, since T is normal, so are T^{-1} and $(T^*T)^{-1}$. But $T^{-1}STS^{-1}=(T^*T)^{-1}$ and T^{-1} commutes with $(T^*T)^{-1}$. An application of the Marcus-Thompson theorem yields $(T^*T)^{-1}=I$, i.e., T is unitary.

Actually better results are possible in both instances by replacing the hypothesis T normal by the weaker hypothesis T convexoid. In the Taussky situation, T is similar to a self adjoint and therefore has real spectrum, so that if T is convexoid, $W(T)$ is real, and $T=T^*$. In our situation matters are slightly more complicated. We have

COROLLARY 2.2. *If* $T\in\mathbf{G}_n$ *and one of the assertions* (a) *through* (d) *of Theorem* 2 *holds, then T convexoid implies T unitary.*

Proof. Assume (a), i.e., T is similar to a unitary. Hence T is diagonalizable and $\sigma(T)=\{\alpha_1,\dots,\alpha_k\}$ lies on the unit circle. Because T is convexoid, $W(T)$ is contained in the unit disk, so that the α_j lie on the boundary of $W(T)$. Consequently $\ker(T-\alpha_j)=\ker(T^*-\bar{\alpha}_j)$, $j=1,\dots,k$, (see e.g., [6], pg. 232). Hence, these subspaces are mutually orthogonal in $\mathbf{C}^n$ and, as T is diagonalizable, it follows that they reduce T and span $\mathbf{C}^n$. Therefore, T is unitarily equivalent to a diagonal unitary matrix and so is, in fact, unitary. ■

We now consider a result which bears a resemblance to Corollary 2.1, in which the hypothesis on T is dropped, but a stronger hypothesis is imposed on $\Delta(T)$, namely

COROLLARY 2.3. *Let* $T\in\mathbf{D}_n$ *and suppose* $V\in\Delta(T)$, *where V is a cramped unitary, then T is unitary.*

Proof. If $V\in\Delta(T)$ and is unitary, then (1.7) implies $V\in\Delta(T^*)$. Hence

$T^*VT=V$ and $TVT^*=V$ so that $TT^*VTT^*=V$. Since TT^* is normal and $0\notin W(V)$, Corollary 2.1 implies TT^* is unitary. But $TT^*>0$. Consequently $\sigma(TT^*)=\{1\}$ so that $TT^*=1$ and T is unitary. ∎

Corollary 2.3 has an alternate proof which does not make use of Theorem 2. Let V be a cramped unitary in $\Delta(T)$, then for $|\alpha|=1$, αV has the same property. Thus we may assume that $-\propto/2<\arg\sigma(V)\leqslant\pi/2$. Consequently, in virtue of the uniqueness of the square root of V^2 constructed in Lemma 1, we have $V=\tilde{V}^2$. On the other hand by (1.7) $T^*VT=V=TVT^*$, from which it follows that $TV^2=V^2T$. Hence by (iii) of Lemma 1, $TV=VT$ so that $T^*T=I$.

We remark that an entirely similar proof with obvious modifications yields a result due to Berberian [1]: If T is invertible and unitarily equivalent to T^* via a cramped unitary, then $T=T^*$.

We conclude this section with some corollaries to Theorem 3. For this purpose we say that $A\in\mathbf{M}_n$ is *conjunctive* with $B\in\mathbf{M}_n$ if $A=S^*BS$ for some $S\in\mathbf{G}_n$. The relation of conjunctivity is obviously symmetric.

COROLLARY 3.1. *$A\in\mathbf{G}_n$ is conjunctive with a diagonal unitary matrix if and only if any one of the assertions (a′) through (d′) of Theorem 3 holds.*

Proof. If D is a diagonal unitary and $A=S^*DS$ with $S\in\mathbf{G}_n$, then $\Phi(A)=S^{-1}D^{*2}S$. Since D^{*2} is unitary, (a′) of Theorem 3 holds. To prove the converse, observe first of all that any non-singular normal matrix is conjunctive with a diagonal unitary. This is readily seen by using a polar decomposition. Hence if (b′) of Theorem 3 is assumed, QAQ is a normal matrix in $\mathbf{G}_n$ with $Q>0$. It then follows that A is conjunctive with a diagonal unitary. ∎

COROLLARY 3.2. $\{A\in\mathbf{M}_n\,|\,0\notin W(A)\}=\{S^*VS\,|\,S\in\mathbf{G}_n$ *and V cramped unitary*$\}$.

