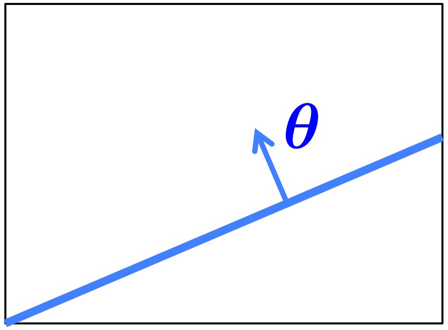
Linear Classification: The Perceptron

Linear Classifiers

- A **hyperplane** partitions $\mathbb{R}^d$ into two half-spaces
  - Defined by the normal vector $\boldsymbol{\theta} \in \mathbb{R}^d$
    - $\boldsymbol{\theta}$ is orthogonal to any vector lying on the hyperplane



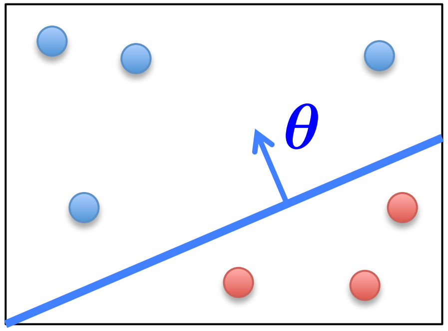
  - Assumed to pass through the origin
    - This is because we incorporated bias term θ_0 into it by $x_0 = 1$

- Consider classification with +1, -1 labels ...

Linear Classifiers

- **Linear classifiers**: represent decision boundary by hyperplane

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_d \end{bmatrix} \quad \boldsymbol{x}^\intercal = \begin{bmatrix} 1 & x_1 & \ldots & x_d \end{bmatrix}$$



$$h(\boldsymbol{x}) = \text{sign}(\boldsymbol{\theta}^\intercal \boldsymbol{x}) \quad \text{where} \quad \text{sign}(z) = \begin{cases} 1 & \text{if } z \geq 0 \\ -1 & \text{if } z < 0 \end{cases}$$

– Note that: $\boldsymbol{\theta}^\intercal \boldsymbol{x} > 0 \implies y = +1$

$\boldsymbol{\theta}^\intercal \boldsymbol{x} < 0 \implies y = -1$

The Perceptron

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- The perceptron uses the following update rule each time it receives a new training instance $(\boldsymbol{x}^{(i)}, y^{(i)})$

$$\theta_j \leftarrow \theta_j - \frac{\alpha}{2} \underbrace{\left(h_{\boldsymbol{\theta}}\left(\boldsymbol{x}^{(i)}\right) - y^{(i)}\right)}_{\text{either 2 or -2}} x_j^{(i)}$$

  - If the prediction matches the label, make no change
  - Otherwise, adjust $\boldsymbol{\theta}$

The Perceptron

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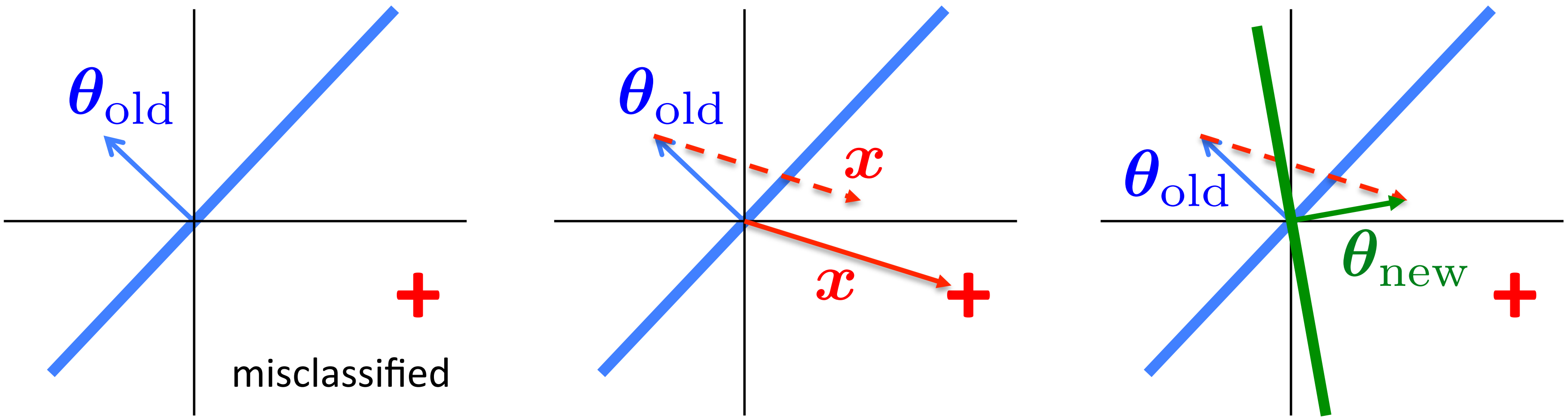
either 2 or -2

