# Fall 2024 CIS 515

# Fundamentals of Linear Algebra and Optimization
# Jean Gallier

## Homework 3

February 28, 2024; Due March 21, 2024

**Problem B1 (10 pts).** Let $f\colon E \to F$ be a linear map which is also a bijection (it is injective and surjective). Prove that the inverse function $f^{-1}\colon F \to E$ is linear.

**Problem B2 (10 pts).** Given two vectors spaces $E$ and $F$, let $(u_i)_{i\in I}$ be any basis of $E$ and let $(v_i)_{i\in I}$ be any family of vectors in $F$. Prove that the unique linear map $f\colon E \to F$ such that $f(u_i) = v_i$ for all $i \in I$ is surjective iff $(v_i)_{i\in I}$ spans $F$.

**Problem B3 (40 pts).** (1) Let $f\colon E \to F$ be a linear map with $\dim(E) = n$ and $\dim(F) = m$. Prove that $f$ has rank 1 iff $f$ is represented by an $m \times n$ matrix of the form

$$A = uv^\top$$

with $u$ a nonzero column vector of dimension $m$ and $v$ a nonzero column vector of dimension $n$.

In the rest of this problem *we assume that* $m = n \geq 1$.

(2) Prove that if $v^\top u \neq 1$, then $M = I - uv^\top$ is invertible and that its inverse is given by

$$M^{-1} = I + (1 - v^\top u)^{-1} uv^\top.$$

(3) Consider the $(n+1) \times (n+1)$ matrix

$$H = \begin{pmatrix} I & u \\ v^\top & 1 \end{pmatrix}.$$

Prove that

$$\begin{pmatrix} I & 0 \\ -v^\top & 1 \end{pmatrix} H = \begin{pmatrix} I & u \\ 0 & 1 - v^\top u \end{pmatrix}.$$

Then prove that

$$\begin{pmatrix} I & u \\ 0 & 1 - v^\top u \end{pmatrix}^{-1} = \begin{pmatrix} I & -u(1 - v^\top u)^{-1} \\ 0 & (1 - v^\top u)^{-1} \end{pmatrix},$$

and that

$$H^{-1} = \begin{pmatrix} I + u(1 - v^\top u)^{-1} v^\top & -u(1 - v^\top u)^{-1} \\ -(1 - v^\top u)^{-1} v^\top & (1 - v^\top u)^{-1} \end{pmatrix}.$$

(4) Prove that

$$\begin{pmatrix} I & -u \\ 0 & 1 \end{pmatrix} H = \begin{pmatrix} I - uv^\top & 0 \\ v^\top & 1 \end{pmatrix}.$$

Then prove that

$$\begin{pmatrix} I - uv^\top & 0 \\ v^\top & 1 \end{pmatrix}^{-1} = \begin{pmatrix} (I - uv^\top)^{-1} & 0 \\ -v^\top (I - uv^\top)^{-1} & 1 \end{pmatrix}$$

and that

$$H^{-1} = \begin{pmatrix} M^{-1} & -M^{-1}u \\ -v^\top M^{-1} & 1 + v^\top M^{-1} u \end{pmatrix},$$

where $M = I - uv^\top$ is the matrix form Part (2).

From the two expressions for $H^{-1}$, deduce again that

$$M^{-1} = I + (1 - v^\top u)^{-1} uv^\top.$$

**Problem B4 (60 pts).** (1) Let $U$ and $V$ be $n \times k$ matrices, with $k \leq n$. We know from HW1, Problem B5, that $I_n - UV^\top$ is invertible iff $I_k - V^\top U$ is invertible. If $I_k - V^\top U$ is invertible, then prove that

$$(I_n - UV^\top)^{-1} = I_n + U(I_k - V^\top U)^{-1} V^\top.$$

If $k$ is a lot smaller than $n$, this formula provides a much cheaper way of computing $(I_n - UV^\top)^{-1}$.