Proof. If A is conjunctive with a cramped unitary V, then $0\notin W(V)$ and, by a remark following (2.1), $0\notin W(A)$. Conversely, if $0\notin W(A)$ then, $A\in\mathbf{G}_n$ and (d′) of Theorem 3 is satisfied. Hence by Corollary 3.1, A is conjunctive with a unitary V. Since $0\notin W(A)$, $0\notin W(V)$ so that V is cramped. ∎

3. $F^{-1}(T)$

We now treat the problem of determining all $A\in\mathbf{G}_n$ for which $A^{-1}A^*$

$=T$ for a given $T\in\mathbf{F}_n$. A principal result

THEOREM 4. *If $T\in\mathbf{F}_n$, then $\Phi^{-1}(T)=$*

Proof. By the second construction in t
implies $\tilde{T}^*H\in\Phi^{-1}(T)$. Conversely, sup
$=\Delta(\tilde{T})$, and we need only show that
$(\tilde{T}^*A\tilde{T})\tilde{T}=\tilde{T}^*(A\tilde{T})\tilde{T}=\tilde{T}^*AT=\tilde{T}^*A^*=(A$
$\Delta_*(T)$.

In particular, if $T=V$ is unitary, then I
with $H=H^*\in\mathbf{G}_n$. Therefore, we have

COROLLARY 4.1. *For V unitary $\Phi^{-1}($*
$=VH\}$.

We may also characterize $\Phi^{-1}(T)$ fo
simplicity we assume $1\notin\sigma(T)$.

COROLLARY 4.2. *If $T\in\mathbf{F}_n$ and $1\notin$*
$=\{i(1-T^*)H\,|\,H\in\Delta_*(T)\}$.

Proof. By Theorem 4, we need
$T^*)^{-1}\tilde{T}^*H\in\Delta_*(T)$ whenever $H\in\Delta_*(T)$.
where $H\in\Delta_*(T)$ so that $S\in\Delta(\tilde{T})=\Delta(T)$.
that $S^{-1}=iH^{-1}\tilde{T}^{*-1}(1-T^*)=i\tilde{T}H^{-1}(1-$
Hence, $S=S^*\in\Delta_*(T)$ as was to be verifie

In [12] it is observed that if V is unita
BA^{-1} is similar to its adjoint via both A^* a
its adjoint via both B^{-1} and B^{*-1}. By a
that BA^{-1} and $A^{-1}B$ may each be written
self-adjoint matrices. In the general case
description of $\mathbf{F}^{-1}(T)$ in terms of any fixed

THEOREM 5. *Let $A\in\Phi^{-1}(T)$ and $B\in$*
there exist $H_i\in\Delta_*(T)$, $i=1,2$, such th
$=H_2H_1^{-1}$.

Proof. If $T=A^{-1}A^*=B^{-1}B^*$, then
$H_i\in\Delta_*(T)$, such that $A=\tilde{T}^*H_1$ and $B=\tilde{T}$
$BA^{-1}=A(A^{-1}B)A^{-1}=\tilde{T}^*H_2H_1^{-1}\tilde{T}^{*-1}=$

$TT^*VTT^* = V$. Since TT^* is normal and
TT^* is unitary. But $TT^* > 0$. Consequently
T is unitary. ■

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at $-\alpha/2 < \arg\sigma(V) \leqslant \pi/2$. Consequently,
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is conjunctive with a diagonal unitary. ■

$W(A)\} = \{S^*VS \mid S \in \mathbf{G}_n$ *and* V *cramped*

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■

determining all $A \in \mathbf{G}_n$ for which $A^{-1}A^*$

$= T$ for a given $T \in \mathbf{F}_n$. A principal result is

THEOREM 4. *If* $T \in \mathbf{F}_n$, *then* $\Phi^{-1}(T) = \{\tilde{T}^*H \mid H \in \Delta_*(T)\}$.

