7.5 Correlation on the Space of Feature Fields $\mathrm{L}^2(\mathbb{R}^d, \widehat{H})$

A typical CNN consists of layers, starting with a lifting layer followed by group correlation layers (often called group convolution layers).

The last layer is typically a projection layer involving some pooling process.

This is a simpler process that we will not discuss here.

The lifting layer takes as input a function $f_{\mathrm{in}} \in \mathrm{L}^2(\mathbb{R}^d)$ and produces an output function $f_{\mathrm{out}} \in \mathrm{L}^2(\mathbb{R}^d \rtimes H)$ given by a lifted correlation, with

$$f_{\mathrm{out}}(x, h) = (k \,\widetilde{\star}\, f_{\mathrm{in}})(x, h),$$

where

$$(k \,\widetilde{\star}\, f_{\mathrm{in}})(x, h) = \int_{\mathbb{R}^d} f_{\mathrm{in}}(t) k(h^{-1} \cdot (t - x)) \, dt, \; (x, h) \in \mathbb{R}^d \times H.$$

Computing $(k \mathbin{\widetilde{\star}} f_{\text{in}})(x, h)$ requires discretizing the group H, which is not possible in practice if $d > 2$.

If the kernel k can be expressed in terms of an H-steerable family Y of L functions in $\mathrm{L}^2(\mathbb{R}^d)$ and a representation $\Sigma\colon H \to \mathbf{U}(L)$, then $f_{\text{out}}(x, h)$ can be computed a lot cheaply in terms of a feature field $\widehat{f_{\text{out}}}\colon \mathbb{R}^d \to \mathrm{M}_L(\mathbb{C})$ defined from f_{in} and Y as

$$f_{\text{out}}(x, h) = (k \mathbin{\widetilde{\star}} f_{\text{in}})(x, h) = \operatorname{tr}\left(\widehat{f_{\text{out}}}(x)\,\Sigma(h)^{\top}\right),$$

where $\widehat{f_{\text{out}}}(x)$ is a matrix of Fourier coefficients.

A group correlation layer takes as input a function $f_{\text{in}} \in \mathrm{L}^2(\mathbb{R}^d \rtimes H)$ and produces as output a function $f_{\text{out}} \in \mathrm{L}^2(\mathbb{R}^d \rtimes H)$ using a group correlation

$$\begin{aligned} f_{\text{out}}(s) &= (k \star f_{\text{in}})(s) \\ &= \int_G f_{\text{in}}(t) k(s^{-1}t)\, d\lambda_G(t), \quad s \in G = \mathbb{R}^d \rtimes H. \end{aligned}$$

We saw in the previous section that a G-feature map $f \in \mathrm{L}^2(\mathbb{R}^d \rtimes H)$ yields a family $\widehat{f} = (\widehat{f}_\rho)_{\rho \in R(H)}$ of feature fields $\widehat{f}_\rho \in \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_\rho$ and that f can be recovered pointwise by Fourier inversion, namely

$$f(x,h) = \sum_{\rho \in R(H)} n_\rho \operatorname{tr}\left(\widehat{f}_\rho(x) M_\rho(h)\right).$$

We know how to transform G-feature maps using group correlation.

This defines a transform Φ on $\mathrm{L}^2(G)$ (where $G = \mathbb{R}^d \rtimes H$) given by $f_{\text{out}} = \Phi(f_{\text{in}}) = k \star f_{\text{in}}$.

We can summarize the situation by the following diagram:

$$\begin{tikzcd}
\mathrm{L}^2(G) \arrow[r, "\Phi"] \arrow[d, shift right=2ex, "\mathcal{F}^{\mathcal{T}}"'] & \mathrm{L}^2(G) \arrow[d, shift right=2ex, "\mathcal{F}^{\mathcal{T}}"'] \\
\mathrm{L}^2(\mathbb{R}^d, \widehat{H}) \arrow[r, dotted, "?"'] \arrow[u, shift right=2ex, "\overline{\mathcal{F}^{\mathcal{T}}}"'] & \mathrm{L}^2(\mathbb{R}^d, \widehat{H}). \arrow[u, shift right=2ex, "\overline{\mathcal{F}^{\mathcal{T}}}"']
\end{tikzcd}$$

Since it is too expensive to compute $\Phi(f_{\text{in}}) = k \star f_{\text{in}}$, it would be nice if we could define the missing map, a *notion of correlation*

$$\widehat{\Phi}\colon \mathrm{L}^2(\mathbb{R}^d, \widehat{H}) \to \mathrm{L}^2(\mathbb{R}^d, \widehat{H})$$

on feature fields, and then we would recover $k \star f_{\text{in}}$ by Fourier inversion.