- Re-write as $\theta_j \leftarrow \theta_j + \alpha y^{(i)} x_j^{(i)}$ (only upon misclassification)
  – Can eliminate α in this case, since its only effect is to scale $\boldsymbol{\theta}$ by a constant, which doesn't affect performance

Perceptron Rule: If $\boldsymbol{x}^{(i)}$ is misclassified, do $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + y^{(i)}\boldsymbol{x}^{(i)}$

Why the Perceptron Update Works



Why the Perceptron Update Works

- Consider the misclassified example (y = +1)
  - Perceptron wrongly thinks that $\boldsymbol{\theta}_{\text{old}}^\intercal \boldsymbol{x} < 0$
- Update:

$$\boldsymbol{\theta}_{\text{new}} = \boldsymbol{\theta}_{\text{old}} + y\boldsymbol{x} = \boldsymbol{\theta}_{\text{old}} + \boldsymbol{x} \qquad (\text{since } y = +1)$$

- Note that

$$\begin{aligned} \boldsymbol{\theta}_{\text{new}}^\intercal \boldsymbol{x} &= (\boldsymbol{\theta}_{\text{old}} + \boldsymbol{x})^\intercal \boldsymbol{x} \\ &= \boldsymbol{\theta}_{\text{old}}^\intercal \boldsymbol{x} + \boldsymbol{x}^\intercal \boldsymbol{x} \end{aligned}$$

$\|\boldsymbol{x}\|_2^2 > 0$

- Therefore, $\boldsymbol{\theta}_{\text{new}}^\intercal \boldsymbol{x}$ is less negative than $\boldsymbol{\theta}_{\text{old}}^\intercal \boldsymbol{x}$
  - So, we are making ourselves <u>more correct</u> on this example!

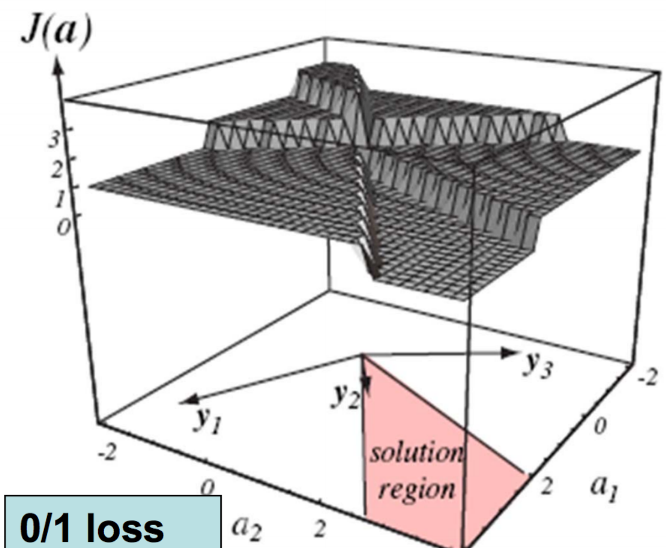
The Perceptron Cost Function

- Prediction is correct if $y^{(i)}\boldsymbol{x}^{(i)}\boldsymbol{\theta} > 0$

- Could have used 0/1 loss

$$J_{0/1}(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n} \ell(sign(x^{(i)}\boldsymbol{\theta}), y^{(i)})$$