Proof. By the second construction in the proof of Theorem 1, $H \in \Delta_*(T)$ implies $\tilde{T}^*H \in \Phi^{-1}(T)$. Conversely, suppose $T = A^{-1}A^*$, then $A \in \Delta(T) = \Delta(\tilde{T})$, and we need only show that $\tilde{T}^{*-1}A = A\tilde{T} \in \Delta_*(T)$. But $A\tilde{T} = (\tilde{T}^*A\tilde{T})\tilde{T} = \tilde{T}^*(A\tilde{T})\tilde{T} = \tilde{T}^*AT = \tilde{T}^*A^* = (A\tilde{T})^*$. Hence $A\tilde{T} = (A\tilde{T})^* \in \Delta_*(\tilde{T}) = \Delta_*(T)$. ■

In particular, if $T = V$ is unitary, then $H \in \Delta_*(V)$ if and only if $VH = HV$ with $H = H^* \in \mathbf{G}_n$. Therefore, we have

COROLLARY 4.1. *For* V *unitary* $\Phi^{-1}(V) = \{\tilde{V}^*H \mid H = H^* \in \mathbf{G}_n$ *and* $HV = VH\}$.

We may also characterize $\Phi^{-1}(T)$ for $T \in \mathbf{F}_n$ in terms of (1.8). For simplicity we assume $1 \notin \sigma(T)$.

COROLLARY 4.2. *If* $T \in \mathbf{F}_n$ *and* $1 \notin \sigma(T)$, *then* $\Phi^{-1}(T) = \{i(1 - T^*)H \mid H \in \Delta_*(T)\}$.

Proof. By Theorem 4, we need only verify that $S = -i(1 - T^*)^{-1}\tilde{T}^*H \in \Delta_*(T)$ whenever $H \in \Delta_*(T)$. But $\tilde{T}^*S\tilde{T} = -i\tilde{T}^*(1 - \tilde{T}^*)^{-1}H = S$ where $H \in \Delta_*(T)$ so that $S \in \Delta(\tilde{T}) = \Delta(T)$. By (1.7) $H^{-1} \in \Delta_*(T^*) = \Delta_*(\tilde{T}^*)$ so that $S^{-1} = iH^{-1}\tilde{T}^{*-1}(1 - T^*) = i\tilde{T}H^{-1}(1 - T^*) = -i(1 - T)\tilde{T}^{-1}H^{-1} = S^{*-1}$. Hence, $S = S^* \in \Delta_*(T)$ as was to be verified. ■

In [12] it is observed that if V is unitary and $V = A^{-1}A^* = B^{-1}B^*$ then BA^{-1} is similar to its adjoint via both A^* and A and that $A^{-1}B$ is similar to its adjoint via both B^{-1} and B^{*-1}. By a theorem of Carlson [2] it follows that BA^{-1} and $A^{-1}B$ may each be written as a product of two non-singular self-adjoint matrices. In the general case of $\mathbf{F}_n$, we obtain the following description of $\mathbf{F}^{-1}(T)$ in terms of any fixed $A \in \Phi^{-1}(T)$:

THEOREM 5. *Let* $A \in \Phi^{-1}(T)$ *and* $B \in \mathbf{G}_n$. $B \in \Phi^{-1}(T)$ *if and only if there exist* $H_i \in \Delta_*(T)$, $i = 1, 2$, *such that* $A^{-1}B = H_1^{-1}H_2$ *and* $BA^{-1} = H_2H_1^{-1}$.

Proof. If $T = A^{-1}A^* = B^{-1}B^*$, then, by Theorem 5, there exist $H_i \in \Delta_*(T)$, such that $A = \tilde{T}^*H_1$ and $B = \tilde{T}^*H_2$. Clearly $A^{-1}B = H_1^{-1}H_2$ and $BA^{-1} = A(A^{-1}B)A^{-1} = \tilde{T}^*H_2H_1^{-1}\tilde{T}^{*-1} = H_2\tilde{T}^{-1}H_1^{-1}\tilde{T}^{*-1} = H_2H_1^{-1}$ since

$H_i \in \Delta_*(\tilde{T})$. On the other hand, if $B = H_2H_1^{-1}A = AH_1^{-1}H_2$ with $H_i \in \Delta_*(T)$ then $B^{-1}B^* = A^{-1}H_1H_2^{-1}H_2H_1^{-1}A^* = A^{-1}A$. ■

We remark that when $H_i \in \Delta_*(T)$, $i = 1,2$, T commutes with $H_1^{-1}H_2$. In fact it follows readily from (1.7) that if $T \in \mathbf{D}_n$ and $S_i \in \Delta(T)$, $i = 1,2$, then $S_1^{-1}S_2$ commutes with T and $S_2S_1^{-1}$ commutes with T^*.