In theory this is possible; we simply define $\widehat{\Phi}$ as

$$\widehat{\Phi} = \mathcal{F}^{\tau} \circ \Phi \circ \overline{\mathcal{F}^{\tau}},$$

using $\mathcal{F}^{\tau}$ and $\overline{\mathcal{F}^{\tau}}$.

We can push this approach further using the fact that the Fourier transform $\mathcal{F}^{\tau}_{\rho}$ is continuous and that Φ is a continuous linear map.

The following proposition is needed.

Proposition 7.5. *If E and F are two normed vector spaces and if $\Phi\colon E \to F$ is a continuous linear map, then the following properties hold:*

(1) For any convergent series $\sum_{n=1}^{\infty} u_n$ (with $u_n \in E$), the series $\sum_{n=1}^{\infty} \Phi(u_n)$ converges in F and

$$\Phi\left(\sum_{n=1}^{\infty} u_n\right) = \sum_{n=1}^{\infty} \Phi(u_n).$$

(2) For any countable index set Λ, for any summable series $\sum_{\ell\in\Lambda} u_\ell$ (with $u_\ell \in E$), the series $\sum_{\ell\in\Lambda} \Phi(u_\ell)$ is summable in F and

$$\Phi\left(\sum_{\ell\in\Lambda} u_\ell\right) = \sum_{\ell\in\Lambda} \Phi(u_\ell).$$

See Vol I, Definition @@@D.6, for the definition of a summable series.

Then for any family $(\widehat{f}_{\rho_1})_{\rho_1 \in R(H)}$ of feature fields in $\mathrm{L}^2(\mathbb{R}^d, \widehat{H})$, we have

$$
\begin{aligned}
&\widehat{\Phi}((\widehat{f}_{\rho_1})_{\rho_1 \in R(H)}) \\
&= \mathcal{F}^\tau\left(\sum_{\rho_1 \in R(H)} \Phi(\overline{\mathcal{F}^\tau}_{\rho_1}(\widehat{f}_{\rho_1}))\right) && \text{by Proposition 7.5 for } \Phi \\
&= \left(\mathcal{F}^\tau_{\rho_2}\left(\sum_{\rho_1 \in R(H)} \Phi(\overline{\mathcal{F}^\tau}_{\rho_1}(\widehat{f}_{\rho_1})\right)\right)_{\rho_2 \in R(H)} && \text{by definition of } \mathcal{F}^\tau \\
&= \left(\sum_{\rho_1 \in R(H)} \mathcal{F}^\tau_{\rho_2}(\Phi(\overline{\mathcal{F}^\tau}_{\rho_1}(\widehat{f}_{\rho_1})))\right)_{\rho_2 \in R(H)} && \text{by Proposition 7.5 for } \mathcal{F}
\end{aligned}
$$

Define $\widehat{\Phi}_{\rho_1}$ and $\widehat{\Phi}_{\rho_2,\rho_1}$ as

$$\widehat{\Phi}_{\rho_2,\rho_1}(\widehat{f}_{\rho_1}) = \mathcal{F}^{\tau}_{\rho_2}(\Phi(\overline{\mathcal{F}^{\tau}}_{\rho_1}(\widehat{f}_{\rho_1}))) \qquad (\widehat{\Phi}_{\rho_2,\rho_1})$$

$$\widehat{\Phi}_{\rho_1}(\widehat{f}_{\rho_1}) = \sum_{\rho_1 \in R(H)} \widehat{\Phi}_{\rho_2,\rho_1}(\widehat{f}_{\rho_1}), \qquad (\widehat{\Phi}_{\rho_1})$$

so that

$$\widehat{\Phi}((\widehat{f}_{\rho_1})_{\rho_1 \in R(H)}) = \big(\widehat{\Phi}_{\rho_1}(\widehat{f}_{\rho_1})\big)_{\rho_2 \in R(H)}. \qquad (\widehat{\Phi})$$

It is an interesting and useful fact that the transforms $\widehat{\Phi}_{\rho_2,\rho_1}$ are equivariant with respect to the represention $\mathrm{Ind}_H^G \sigma_{\rho_1}$ and $\mathrm{Ind}_H^G \sigma_{\rho_2}$.