(2) Let $A$ be an invertible $n \times n$ matrix. Again, show that HW1, Problem B5, implies that $A - UV^\top$ is invertible iff $I_k - V^\top A^{-1} U$ is invertible. If $A - UV^\top$ is invertible, prove that

$$(A - UV^\top)^{-1} = A^{-1} + A^{-1} U (I_k - V^\top A^{-1} U)^{-1} V^\top A^{-1}.$$

This is the *Sherman–Morrison–Woodburry formula*.

(3) Prove that the $(n + k) \times (n + k)$ matrix

$$H = \begin{pmatrix} A & U \\ V^\top & I_k \end{pmatrix}$$

is invertible iff the matrix $A - UV^\top$ is invertible.

*Hint*. Examine the nullspaces of these two matrices.

(4) Check that

$$\begin{pmatrix} I_n & 0 \\ -V^\top A^{-1} & I_k \end{pmatrix} H = \begin{pmatrix} A & U \\ 0 & I_k - V^\top A^{-1} U \end{pmatrix}.$$

Also check that

$$\begin{pmatrix} I_n & -U \\ 0 & I_k \end{pmatrix} H = \begin{pmatrix} A - UV^\top & 0 \\ V^\top & I_k \end{pmatrix}.$$

Let $C = I_k - V^\top A^{-1}U$ and $M = A - UV^\top$. Check that

$$\begin{pmatrix} A & U \\ 0 & C \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}UC^{-1} \\ 0 & C^{-1} \end{pmatrix},$$

and that

$$\begin{pmatrix} M & 0 \\ V^\top & I_k \end{pmatrix}^{-1} = \begin{pmatrix} M^{-1} & 0 \\ -V^\top M^{-1} & I_k \end{pmatrix}.$$

Deduce from the above equations that

$$H^{-1} = \begin{pmatrix} A^{-1} + A^{-1}UC^{-1}V^\top A^{-1} & -A^{-1}UC^{-1} \\ -C^{-1}V^\top A^{-1} & C^{-1} \end{pmatrix} = \begin{pmatrix} M^{-1} & -M^{-1}U \\ -V^\top M^{-1} & I_k + V^\top M^{-1}U \end{pmatrix}.$$

Use the above to derive again the formula in (2).

(5) Prove that $UV^\top$ has rank at most $k$. Prove that $UV^\top$ has rank $k$ iff both $U$ and $V$ have rank $k$.

(6) Suppose $M = A - UV^\top$ is invertible. Here is a method to solve the linear system $My = b$ (where $b \in \mathbb{R}^n$) without actually using $M$, but instead using $I_k - V^\top A^{-1}U$, which is a much smaller matrix than $M$ if $k \ll n$.

(1) Let $Z$ be an $n \times k$ matrix with columns $Z^1, \ldots, Z^k$. Solve the system $Ax = b$ ($x \in \mathbb{R}^n$) and the $k$ linear systems $AZ^i = U^i$, where $U^i$ is the $i$th column of $U$ for $i = 1, \ldots, k$, which is equivalent to solving $AZ = U$.

(2) Let $C = I_k - V^\top Z$, and solve the system $Cw = V^\top x$ ($w \in \mathbb{R}^k$).

Note that no matrix inversion is necessary, only Gaussian elimination is needed.

We claim that the solution $y$ ($y \in \mathbb{R}^n$) to the system $My = b$ is

$$y = x + Zw.$$

Prove the above claim by using the equation of Part (2).

**Problem B5 (20 pts).** Prove that for every vector space $E$, if $f\colon E \to E$ is an idempotent linear map, i.e., $f \circ f = f$, then we have a direct sum

$$E = \operatorname{Ker} f \oplus \operatorname{Im} f,$$

so that $f$ is the projection onto its image $\operatorname{Im} f$.

**Problem B6 (40 pts).** Given any vector space $E$, a linear map $f\colon E \to E$ is an *involution* if $f \circ f = \mathrm{id}$.