Theorem 5 yields some necessary conditions for $\Phi(A) = \Phi(B)$ which are in general not sufficient. For example,

COROLLARY 5.1. *If $\Phi(A) = \Phi(B)$, the following hold*

$$i[A^{-1},B] \equiv i(A^{-1}B - BA^{-1}) \text{ is self-adjoint.} \tag{3.1}$$

$$A^{-1}B^2A > 0. \tag{3.2}$$

$$i[A,B^*] \text{ is self-adjoint.} \tag{3.3}$$

Since Theorem 4 reveals an intimate relationship between $\Phi^{-1}(T)$ and $\Delta_*(T)$, it seems appropriate to conclude this section with some remarks on the structure of $\Delta_*(T)$ for $T \in \mathbf{D}_n$. We shall confine our comments to the case in which T is diagonalizable, i.e., when there exists $R \in \mathbf{G}_n$ and a diagonal matrix D_T for which $TR = RD_T$. Other facts pertaining to $\Delta(T)$ are contained in [13].

For $T \in \mathbf{D}_n$ we have already observed that (1.7) holds. An easy computation shows that $T \in \mathbf{D}_n$ if and only if $R^{-1}TR \in \mathbf{D}_n$ for all $R \in \mathbf{G}_n$; moreover,

$$\Delta_*(R^{-1}TR) = R^*\Delta_*(T)R. \tag{3.4}$$

If $T_1 \in \mathbf{D}_{n_1}$ and $T_2 \in \mathbf{D}_{n_2}$ with $n = n_1 + n_2$, then the direct sum $T = T_1 \oplus T_2 \in \mathbf{D}_n$ and if in addition $\sigma(T_1) \cap \sigma(T_2) = \emptyset$, it is not hard to see that

$$\Delta_*(T_1 \oplus T_2) = \Delta_*(T_1) \oplus \Delta_*(T_2). \tag{3.5}$$

A description of $\Delta_*(T)$ is readily obtainable in terms of any fixed element H_0 of $\Delta_*(T)$:

$$\Delta_*(T) = \{H_0C \mid C \in \mathbf{G}_n, TC = CT, H_0C = C^*H_0\}. \tag{3.6}$$

Let us now assume $T \in \mathbf{D}_n$ is diagonalizable. In view of Proposition 3, there exists $R \in \mathbf{G}_n$, such that

$$R^{-1}TR = D_0 \oplus L_1 \oplus \cdots \oplus L_s, \tag{3.7}$$

where D_0 is a unitary diagonal matrix in

$L_j = \begin{pmatrix} \lambda_j I_j & \\ 0 & \end{pmatrix}$

with λ_j, $j = 1,\ldots,s$, the distinct eigenval
identity matrix in $\mathbf{M}_{m_j}$ with $m_j = \dim\ker($
With these facts and notation we may no

THEOREM 6. *If $T \in \mathbf{D}_n$ and is diagona*

$\Delta_*(T) = S^{*-1}(D \oplus K_1 \oplus$

where D is any self adjoint diagonal matr

$K_j = \begin{pmatrix} 0 & \\ B_j^* & \end{pmatrix}$

$j = 1,\ldots,s$, with B_j arbitrary elements of $\mathbf{G}$

Proof. In view of (3.4), (3.5), and (3.7
and $\Delta_*(L_j)$, $j = 1,\ldots,s$. Consider the latter

$H_j = \begin{pmatrix} 0 & I_j \\ I_j & 0 \end{pmatrix} \in$

From the representation (3.6), it follows th
the form asserted in the theorem. To comp
applied to yield $\Delta_*(D_0) = \{H = H^* \in$
$= D^* \in \mathbf{G}_k$ and is diagonal, U' is unitar
$U = U' \oplus I_{2m}$, $2m = n - k$, we see that U is
D_T so that $R \in \mathbf{G}_n$ diagonalizes T if and
theorem now follows from (3.7).