Consider the diagram

$$\begin{tikzcd}
\mathrm{L}^2(\mathbb{R}^d, \widehat{H})_{\rho_1} \arrow[r, "\overline{\mathcal{F}^\tau}_{\rho_1}"] \arrow[d, "(\mathrm{Ind}_H^G \sigma_{\rho_1})_{(x,h)}"'] & \mathrm{L}^2(G) \arrow[r, "\Phi"] \arrow[d, "\mathbf{R}_{(x,h)}"'] & \mathrm{L}^2(G) \arrow[r, "\mathcal{F}^\tau_{\rho_2}"] \arrow[d, "\mathbf{R}_{(x,h)}"] & \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_{\rho_2} \arrow[d, "(\mathrm{Ind}_H^G \sigma_{\rho_2})_{(x,h)}"] \\
\mathrm{L}^2(\mathbb{R}^d, \widehat{H})_{\rho_1} \arrow[r, "\overline{\mathcal{F}^\tau}_{\rho_1}"'] & \mathrm{L}^2(G) \arrow[r, "\Phi"'] & \mathrm{L}^2(G) \arrow[r, "\mathcal{F}^\tau_{\rho_2}"'] & \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_{\rho_2}.
\end{tikzcd}$$

Since the three squares commute, the outer square also commutes, so we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{L}^2(\mathbb{R}^d, \widehat{H})_{\rho_1} \arrow[r, "\widehat{\Phi}_{\rho_2,\rho_1}"] \arrow[d, "(\mathrm{Ind}_H^G \sigma_{\rho_1})_{(x,h)}"'] & \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_{\rho_2} \arrow[d, "(\mathrm{Ind}_H^G \sigma_{\rho_2})_{(x,h)}"] \\
\mathrm{L}^2(\mathbb{R}^d, \widehat{H})_{\rho_1} \arrow[r, "\widehat{\Phi}_{\rho_2,\rho_1}"'] & \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_{\rho_2},
\end{tikzcd}$$

which shows that $\widehat{\Phi}_{\rho_2,\rho_1}$ is equivariant with respect to the representations $\mathrm{Ind}_H^G \sigma_{\rho_1}$ and $\mathrm{Ind}_H^G \sigma_{\rho_2}$.

Suppose the group correlation $\Phi\colon \mathrm{L}^2(G) \to \mathrm{L}^2(G)$ is given by a kernel k as

$$\Phi(f)(x,h) = \int_{\mathbb{R}^d \rtimes H} k(h^{-1}\cdot(x_1 - x), h^{-1}h_1) \\ f(x_1,h_1)\, d\lambda_H(h_1)\, dx_1.$$

Since

$$[\overline{\mathcal{F}^\tau}_{\rho_1}(\widehat{f}_{\rho_1})](x_1,h_1) = n_{\rho_1}\mathrm{tr}\Big(\widehat{f}_{\rho_1}(x_1) M_{\rho_1}(h_1)\Big),$$

we have

$$\Phi(\overline{\mathcal{F}^\tau}_{\rho_1}(\widehat{f}_{\rho_1}))(x,h) = \int_{\mathbb{R}^d \rtimes H} k(h^{-1}\cdot(x_1 - x), h^{-1}h_1) \\ n_{\rho_1}\mathrm{tr}\Big(\widehat{f}_{\rho_1}(x_1) M_{\rho_1}(h_1)\Big)\, d\lambda_H(h_1)\, dx_1,$$

and then using Fubini we have

$$
\begin{aligned}
&\mathcal{F}^{\tau}_{\rho_2}[\Phi(\overline{\mathcal{F}^{\tau}}_{\rho_1}(\widehat{f}_{\rho_1}))](x)\\
&= \int_H \int_{\mathbb{R}^d \rtimes H} k(h^{-1}\cdot(x_1 - x), h^{-1}h_1)\\
&\qquad n_{\rho_1}\mathrm{tr}\Big(\widehat{f}_{\rho_1}(x_1)M_{\rho_1}(h_1)\Big)\, d\lambda_H(h_1)\, dx_1\, M_{\rho_2}(h)^*\, d\lambda_H(h)\\
&= \int_{\mathbb{R}^d}\int_H\int_H n_{\rho_1}\mathrm{tr}\Big(\widehat{f}_{\rho_1}(x_1)M_{\rho_1}(h_1)\Big)\\
&\qquad k(h^{-1}\cdot(x_1 - x), h^{-1}h_1) M_{\rho_2}(h)^*\, d\lambda_H(h)\, d\lambda_H(h_1)\, dx_1.
\end{aligned}
$$

