Fourier

1. When can we say that the Fourier transform of a real signal is real? When can we say that it is purely imaginary?

Solution:
For each real signal, we can decompose into two parts, even and odd, \( f(x) = e(x) + o(x) \). And its corresponding Fourier transformation can be obtained like below:

\[
F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx = \int_{-\infty}^{\infty} e(x) e^{-j\omega x} dx + \int_{-\infty}^{\infty} o(x) e^{-j\omega x} dx
\]

\[
= \int_{-\infty}^{\infty} e(x) \cos(\omega x) dx - j \int_{-\infty}^{\infty} e(x) \sin(\omega x) dx + \int_{-\infty}^{\infty} o(x) \cos(\omega x) dx - j \int_{-\infty}^{\infty} o(x) \sin(\omega x) dx
\]

\[
= 2 \int_{0}^{\infty} e(x) \cos(\omega x) dx - 2j \int_{0}^{\infty} o(x) \sin(\omega x) dx
\]

Such that if the real signal is an even function, its Fourier transform is real. In addition, when the real signal is an odd function, the corresponding Fourier transform is purely imaginary.

2. Do we really need the negative frequencies in the continuous Fourier transform? When can we “predict” the negative frequencies from the positive ones. Give an example of a function where we cannot predict the negative frequency components from the positive ones.

Solution:
Actually, it depends on the type of continuous signal.

For real signals, we don’t need the negative part since the negative frequency coefficients can be obtained as the conjugate of the correspondent positive frequency coefficient.

For example, suppose we have a real signal

\[
\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \rightarrow \delta(\omega + \omega_0) + \delta(\omega - \omega_0)
\]

This real signal contains two equal-amplitude complex exponentials, rotating in opposite directions, so that the real parts combine and imaginary parts cancel out. This is why the spectrum of a sine wave always has 2 spikes, one positive and one negative. but if we already know that’s a real signal, the other side of the spectrum doesn’t provide any extra information, and can be inferred from the positive part.
For complex signals, they require the sign of the frequency to get an idea of the direction/phase, so we need the negative frequencies in the continuous Fourier transformation.

For example, suppose we have a complex signal like below

\[ ae^{-j\omega t} \rightarrow a\delta(\omega - \omega_0) \]

Here, \( a \) is a scalar factor and its Fourier transform only contains one positive frequency, in this case, since the original signal is complex, we cannot infer some negative frequency information based on the only frequency component.

3. The Fourier transform of a discrete time signal is periodic. Prove true or false (counterexample).

Solution:
True.

Actually, the sampling operation is multiplication with a comb function in the time domain, which is convolution with a comb in the frequency domain such that periodizes the spectrum.

More specifically, in the time domain, for the sampling operation, we have

\[ f(t) \times \sum_{k=0}^{\inf} \sigma(t-k) \rightarrow F(\omega) \ast [1 + e^{-j\omega k_1} + e^{-j\omega k_2} + ... + e^{-j\omega k_N}] \]

So in the frequency domain, the spectrum convolves with the exponential function plays as the shifting (translation) operation. Therefore, in the frequency domain, it’s the sum of all shifted spectrums and all of them has equal interval, and that is what we call the periodic.
4. The Fourier transform of a periodic signal is discrete. Prove true or false (counterexample).

**Solutions:**
True.

It follows the same idea as the previous question. Periodic signal in time domain can be expressed as a single period of the signal convolved with the dirac comb to produce a continuous signal having the said period.

Thus, the fourier transform of such a signal is basically the product of fourier transform of the single period signal with the fourier transform of the dirac comb (which is also a dirac comb). Multiplying a continuous signal by a dirac comb will yield a discrete signal.