4. SQUARE ROOTS.

The existence of square roots of a non
(e.g., [5], pp. 231–234). However, the part
which possesses the property that $\Delta(T) = \Delta$

$3 = H_2H_1^{-1}A = AH_1^{-1}H_2$ with $H_i \in \Delta_*(T)$
$= A^{-1}A$. ■

$)$, $i=1,2$, T commutes with $H_1^{-1}H_2$. In
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$n_1 + n_2$, then the direct sum $T = T_1 \oplus$
$T_2) = 0$, it is not hard to see that

$= \Delta_*(T_1) \oplus \Delta_*(T_2)$. (3.5)

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iagonalizable. In view of Proposition 3,

$\oplus L_1 \oplus \cdots \oplus L_s$, (3.7)

where D_0 is a unitary diagonal matrix in $\mathbf{M}_k$,

$$L_j = \begin{pmatrix} \lambda_j I_j & 0 \\ 0 & \bar{\lambda}_j^{-1} I_j \end{pmatrix}$$

with λ_j, $j=1,\ldots,s$, the distinct eigenvalues of T with $|\lambda_j|<1$ and I_j the identity matrix in $\mathbf{M}_{m_j}$ with $m_j = \dim\ker(T-\lambda_j)$. Note that $n = k + 2\sum_{j=1}^{s} m_j$. With these facts and notation we may now prove

THEOREM 6. *If $T \in \mathbf{D}_n$ and is diagonalizable, then*

$$\Delta_*(T) = S^{*-1}(D \oplus K_1 \oplus \cdots \oplus K_s)S^{-1}, \tag{3.8}$$

where D is any self adjoint diagonal matrix in $\mathbf{G}_k$,

$$K_j = \begin{pmatrix} 0 & B_j \\ B_j^* & 0 \end{pmatrix},$$

$j=1,\ldots,s$, with B_j arbitrary elements of $\mathbf{G}_{m_j}$, and $S \in \mathbf{G}_n$ diagonalizes T.

Proof. In view of (3.4), (3.5), and (3.7) we need only determine $\Delta_*(D_0)$ and $\Delta_*(L_j)$, $j=1,\ldots,s$. Consider the latter first and note that

$$H_j = \begin{pmatrix} 0 & I_j \\ I_j & 0 \end{pmatrix} \in \Delta_*(L_j).$$

From the representation (3.6), it follows that $K_j \in \Delta_*(L_j)$ if and only if K_j has the form asserted in the theorem. To compute $\Delta_*(D_0)$, Corollary 4.1 may be applied to yield $\Delta_*(D_0) = \{H = H^* \in \mathbf{G}_k \mid D_0H = HD_0\} = \{U'DU'^* \mid D = D^* \in \mathbf{G}_k$ and is diagonal, U' is unitary in $\mathbf{G}_k$, $U'D_0 = D_0U'\}$. Setting $U = U' \oplus I_{2m}$, $2m = n-k$, we see that U is unitary in $\mathbf{M}_n$ and commutes with D_T so that $R \in \mathbf{G}_n$ diagonalizes T if and only if RU diagonalizes T. The theorem now follows from (3.7). ■

4. SQUARE ROOTS.

The existence of square roots of a non-singular matrix T is well known (e.g., [5], pp. 231–234). However, the particular square root we need is one which possesses the property that $\Delta(T) = \Delta(\tilde{T})$ if $\tilde{T}^2 = T$. For this reason we have stated Lemma 1 in §1. Since we find it difficult to quote a particular reference in this matter and since the lemma as stated may be of independent interest, we proceed to prove it.

For $T \in \mathbf{G}_n$ and $\lambda \in \sigma(T)$ let us agree to set $\lambda = |\lambda| e^{i\theta}$ with $-\pi < \theta \leqslant \pi$. For $-\pi < \arg z \leqslant \pi$, set $z^{1/2} = |z|^{1/2}\exp(\frac{1}{2}\arg z)$. Then the function $f(z) = z^{1/2}$ is holomorphic for $|\arg z| < \pi$. For convenience, we restate the lemma.

LEMMA 1. *Let $T \in \mathbf{G}_n$. There exists a unique $\tilde{T} \in \mathbf{G}_n$ satisfying*

$$\text{(i)} \quad \tilde{T}^2 = T. \qquad \text{(ii)} \quad \frac{-\pi}{2} < \arg\sigma(\tilde{T}) \leqslant \frac{\pi}{2},$$

$$\text{(iii)} \quad TC = CT \Rightarrow \tilde{T}C = C\tilde{T}.$$

Moreover $\Delta(T) = \Delta(\tilde{T})$.