This suggests defining

$\Phi_{\rho_2,\rho_1} \colon \mathbb{R}^d \times \mathrm{M}_{n_{\rho_1}}(\mathbb{C}) \to \mathrm{M}_{n_{\rho_2}}(\mathbb{C})$ by

$$
\begin{aligned}
&\Phi_{\rho_2,\rho_1}(x_1 - x, A)\\
&\qquad = \int_H\int_H n_{\rho_1}\mathrm{tr}\Big(AM_{\rho_1}(h_1)\Big) k(h^{-1}\cdot(x_1 - x), h^{-1}h_1)\\
&\qquad\qquad M_{\rho_2}(h)^*\, d\lambda_H(h)\, d\lambda_H(h_1),
\end{aligned}
\tag{Φ_{ρ_2,ρ_1}}
$$

where $A \in \mathrm{M}_{n_{\rho_1}}(\mathbb{C})$, so that

$$\begin{aligned}[\widehat{\Phi}_{\rho_2,\rho_1}(\widehat{f}_{\rho_1})](x) &= \mathcal{F}^{\tau}_{\rho_2}[\Phi(\overline{\mathcal{F}^{\tau}}_{\rho_1}(\widehat{f}_{\rho_1}))](x) \\ &= \int_{\mathbb{R}^d} \Phi_{\rho_2,\rho_1}(x_1 - x, \widehat{f}_{\rho_1}(x_1))\, dx_1. \end{aligned} \tag{$\widehat{\Phi}^{\mathrm{bis}}_{\rho_2,\rho_1}$}$$

In order to go further we need to express the kernel $\Phi_{\rho_2,\rho_1}(x, A)$ in terms of H-steerable functions on $\mathrm{L}^2(\mathbb{R}^d \rtimes H)$.

Next we show how to proceed with $H = \mathbf{SO}(d)$.

By (f^ρ) and (str17) in Example 7.6, the Hilbert space $\mathrm{L}^2(\mathbf{SE}(d))$ has a Hilbert basis consisting of functions of the form

$$\left(\overline{m^{(\rho)}_{k_\rho \ell_\rho}(h_1)}\, w_{\rho,k_\rho,\ell_\rho}(h_1^{-1}x)\right)_{1\leq k_\rho,\ell_\rho\leq n_\rho,\, \rho\in R(\mathbf{SO}(d))}, \quad \text{(str20)}$$

with $h_1 \in \mathbf{SO}(d)$ and $x \in \mathbb{R}^d$, where $w_{\rho,k_\rho,\ell_\rho}$ is the sum of a series in the functions

$$e^{-\frac{\|x\|^2}{2}} H_{k_1}(x_1)\cdots H_{k_n}(x_d). \quad \text{(str21)}$$

Thus the kernel $k(x_1, h_1)$ can be expressed as the sum of a series

$$k(x_1, h_1) = \sum_{1\leq k_\rho,\ell_\rho\leq n_\rho,\, \rho\in R(\mathbf{SO}(d))} \overline{m^{(\rho)}_{k_\rho \ell_\rho}(h_1)}\, w_{\rho,k_\rho,\ell_\rho}(h_1^{-1}x_1). \quad (k(x_1, h_1))$$

The result to be presented next makes use of the $n_{\rho_2} \times n_{\rho_2}$ matrix $W_{\rho_2}(x_1)$ *whose* $(k_{\rho_2}, \ell_{\rho_2})$ *entry is* $w_{\rho_2,k_{\rho_2},\ell_{\rho_2}}(x_1)$.

We need to find an expession for $k(h^{-1}(x_1 - x), h^{-1}h_1)$.