where $\ell()$ is 0 if the prediction is correct, 1 otherwise



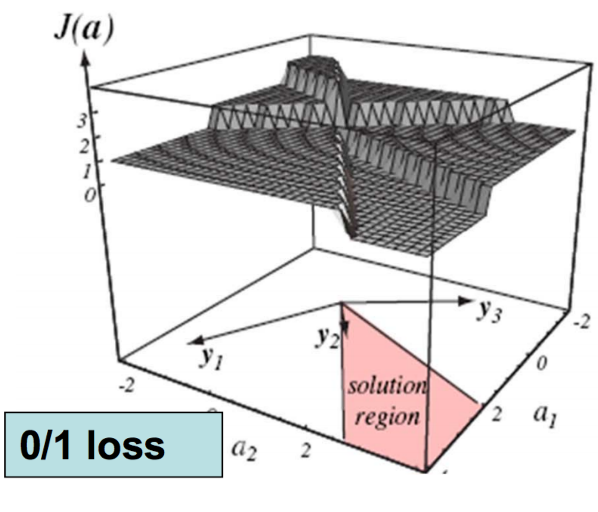
Doesn't produce a useful gradient

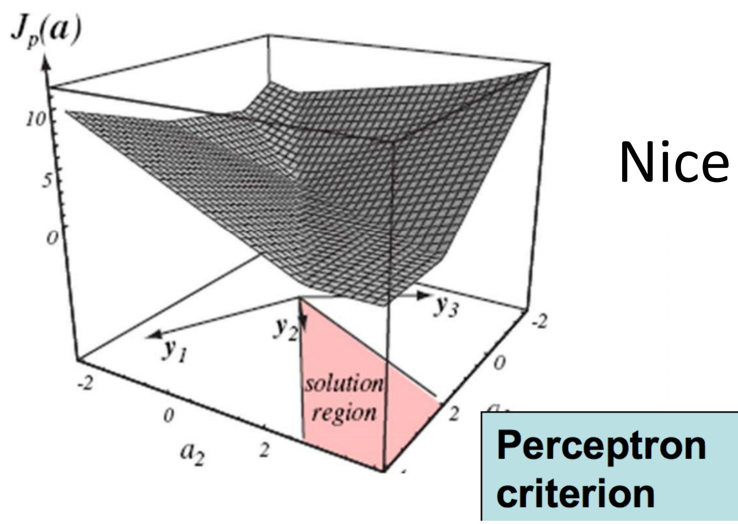
The Perceptron Cost Function

- The perceptron uses the following cost function

$$J_p(\boldsymbol{\theta}) = \frac{1}{n} \sum_{i=1}^{n} \max(0, -y^{(i)} x^{(i)} \boldsymbol{\theta})$$

  - $\max(0, -y^{(i)} x^{(i)} \boldsymbol{\theta})$ is 0 if the prediction is correct
  - Otherwise, it is the confidence in the misprediction