Proof. First the uniqueness. Suppose both $\tilde{T}$ and B satisfy (i), (ii), (iii). By (i) both are non-singular, by (iii) the fact that $[\tilde{T}, T] = 0$ implies $[\tilde{T}, B] = 0$. Consequently $0 = \tilde{T}^2 - B^2 = (\tilde{T} - B)(\tilde{T} + B)$ and $\tilde{T} = B$ if $\tilde{T} + B \in \mathbf{G}_n$, i.e., if $-1 \notin \sigma(\tilde{T}B^{-1}) \subset \sigma(\tilde{T})/\sigma(B)$ since $[\tilde{T}, B] = 0$. But from (ii) we see that $\pi \notin \arg\sigma(\tilde{T}) - \arg\sigma(B)$. Hence $-1 \notin \sigma(\tilde{T})/\sigma(B)$ so that $-1 \notin \sigma(\tilde{T}B^{-1})$.

There are several possible constructions for $\tilde{T}$. We find the representation as a Cauchy integral convenient. Assume first that $\pi \notin \arg\sigma(T)$. Let Γ be the oriented Jordan curve consisting of circular arcs and line segments shown in the figure and containing $\sigma(T)$ in its interior domain Δ (Fig. 1). Set

$$\tilde{T} = \frac{1}{2\pi i}\int_\Gamma z^{1/2}(z - T)^{-1}\,dz. \tag{4.1}$$

Referring to [14] (pg. 287ff) it is a simple matter to verify (i) and a stronger version of (ii), namely (ii)′ $-\pi/2 < \arg\sigma(\tilde{T}) < \pi/2$, since $z^{1/2} - \zeta \neq 0$ for $\operatorname{Re}\zeta \leqslant 0$ and $z \in \bar{\Delta}$. From this it also follows that $\tilde{T} \in \mathbf{G}_n$. Since $TC = CT$ implies $(z - T)^{-1}C = C(z - T)^{-1}$ for $z \in \Gamma$, (4.1) shows $\tilde{T}C = C\tilde{T}$. Clearly $\Delta(\tilde{T}) \subseteq \Delta(T)$ since $\tilde{T}^2 = T$. On the other hand, if $T^*ST = S \in \mathbf{G}_n$ then $\sigma(T) = \sigma(T^{*-1})$ and $S(z - T)^{-1} = (z - T^{*-1})^{-1}S$. But (4.1) with T replaced by T^{*-1} defines $(\widetilde{T^{*-1}})$ so that $ST = (\widetilde{T^{*-1}})S$. It is easy to check that both $(\widetilde{T^{*-1}})$ and $\tilde{T}^{*-1}$ satisfy (i), (ii), (iii) of the lemma relative to T^{*-1}. By uniqueness it follows that $(\widetilde{T^{*-1}}) = \tilde{T}^{*-1}$ and $\tilde{T}^*S\tilde{T} = S$. Hence $\Delta(\tilde{T}) = \Delta(T)$.

Finally, to remove the restriction $\pi \notin \arg\sigma(T)$, we need only observe for sufficiently small positive ϵ that $T_\epsilon = e^{-2i\epsilon}T$ satisfies $\pi \notin \arg\sigma(T_\epsilon)$ and $-\pi < \arg\sigma(T) - 2\epsilon = \arg\sigma(T_\epsilon) < \pi$. Therefore $\tilde{T}_\epsilon$ satisfies (i), (ii)′, (iii) relative to T_ϵ. Consequently $e^{i\epsilon}\tilde{T}_\epsilon$ satisfies (i), (ii), (iii), relative to T. Define $\tilde{T}$ by $e^{i\epsilon}\tilde{T}_\epsilon$. Since $\Delta(\alpha T) = \Delta(T)$ for $|\alpha| = 1$, it follows that $\Delta(\tilde{T}) = \Delta(\tilde{T}_\epsilon) = \Delta(T_\epsilon) = \Delta(T)$. This completes the proof of the lemma. ■

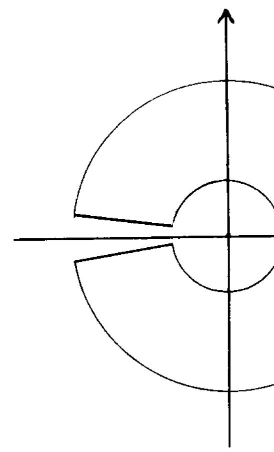

FIG. 1.