After some computations(!) we get

$$\Phi_{\rho_2,\rho_1}(x_1 - x, A) = \sum_{\substack{1 \le k_\rho, \ell_\rho, j_\rho \le n_\rho \\ \rho \in R(\mathbf{SO}(d))}} \int_H \int_H n_{\rho_1} \operatorname{tr}\Big(A M_{\rho_1}(h_1)\Big)(1/n_\rho)\, \overline{m^{(\rho)}_{j_\rho \ell_\rho}(h_1)}$$
$$w_{\rho,k_\rho,\ell_\rho}(h_1^{-1}(x_1 - x))\, m^{(\rho)}_{j_\rho k_\rho}(h)\, M_{\rho_2}(h)^*\, d\lambda_H(h)\, d\lambda_H(h_1).$$

Since the functions $m^{(\rho)}_{j_\rho k_\rho}$ and $m^{(\rho_2)}_{j_{\rho_2} k_{\rho_2}}$ are orthogonal for $\rho \neq \rho_2$ by Theorem 4.4(1), only the terms for which $\rho = \rho_2$ survive, so we get

$$\begin{aligned}
&\Phi_{\rho_2,\rho_1}(x_1 - x, A) = \\
&\quad \sum_{1 \leq k_{\rho_2}, \ell_{\rho_2}, j_{\rho_2} \leq n_{\rho_2}} \int_H n_{\rho_1} \operatorname{tr}\Big(A M_{\rho_1}(h_1)\Big) \overline{m^{(\rho_2)}_{j_{\rho_2} \ell_{\rho_2}}(h_1)} \\
&\qquad w_{\rho_2, k_{\rho_2}, \ell_{\rho_2}}(h_1^{-1}(x_1 - x)) \\
&\qquad \int_H (1/n_{\rho_2})\, m^{(\rho_2)}_{j_{\rho_2} k_{\rho_2}}(h)\, M_{\rho_2}(h)^* \, d\lambda_H(h)\, d\lambda_H(h_1).
\end{aligned}$$

Now the $(k'_{\rho_2}, j'_{\rho_2})$-entry in the matrix $M_{\rho_2}(h)^*$ is $\overline{m^{(\rho_2)}_{j'_{\rho_2}k'_{\rho_2}}(h)}$, and since by Theorem 4.4(1,3) the functions $m^{(\rho_2)}_{j'_{\rho_2}k'_{\rho_2}}$ and $m^{(\rho_2)}_{j_{\rho_2}k_{\rho_2}}$ are orthogonal unless $k'_{\rho_2} = k_{\rho_2}$ and $j'_{\rho_2} = j_{\rho_2}$, in which case $\langle m^{(\rho_2)}_{j_{\rho_2}k_{\rho_2}}, m^{(\rho_2)}_{j_{\rho_2}k_{\rho_2}}\rangle = n_{\rho_2}$, the inner integral evaluates to

$$\int_H (1/n_{\rho_2})\, m^{(\rho_2)}_{j_{\rho_2}k_{\rho_2}}(h)\, M_{\rho_2}(h)^*\, d\lambda_H(h) = E_{k_{\rho_2}j_{\rho_2}},$$

the matrix with 1 in the (k_{ρ_2}, j_{ρ_2}) entry and 0 otherwise, so (after some computation!) we get

$$\Phi_{\rho_2,\rho_1}(x_1 - x, A) = \int_H n_{\rho_1} \operatorname{tr}\Big(AM_{\rho_1}(h_1)\Big) \\ W_{\rho_2}(h_1^{-1}(x_1 - x))M^*_{\rho_2}(h_1)\, d\lambda_H(h_1), \tag{$*_{\Phi_{\rho_2,\rho_1}}$}$$

and

$$\begin{aligned}[\widehat{\Phi}_{\rho_2,\rho_1}(\widehat{f}_{\rho_1})](x) &= \mathcal{F}^{\tau}_{\rho_2}[\Phi(\overline{\mathcal{F}^{\tau}}_{\rho_1}(\widehat{f}_{\rho_1}))](x) \\ &= \int_{\mathbb{R}^d} \Phi_{\rho_2,\rho_1}(x_1 - x, \widehat{f}_{\rho_1}(x_1))\, dx_1,\end{aligned} \tag{$*_{\widehat{\Phi}_{\rho_2,\rho_1}}$}$$

where $W_{\rho_2}(x_1)$ is the $n_{\rho_2} \times n_{\rho_2}$ matrix whose $(k_{\rho_2}, \ell_{\rho_2})$ entry is $w_{\rho_2,k_{\rho_2},\ell_{\rho_2}}(x_1)$ introduced just after $(k(x_1, h_1))$.

It is not hard to show that the above results can be generalized to the situation where H is a compact matrix group acting on $\mathbb{R}^d$ by multiplication.