Nice gradient

Online Perceptron Algorithm

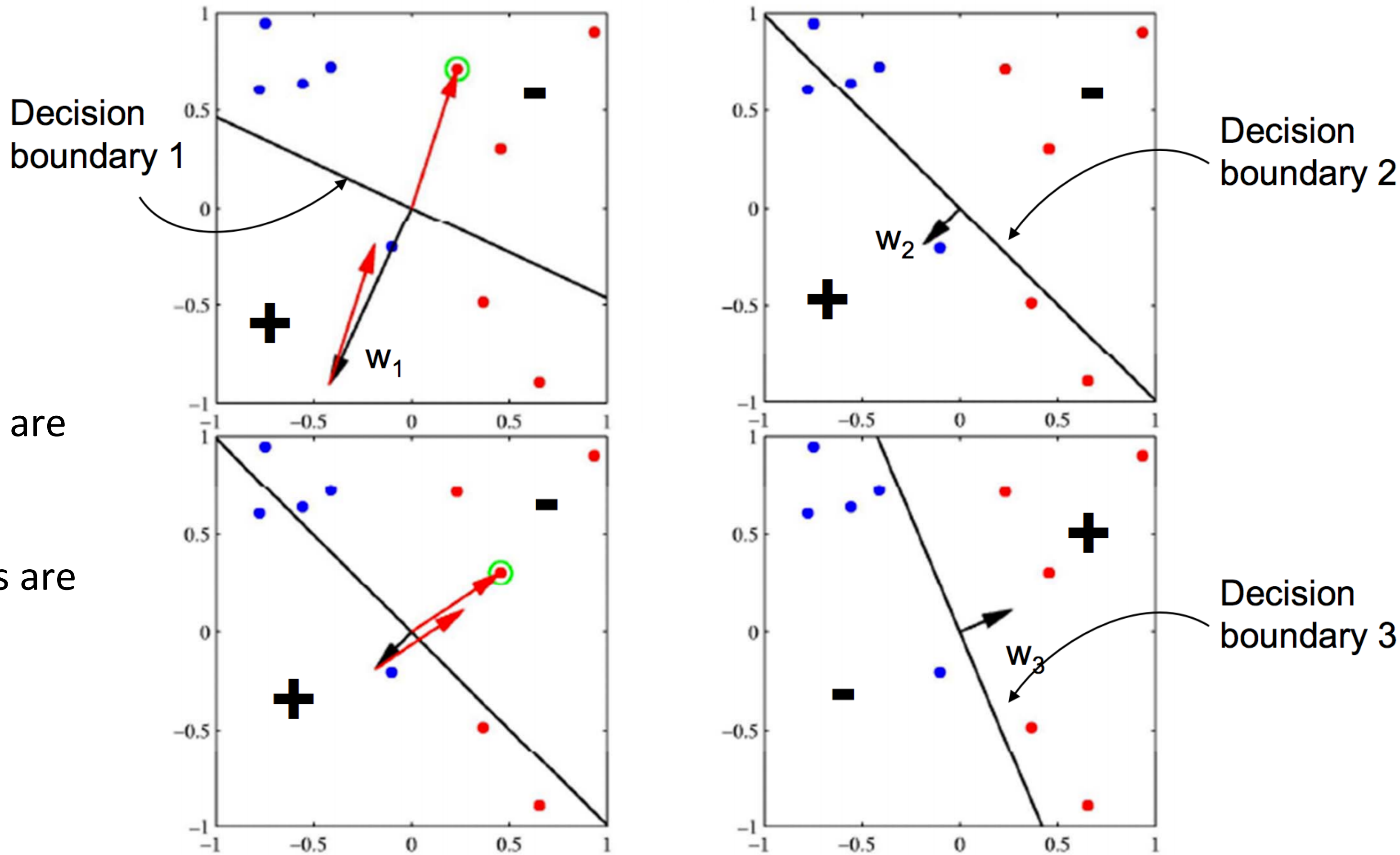
Let $\boldsymbol{\theta} \leftarrow [0, 0, \ldots, 0]$
Repeat:
  Receive training example $(\boldsymbol{x}^{(i)}, y^{(i)})$
  if $y^{(i)}\boldsymbol{x}^{(i)}\boldsymbol{\theta} \leq 0$ // prediction is incorrect
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + y^{(i)}\boldsymbol{x}^{(i)}$

Online learning – the learning mode where the model update is performed each time a single observation is received

Batch learning – the learning mode where the model update is performed after observing the entire training set

Online Perceptron Algorithm

When an error is made, moves the weight in a direction that corrects the error



Red points are labeled +

Blue points are labeled -

See the perceptron in action: www.youtube.com/watch?v=vGwemZhPlsA

Batch Perceptron

Given training data $\left\{(\boldsymbol{x}^{(i)}, y^{(i)})\right\}_{i=1}^{n}$
Let $\boldsymbol{\theta} \leftarrow [0, 0, \ldots, 0]$
Repeat:
    Let $\boldsymbol{\Delta} \leftarrow [0, 0, \ldots, 0]$
    for $i = 1 \ldots n$, do
        if $y^{(i)} \boldsymbol{x}^{(i)} \boldsymbol{\theta} \leq 0$ // prediction for ith instance is incorrect
            $\boldsymbol{\Delta} \leftarrow \boldsymbol{\Delta} + y^{(i)} \boldsymbol{x}^{(i)}$
    $\boldsymbol{\Delta} \leftarrow \boldsymbol{\Delta}/n$ // compute average update
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha \boldsymbol{\Delta}$
Until $\|\boldsymbol{\Delta}\|_2 < \epsilon$

- Simplest case: $\alpha = 1$ and don't normalize, yields the fixed increment perceptron
- Guaranteed to find a separating hyperplane if one exists

Based on slide by Alan Fern

Improving the Perceptron

- The Perceptron produces many $\boldsymbol{\theta}$'s during training
- The standard Perceptron simply uses the final $\boldsymbol{\theta}$ at test time
  - This may sometimes not be a good idea!
  - Some other $\boldsymbol{\theta}$ may be correct on 1,000 consecutive examples, but one mistake ruins it!

- **Idea:** Use a combination of multiple perceptrons
  - (i.e., neural networks!)
- **Idea:** Use the intermediate $\boldsymbol{\theta}$'s
  - **Voted Perceptron**: vote on predictions of the intermediate $\boldsymbol{\theta}$'s
  - **Averaged Perceptron**: average the intermediate $\boldsymbol{\theta}$'s

Based on slide by Piyush Rai