A few observations may be made. If V
$\tilde{V}^*I\tilde{V} = I = \tilde{V}^*\tilde{V}$ so that $\tilde{V}$ is a cramped uni
$\tilde{Q} = Q^{1/2}$, the positive square root of Q. If
(iii) $TT^* = T^*T$. Hence $\tilde{T}T^* = \tilde{T}^*T$ and ag
normal. For $H = H^* \in \mathbf{G}_n$, $H = H_+ - H_-$
$= H_-H_+ = 0$. Therefore $\tilde{H} = H_+^{1/2} + iH_-^{1/2}$

Actually, all square roots of $T \in \mathbf{G}_n$ m
principal square root $\tilde{T}$, namely

PROPOSITION 5. *Let $T, C \in \mathbf{G}_n$, then*
some $J \in \mathbf{M}_n$ with $J^2 = I$ and $TJ = JT$.

Proof. If $C = J\tilde{T}$ for such J, then $J\tilde{T} =$
$C^2 = T$. On the other hand if $C^2 = T$, then
1, $\tilde{T}C = C\tilde{T}$; therefore $(C\tilde{T}^{-1})^2 = I$ and TC

We should like to express our gratitud
useful comments and suggestions during t

Note. After this paper was accepted
informed that M. D. Choi independently

nce we find it difficult to quote a particular
nce the lemma as stated may be of indepen-
ove it.

t us agree to set $\lambda=|\lambda|e^{i\theta}$ with $-\pi<\theta\leqslant\pi$.
$|^{1/2}\exp(\frac{1}{2}\arg z)$. Then the function $f(z)=z^{1/2}$
For convenience, we restate the lemma.

re exists a unique $\tilde{T}\in\mathbf{G}_n$ *satisfying*

(ii) $\frac{-\pi}{2}<\arg\sigma(\tilde{T})\leqslant\frac{\pi}{2}$,

$TC=CT\Rightarrow\tilde{T}C=C\tilde{T}$.

ppose both $\tilde{T}$ and B satisfy (i), (ii), (iii). By (i)
the fact that $[\tilde{T},T]=0$ implies $[\tilde{T},B]=0$.
$-B)(\tilde{T}+B)$ and $\tilde{T}=B$ if $\tilde{T}+B\in\mathbf{G}_n$, i.e., if
nce $[\tilde{T},B]=0$. But from (ii) we see that $\pi\notin$
$\notin\sigma(\tilde{T})/\sigma(B)$ so that $-1\notin\sigma(\tilde{T}B^{-1})$.
nstructions for $\tilde{T}$. We find the representation
t. Assume first that $\pi\notin\arg\sigma(T)$. Let Γ be the
g of circular arcs and line segments shown in
in its interior domain Δ (Fig. 1). Set

$$\frac{1}{2\pi i}\int_\Gamma z^{1/2}(z-T)^{-1}dz. \tag{4.1}$$

s a simple matter to verify (i) and a stronger
$\pi/2<\arg\sigma(\tilde{T})<\pi/2$, since $z^{1/2}-\zeta\neq0$ for
s it also follows that $\tilde{T}\in\mathbf{G}_n$. Since $TC=CT$
^{1} for $z\in\Gamma$, (4.1) shows $\tilde{T}C=C\tilde{T}$. Clearly
the other hand, if $T^*ST=S\in\mathbf{G}_n$ then $\sigma(T)$
$-T^{*-1})^{-1}S$. But (4.1) with T replaced by
$ST=(\widetilde{T^{*-1}})S$. It is easy to check that both
(ii), (iii) of the lemma relative to T^{*-1}. By
$^{1})=\tilde{T}^{*-1}$ and $\tilde{T}^*S\tilde{T}=S$. Hence $\Delta(\tilde{T})=\Delta(T)$.
ction $\pi\notin\arg\sigma(T)$, we need only observe for
t $T_\epsilon=e^{-2i\epsilon}T$ satisfies $\pi\notin\arg\sigma(T_\epsilon)$ and $-\pi$
Therefore $\tilde{T}_\epsilon$ satisfies (i), (ii)′, (iii) relative to
(i), (ii), (iii), relative to T. Define $\tilde{T}$ by $e^{i\epsilon}\tilde{T}_\epsilon$.
it follows that $\Delta(\tilde{T})=\Delta(\tilde{T}_\epsilon)=\Delta(T_\epsilon)=\Delta(T)$.
lemma. ■