In the special case where $d = 2$ and $H = \mathbf{SO}(2)$ (*harmonic nets*) we can use polar coordinates and view the functions in $\mathrm{L}^2(\mathbf{SE}(2))$ as functions $f((\|x\|, \alpha), \theta)$. In this case, by (str14) from Example 7.4, a Hilbert basis consists of the functions of the form

$$e^{-im\theta} e^{ik(\theta - \alpha_x)} w_{m,k}(\|x\|), \quad m, k \in \mathbb{Z}.$$

In this special case $\ell_\rho = \underline{\rho \in \mathbb{Z},}$ there is no index k_ρ since $n_\rho = 1$, $h_1 = e^{i\theta'}$, $m^{(\rho)}_{k_\rho \ell_\rho}(h_1) = e^{-i\rho\theta'}$, $M_\rho(\theta') = e^{i\rho\theta'}$, $m = \rho_2$, and by (str20) and (str21) the matrix $W_{\rho_2}(\|x_1 - x\|, \alpha_{x_1-x} - \theta')$ consists of the series

$$\sum_{k=\infty}^{\infty} e^{-ik(\theta'-\alpha_{x_1-x})} w_{\rho_2,k}(\|x_1 - x\|).$$

It follows that we need to evaluate the integral $(*_{\Phi_{\rho_2,\rho_1}})$;

$$\begin{aligned}
&\sum_{k=\infty}^{\infty} \int_{\mathbf{SO}(2)} e^{i\rho_1\theta'} e^{-ik(\theta'-\alpha_{x_1-x})} w_{\rho_2,k}(\|x_1 - x\|) e^{-i\rho_2\theta'}\, d\theta' \\
&\quad = \sum_{k=\infty}^{\infty} e^{ik\alpha_{x_1-x}} w_{\rho_2,k}(\|x_1 - x\|) \int_{\mathbf{SO}(2)} e^{i(\rho_1-\rho_2-k)\theta'}\, d\theta' \\
&= e^{-i(\rho_2-\rho_1)\alpha_{x_1-x}} w_{\rho_2,\rho_1-\rho_2}(\|x_1 - x\|).
\end{aligned}$$

In conclusion we obtain the kernel

$$\Phi_{\rho_2,\rho_1}(x_1 - x, A) = Ae^{-i(\rho_2-\rho_1)\alpha_{x_1-x}} w_{\rho_2,\rho_1-\rho_2}(\|x_1 - x\|).$$

Since this is a scalar kernel that simply multiplies by A, we can express it as

$$\Phi_{\rho_2,\rho_1}(x_1 - x) = e^{-i(\rho_2-\rho_1)\alpha_{x_1-x}} w_{\rho_2,\rho_1-\rho_2}(\|x_1 - x\|).$$

We derive this formula in full detail in the next section on *harmonic nets*.

The second index $\rho_1 - \rho_2$ is different from what we get in the next section because the computation makes use of polar coordinates early on.

If we index $w_{m,k}$ as $w_{m,k+m}$ we find the same term $w_{\rho_2,\rho_1}(\|x_1 - x\|)$.

7.6 Equivariant Correlation G-Kernels When $G = \mathbb{R}^d \rtimes H$

In Section 7.5 we solved the problem of finding a notion of equivariant group correlation for *feature fields* $\widehat{f}_\rho \in \mathrm{L}^2(\mathbb{R}^d, \widehat{H})$, which are functions $\widehat{f}_\rho \colon \mathbb{R}^d \to \mathrm{M}_{n_\rho}(\mathbb{C})$ that transform under the representation $\sigma_\rho \colon H \to \mathbf{U}(\mathrm{M}_{n_\rho}(\mathbb{C}))$, with $\sigma_\rho = \mathrm{Hom}(M_\rho, \mathrm{id})$ (see Proposition 7.3).

For this we used the Fourier transform $\mathcal{F}^\tau$ and the Fourier cotransform $\overline{\mathcal{F}^\tau}$ defined in Section 7.4.

Recall that given a correlation kernel k on $\mathrm{L}^2(G)$ we have the group correlation Φ on $\mathrm{L}^2(G)$ (where $G = \mathbb{R}^d \rtimes H$) given by $f_{\mathrm{out}} = \Phi(f_{\mathrm{in}}) = k \star f_{\mathrm{in}}$.