(1) Prove that an involution $f$ is invertible. What is its inverse?

(2) Let $E_1$ and $E_{-1}$ be the subspaces of $E$ defined as follows:

$$\begin{aligned} E_1 &= \{u \in E \mid f(u) = u\} \\ E_{-1} &= \{u \in E \mid f(u) = -u\}. \end{aligned}$$

Prove that we have a direct sum

$$E = E_1 \oplus E_{-1}.$$

*Hint*. For every $u \in E$, write

$$u = \frac{u + f(u)}{2} + \frac{u - f(u)}{2}.$$

(3) If $E$ is finite-dimensional and $f$ is an involution, prove that there is some basis of $E$ with respect to which the matrix of $f$ is of the form

$$I_{k,n-k} = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix},$$

where $I_k$ is the $k \times k$ identity matrix (similarly for $I_{n-k}$) and $k = \dim(E_1)$. Can you give a geometric interpretation of the action of $f$ (especially when $k = n - 1$)?

**Problem B7 (60 pts).** Let $E$ be a real vector space of dimension $n \geq 2$ and let $F$ be any real vector space. Pick any basis $(u_1, \ldots, u_n)$ in $E$.

(1) Prove that for any bilinear alternating map $f\colon E \times E \to F$, for any two vectors $x = x_1 u_1 + \cdots + x_n u_n$ and $y = y_1 u_1 + \cdots + y_n u_n$, we have

$$f(x, y) = \sum_{1 \leq i < j \leq n} (x_i y_j - x_j y_i) f(u_i, u_j).$$

Observe that

$$x_i y_j - x_j y_i = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

is the determinant obtained from the $2 \times n$ matrix

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

by choosing two columns of index $i < j$ among the $n$ columns.

*Hint*. Let $v = x_2 u_2 + \cdots + x_n u_n$ and $w = y_2 u_2 + \cdots + y_n u_n$. First prove that

$$\begin{aligned} f(x, y) = (x_1 y_2 - x_2 y_1) f(u_1, u_2) + (x_1 y_3 - x_3 y_1) f(u_1, u_3) + \cdots + (x_1 y_n - x_n y_1) f(u_1, u_n) \\ + f(v, w). \end{aligned}$$

Then use induction.

(2) Prove that for any sequence $(w_{ij})_{1 \leq i < j \leq n}$ of $\binom{n}{2} = n(n-1)/2$ vectors $w_{ij} \in F$, there is a unique bilinear alternating map $f\colon E \times E \to F$ such that

$$f(u_i, u_j) = w_{ij}, \quad 1 \leq i < j \leq n,$$

and in fact,

$$f(x, y) = \sum_{1 \leq i < j \leq n} (x_i y_j - x_j y_i) w_{ij}.$$

Conclude that there is a bijection $\varphi$ between the set $\mathrm{Alt}^2(E; F)$ of bilinear alternating maps $f\colon E \times E \to F$ and the product vector space $F^{n(n-1)/2}$ given by

$$\varphi(f) = (f(u_i, u_j))_{1 \leq i < j \leq n}.$$

**Remark**. Observe that when $F = \mathbb{R}$, if we let $A$ be the $n \times n$ matrix given by $A = (f(e_i, e_j))$ and if we let $X$ be the column vector with entries $(x_1, \ldots, x_n)$ and $Y$ be the column vector with entries $(y_1, \ldots, y_n)$, then $A^\top = -A$ and $f(x, y) = X^\top A Y$.

(3) We define addition and scalar multiplication on the set of bilinear alternating maps as follows. For any two bilinear alternating maps $f\colon E \times E \to F$ and $g\colon E \times E \to F$, for all $x, y \in E$ and all $\lambda \in \mathbb{R}$,

$$(f + g)(x, y) = f(x, y) + g(x, y),$$

and

$$(\lambda f)(x, y) = \lambda f(x, y).$$

Check (quickly) that $f + g$ and $\lambda f$ are bilinear and alternating, and that the set $\mathrm{Alt}^2(E; F)$ of bilinear alternating maps with the above addition and scalar multiplication is a real vector space.