5. The Fourier transform of a periodic signal has finite support. Prove true or false (counterexample).

**Solution:**
False.

We can write a periodic signal $x(t)$ as Fourier series,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t}$$

and apply the Fourier transformation function we can get the spectrum formula

$$X(e^{j\omega}) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_k e^{j(k\omega_0 - \omega)t} dt = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} a_k e^{j(k\omega_0 - \omega)t} dt$$

We can find only when $\omega = k\omega_0$, the integral will not be zero, but the integral from -inf to inf will lead to inf result. Therefore, we cannot say the Fourier transform of a periodic signal has finite support.

Actually, the square wave is a counter example in this case.

6. Compute the Fourier transform of the filter $[1/2 \ 1/2]$ and plot it over $(-\pi, \pi]$.


The fourier transform of a shifted dirac impulse as above will be a complex exponential.
We have here
\[
\delta[n - 1] = 1/2
\]
\[
\delta[n - 2] = 1/2
\]
which after Fourier transform will be
\[
F(\delta[n - 1]) = \sum_{n=-\infty}^{\infty} \delta[n - 1]e^{-j\omega n}
\]
\[
= 1/2e^{-j\omega n}|_{n=1}
\]
\[
= 1/2e^{-j\omega}
\]
\[
F(\delta[n - 2]) = \sum_{n=-\infty}^{\infty} \delta[n - 2]e^{-j\omega n}
\]
\[
= 1/2e^{-j\omega n}|_{n=2}
\]
\[
= 1/2e^{-j2\omega}
\]
\[
F(\delta) = F(\delta[n - 1]) + F(\delta[n - 2])
\]
\[
= 1/2e^{-j\omega} + 1/2e^{-j2\omega}
\]
\[
= 1/2(cos(\omega) - jsin(\omega) + cos(2\omega) - jsin(2\omega))
\]
\[
= 1/2((cos(\omega) + cos(2\omega)) - j(sin(\omega) + sin(2\omega)))
\]

7. We can always invert the result of a convolution, which means with known h[n] and r[n]
and r[n] = f[n] * h[n] recover f[n]. Prove true or false (counterexample).
False.
Generally, given a convolution of two signals in the time domain, it can be transformed to the
fourier domain where it will be just a multiplication of two frequencies. We can recover the
frequency corresponding to the signal f[n] in the fourier by just a simple mathematical inversion
and then inverse transform of the to get the input signal f[n]. But this won’t hold if for example
the fourier transform H(w) of h[n] yields zero values at some points. In this case, the
mathematical operation does not hold and we won’t be able to recover the f[n].
8. Compute and plot the Fourier transform of a white square of side length \( B \) centered on a black image.

\[
F(u, v) = \iint f(x, y)e^{-j2\pi(ux+vy)}dxdy \\
= \int_{-B/2}^{B/2} e^{-j2\pi ux}dx \int_{-B/2}^{B/2} e^{-j2\pi vy}dy \\
= \left[ \frac{e^{-j2\pi ux}}{-j2\pi u} \right]_{-B/2}^{B/2} \left[ \frac{e^{-j2\pi vy}}{-j2\pi v} \right]_{-B/2}^{B/2} \\
= \frac{1}{-j2\pi u} [e^{-jUB} - e^{jUB}] \frac{1}{-j2\pi v} [e^{-jvB} - e^{jvB}] \\
= B^2 \frac{\sin(\pi Bu)}{\pi Bu} \frac{\sin(\pi Bv)}{\pi Bv} \\
= B^2 \text{sinc}(\pi Bu)\text{sinc}(\pi Bv)
\]

9. Compute and plot the Fourier transform of a white rhombus with vertices at \((A, 0),(0, A),(-A, 0),(0 - A)\) on a black image.

As shown in 8, the Fourier transform of a square with side \( A \) is:

\[
F(u, v) = A^2 \text{sinc}(\pi Au)\text{sinc}(\pi Av)
\]

However, in this case, the square is rotated by \(45^\circ\) by to form a rhombus. Rotating a 2D function rotates its Fourier transform i.e. an anti-clockwise rotation of a function by an angle \( \theta \) implies that its Fourier transform is also rotated anti-clockwise by the same angle.