The correlation $\widehat{\Phi}$ on feature fields in $\mathrm{L}^2(\mathbb{R}^d, \widehat{H})$ is the map that makes the following diagram commute:

$$\begin{tikzcd}
\mathrm{L}^2(G) \arrow[r, "\Phi"] \arrow[d, shift right, "\mathcal{F}^\tau"'] & \mathrm{L}^2(G) \arrow[d, shift right, "\mathcal{F}^\tau"'] \\
\mathrm{L}^2(\mathbb{R}^d, \widehat{H}) \arrow[r, "\widehat{\Phi}"'] \arrow[u, shift right, "\overline{\mathcal{F}^\tau}"'] & \mathrm{L}^2(\mathbb{R}^d, \widehat{H}). \arrow[u, shift right, "\overline{\mathcal{F}^\tau}"']
\end{tikzcd}$$

In Section 7.5 we showed how to construct $\widehat{\Phi}$ by expressing the kernel k in terms of a basis of steerable functions in $\mathrm{L}^2(G)$.

Because the group correlation Φ is equivariant with respect to the left regular representation $\mathbf{R}$ (on $\mathrm{L}^2(G)$), the components $\widehat{\Phi}_{\rho_2,\rho_1}$ of $\widehat{\Phi}$ are equivariant with respect to the representations $\mathrm{Ind}_H^G \sigma_{\rho_1}$ and $\mathrm{Ind}_H^G \sigma_{\rho_2}$, namely the following diagram commutes.

$$\begin{tikzcd}
\mathrm{L}^2(\mathbb{R}^d, \widehat{H})_{\rho_1} \arrow[r, "\widehat{\Phi}_{\rho_2,\rho_1}"] \arrow[d, "(\mathrm{Ind}_H^G \sigma_{\rho_1})_{(x,h)}"'] & \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_{\rho_2} \arrow[d, "(\mathrm{Ind}_H^G \sigma_{\rho_2})_{(x,h)}"] \\
\mathrm{L}^2(\mathbb{R}^d, \widehat{H})_{\rho_1} \arrow[r, "\widehat{\Phi}_{\rho_2,\rho_1}"'] & \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_{\rho_2}.
\end{tikzcd}$$

Practice shows that it is desirable to design more general group correlations that are equivariant with respect to other representations besides the left regular representation and to consider feature fields that transform under representations other than the representations $\mathrm{Hom}(M_\rho, \mathrm{id})$.

A first generalization is to have two feature fields spaces $\mathbf{FF}(\mathbb{R}^d, H, \sigma_{\mathrm{in}} \colon H \to \mathbf{U}(\mathcal{H}_{\mathrm{in}}))$ and $\mathbf{FF}(\mathbb{R}^d, H, \sigma_{\mathrm{out}} \colon H \to \mathbf{U}(\mathcal{H}_{\mathrm{out}}))$ associated with an input representation σ_{in} and an output representation σ_{out}, where $\mathcal{H}_{\mathrm{in}}$ and $\mathcal{H}_{\mathrm{out}}$ are two finite-dimensional vector spaces equipped with a hermitian inner product, and what we are seeking is a linear G-equivariant map $\widehat{\Phi}$ between these spaces.

We assume that feature fields $f \colon \mathbb{R}^d \to \mathcal{H}_{\mathrm{in}}$ are functions in $\mathrm{L}^2(\mathbb{R}^d, \mathcal{H}_{\mathrm{in}})$, and similarly for feature fields $f \colon \mathbb{R}^d \to \mathcal{H}_{\mathrm{out}}$ (see Definition 6.20).

To say that $\widehat{\Phi}$ is G-equivariant means that the following diagrams commute

$$\begin{tikzcd}
\mathbf{FF}(\mathbb{R}^d, H, \sigma_{\mathrm{in}}) \arrow[r, "\widehat{\Phi}"] \arrow[d, "(\mathrm{Ind}_H^G\, \sigma_{\mathrm{in}})_{(x,h)}"'] & \mathbf{FF}(\mathbb{R}^d, H, \sigma_{\mathrm{out}}) \arrow[d, "(\mathrm{Ind}_H^G\, \sigma_{\mathrm{out}})_{(x,h)}"] \\
\mathbf{FF}(\mathbb{R}^d, H, \sigma_{\mathrm{in}}) \arrow[r, "\widehat{\Phi}"'] & \mathbf{FF}(\mathbb{R}^d, H, \sigma_{\mathrm{out}})
\end{tikzcd}$$