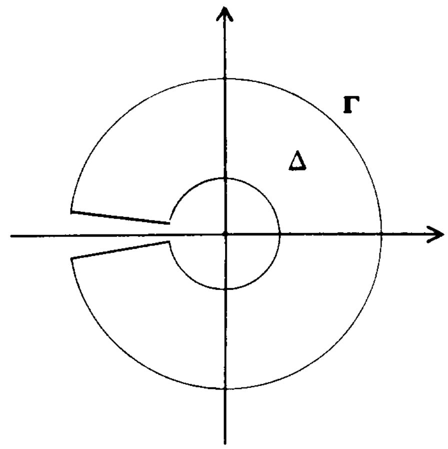

FIG. 1.

A few observations may be made. If V is unitary, $I\in\Delta(V)$ and therefore $\tilde{V}^*I\tilde{V}=I=\tilde{V}^*\tilde{V}$ so that $\tilde{V}$ is a cramped unitary. If $Q>0$, then by uniqueness $\tilde{Q}=Q^{1/2}$, the positive square root of Q. If T is normal and invertible then by (iii) $TT^*=T^*T$. Hence $\tilde{T}T^*=\tilde{T}^*T$ and again by (iii) $\tilde{T}\tilde{T}^*=\tilde{T}^*\tilde{T}$ so that $\tilde{T}$ is normal. For $H=H^*\in\mathbf{G}_n$, $H=H_+-H_-$ with $H>0$, $H_->0$, and $H_+H_-=H_-H_+=0$. Therefore $\tilde{H}=H_+^{1/2}+iH_-^{1/2}$.

Actually, all square roots of $T\in\mathbf{G}_n$ may be determined in terms of the principal square root $\tilde{T}$, namely

PROPOSITION 5. *Let* $T,C\in\mathbf{G}_n$, *then* $C^2=T$ *if and only if* $C=J\tilde{T}$ *for some* $J\in\mathbf{M}_n$ *with* $J^2=I$ *and* $TJ=JT$.

Proof. If $C=J\tilde{T}$ for such J, then $J\tilde{T}=\tilde{T}J$ by (iii) of Lemma 1, so that $C^2=T$. On the other hand if $C^2=T$, then clearly $TC=CT$. By (iii) of Lemma 1, $\tilde{T}C=C\tilde{T}$; therefore $(C\tilde{T}^{-1})^2=I$ and $TC\tilde{T}^{-1}=\tilde{T}C=C\tilde{T}^{-1}T$. ■

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Note. After this paper was accepted for publication, the authors were informed that M. D. Choi independently found and proved Theorem 1.

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ADDENDUM: After we had submitted this paper, there appeared a paper by U. N. Singh and K. Mangla: "Operators with inverses similar to their adjoints", *PAMS* **38**, 258–260 (1973). There is an overlap between §2 of our paper and their paper. In particular, our Proposition 4 is related to their Theorem 1. In addition, the equivalence of assertions (a) and (c) of our Theorem 2 (finite dimensionality is clearly not involved in our proof) coincides with their Corollary 2, as does our Corollary 2.3 with their Theorem 2.

On Strictly Dissipative Matrices*

Ky Fan
University of California, Santa Barbara, C

ABSTRACT

For a square complex matrix A with p
$(A-A^*)/2i$, this paper studies: Hermitian co
lues $\{\alpha_j\}$ of $A^{-1}A^*$ relative to a line in the c
the arguments, real parts and imaginary pa
terminants and singular values.

1. INTRODUCTION

By a *strictly dissipative* matrix we mea
that its imaginary component $(2i)^{-1}(A-$
have seen that certain properties of a stri
necessarily invertible) are closely related
$A^{-1}A^*$ which is similar to a unitary matri
A, we study in the present paper: the mat
distribution of the eigenvalues of $A^{-1}A^*$
plane, arguments, real parts and imagi
$A^{-1}A^*$, and certain inequalities for deter

2. THE MATRIX $A^{-1}A^*$

The fact that for every strictly dissipati
unitary matrix [5], can be given the follow
statement.

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