(4) Prove that the bijection $\varphi\colon \mathrm{Alt}^2(E; F) \to F^{n(n-1)/2}$ in (2) given by

$$\varphi(f) = (f(u_i, u_j))_{1 \leq i < j \leq n}$$

is linear. Conclude that $\varphi$ is an isomorphism of vector spaces, and that if $F$ has dimension $m$, then $\mathrm{Alt}^2(E; F)$ has dimension $mn(n-1)/2$.

**Extra Credit (50 pts).**

(5) Let $p$ be an integer such that $1 \leq p \leq n$. Consider the set $\mathrm{Alt}^p(E; F)$ of multilinear alternating maps $f\colon E^p \to F$. Prove that for any vectors $x_1, \ldots, x_p \in E$, if

$$x_i = x_{i1} u_1 + \cdots + x_{in} u_n, \quad i = 1, \ldots, p,$$

then

$$f(x_1, \ldots, x_p) = \sum_{1 \leq j_1 < j_2 < \cdots < j_p \leq n} \Delta_{j_1, j_2, \ldots, j_p}(x_1, \ldots, x_p) f(u_{j_1}, u_{j_2}, \ldots, u_{j_p}),$$

where $\Delta_{j_1,j_2,\ldots,j_p}(x_1,\ldots,x_p)$ is the determinant (of a $p\times p$ matrix)

$$\Delta_{j_1,j_2,\ldots,j_p}(x_1,\ldots,x_p) = \begin{vmatrix} x_{1j_1} & x_{1j_2} & \cdots & x_{1j_p} \\ x_{2j_1} & x_{2j_2} & \cdots & x_{2j_p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{pj_1} & x_{pj_2} & \cdots & x_{pj_p} \end{vmatrix}.$$

Observe that the above determinant is obtained from the $p\times n$ matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pn} \end{pmatrix},$$

by choosing the columns of index $j_1, j_2, \ldots, j_p$ among the $n$ columns.

*Hint*. First observe that

$$f(x_1,\ldots,x_p) = \sum_{(j_1,\ldots,j_p)\in\{1,\ldots,n\}^{\{1,\ldots,p\}}} x_{1j_1}\cdots x_{pj_p} f(u_{j_1},\ldots,u_{j_p}),$$

where the sum extends over all sequences $(j_1,\ldots,j_p)$ of length $p$ of elements from $\{1,\ldots,n\}$.

You will also need the fact that the notion of signature of a permutation, which was defined for permutations of the set $\{1,\ldots,n\}$, is defined in a similar way for permutations of the set $\{j_1,\ldots,j_p\}$, with $1\leq j_1<\cdots<j_p\leq n$.

(6) Give $\mathrm{Alt}^p(E;F)$ the structure of a vector space as in (3). Prove that the map $\varphi\colon \mathrm{Alt}^p(E;F)\to F^{\binom{n}{p}}$ given by

$$\varphi(f) = \big(f(u_{j_1},u_{j_2},\ldots,u_{j_p})\big)_{1\leq j_1<j_2<\cdots<j_p\leq n}$$

is an isomorphism of vector spaces.

What more can you say when $p=n$? What is the dimension of $\mathrm{Alt}^n(E;F)$?

Suppose $F=\mathbb{R}$. Prove that the dimension of $\mathrm{Alt}^p(E;\mathbb{R})$ is $\binom{n}{p}$ (recall that $1\leq p\leq n$). What is the dimension of $\mathrm{Alt}^n(E;\mathbb{R})$?

(7) Prove that for $p>n$, every multilinear alternating map $f\colon E^p\to F$ is the zero map.

**TOTAL: 240 + 50 points.**