We define a new coordinate system where:
10. Compute and plot the 2D Fourier transform of $\cos \omega_1 x + \cos \omega_2 y$.

$$F(\omega_x, \omega_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos(\omega_1 x) + \cos(\omega_2 y) \, dx \, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos(\omega_1 x) e^{-j\omega_1 y} e^{-j\omega_2 y} \, dx \, dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos(\omega_2 y) e^{-j\omega_2 x} e^{-j\omega_1 y} \, dx \, dy$$

$$= \int_{-\infty}^{\infty} e^{-j\omega_2 y} dy \int_{-\infty}^{\infty} \frac{e^{-j(\omega_1 + \omega_2) x} + e^{j(\omega_1 - \omega_2) x}}{2} e^{-j\omega_1 y} \, dx + \int_{-\infty}^{\infty} e^{-j\omega_1 y} dx \int_{-\infty}^{\infty} \frac{e^{-j(\omega_2 + \omega_1) y} + e^{j(\omega_2 - \omega_1) y}}{2} e^{-j\omega_2 x} \, dy$$

$$= \frac{1}{2} \left( \delta(\omega_x - \omega_1) + \delta(\omega_x + \omega_1) + \delta(\omega_y - \omega_2) + \delta(\omega_y + \omega_2) \right)$$

Easy solution: The Fourier transform of this function is basically a sum of the Fourier transforms of 2 cosines one along the x axis and one along the y axis. This results in 2 deltas each on the x and y axis respectively.
11. Compute and plot the 2D Fourier transform of \( \cos(\omega_1 x) \cos(\omega_2 y) \)

\[
F(\omega_x, \omega_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos(\omega_1 x) \cos(\omega_2 y) \, dx \, dy \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos(\omega_1 x) \cos(\omega_2 y) e^{-j \omega_x x} e^{-j \omega_y y} \, dx \, dy \\
= \int_{-\infty}^{\infty} \frac{e^{-j \omega_2 y} + e^{j \omega_2 y}}{2} e^{-j \omega_y y} dy \int_{-\infty}^{\infty} \frac{e^{-j \omega_1 x} + e^{j \omega_1 x}}{2} e^{-j \omega_x x} \, dx \\
= \int_{-\infty}^{\infty} \frac{e^{-j (\omega_y - \omega_2) y} + e^{j (\omega_y - \omega_2) y}}{2} dy \int_{-\infty}^{\infty} \frac{e^{-j (\omega_x - \omega_1) x} + e^{j (\omega_x - \omega_1) x}}{2} \, dx \\
= \frac{1}{4} (\delta(\omega_x - \omega_1) + \delta(\omega_x + \omega_1))(\delta(\omega_y - \omega_2) + \delta(\omega_y + \omega_2))
\]
Edges, ridges, and waves

1. Compute and plot the convolution of a step edge with the first derivative of a Gaussian.

Suppose \( s(x) \) is the step function and \( g(x) \) is the Gaussian function. Since derivatives commute in convolution, the convolution \( y(x) \) is computed as

\[
y(x) = s(x) \ast \frac{dg(x)}{dx}
\]

\[
= \frac{ds(x)}{dx} \ast g(x)
\]

\[
= \delta(x) \ast g(x)
\]

\[
= g(x)
\]

Hence the result is the Gaussian itself.

2. Compute and plot the convolution of a step edge with the second derivative of a Gaussian.

Suppose \( s(x) \) is the step function and \( g'(x) \) is the first derivative of the Gaussian function. Since derivatives commute in convolution, the convolution \( y(x) \) is computed as

\[
y(x) = s(x) \ast \frac{dg'(x)}{dx}
\]

\[
= \frac{ds(x)}{dx} \ast g'(x)
\]

\[
= \delta(x) \ast g'(x)
\]

\[
= g'(x)
\]

Hence the result is the first derivative of the Gaussian itself.