for all $(x, h) \in G = \mathbb{R}^d \rtimes H$, with

$$\begin{aligned}
[(\mathrm{Ind}_H^G\, \sigma_{\mathrm{in}})_{(x,h)} f_{\mathrm{in}}](t) &= \sigma_{\mathrm{in}}(h)(f_{\mathrm{in}}(h^{-1} \cdot (t - x))), \\
&\qquad t \in \mathbb{R}^d,\ f_{\mathrm{in}} \colon \mathbb{R}^d \to \mathcal{H}_{\mathrm{in}} \\
[(\mathrm{Ind}_H^G\, \sigma_{\mathrm{out}})_{(x,h)} f_{\mathrm{out}}](t) &= \sigma_{\mathrm{out}}(h)(f_{\mathrm{out}}(h^{-1} \cdot (t - x))), \\
&\qquad t \in \mathbb{R}^d,\ f_{\mathrm{out}} \colon \mathbb{R}^d \to \mathcal{H}_{\mathrm{out}},
\end{aligned}$$

as in ($\dagger_2$).

A complete solution to this problem was given in a sequence of remarkable papers by Weiler, Geiger, Weilling, Boomsma and Cohen [41] (for $\mathbf{SE}(3)$), Weiler and Cesa [40] (for $\mathbf{E}(2)$), Lang and Weiler [31] (for a homogeneous space X induced by a transitive action of a compact group H), Cesa, Lang and Weiler [7] (for $\mathbf{E}(3)$), and Cohen, Geiger and Weiler [8] (feature fields on homogeneous spaces).

It is shown by Weiler, Geiger, Weilling, Boomsma and Cohen [41] that in the case where $H = \mathbf{SO}(d)$, such a map is given by a kernel

$$K\colon \mathbb{R}^d \to \mathrm{Hom}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}})$$

via

$$\widehat{\Phi}(f)(t) = \int_{\mathbb{R}^d} K(y-t)(f(y))\, dy, \quad f\colon \mathbb{R}^d \to \mathcal{H}_{\mathrm{in}}, t \in \mathbb{R}^d, \tag{K1}$$

and the kernel K satisfies the *equivariance constraint*

$$K(h \cdot t) = \sigma_{\mathrm{out}}(h) \circ K(t) \circ \sigma_{\mathrm{in}}(h)^{-1},\ h \in \mathbf{SO}(d),\ t \in \mathbb{R}^d. \tag{EC$_1$}$$

Functions $K\colon \mathbb{R}^d \to \mathrm{Hom}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}})$ satisfying the equivariance constraint (EC_1) are called *equivariant convolution kernels* or *G-steerable kernels*.

The above result is often referred to by the slogan *"correlation is all you need."*

It is instructive to give the proof since it is prototypical of this kind of argument.

Proof. The first step is to make use of a result of functional analysis that says that any continuous linear map (actually, a Hilbert–Schmidt operator) $\widehat{\Phi}\colon \mathrm{L}^2(\mathbb{R}^d, \mathcal{H}_{\mathrm{in}}) \to \mathrm{L}^2(\mathbb{R}^d, \mathcal{H}_{\mathrm{out}})$ can be expressed in terms of a so-called kernel $\mathcal{K}\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathrm{Hom}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}})$, as

$$\widehat{\Phi}(f)(t) = \int_{\mathbb{R}^d} \mathcal{K}(t, y)(f(y))\, dy, \quad f \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{H}_{\mathrm{in}}),\ t, y \in \mathbb{R}^d, \qquad (*_{\mathcal{K}_1})$$

where $\mathcal{K}$ is L^1-integrable.

The next step is to find the conditions for a linear continuous map $\widehat{\Phi}$ as above to be equivariant, which means that

$$(\mathrm{Ind}_H^G \sigma_{\mathrm{out}})_{(x,h)} \circ \widehat{\Phi} = \widehat{\Phi} \circ (\mathrm{Ind}_H^G \sigma_{\mathrm{in}})_{(x,h)},$$

for all $g = (x, h) \in \mathbb{R}^d \rtimes H$ (with $H = \mathbf{SO}(d)$).

Since $H = \mathbf{SO}(d)$ and $\mathbf{SO}(d)$ acts on $\mathbb{R}^d$ by multiplication we simply write hy for $h \cdot y$, where $h \in \mathbf{SO}(d)$ and $y \in \mathbb{R}^d$.

The action of $G = \mathbb{R}^d \rtimes \mathbf{SO}(d)$ on $\mathbb{R}^d$ is given by $g \cdot y = hy + x$, where $g = (x, h) \in \mathbb{R}^d \rtimes \mathbf{SO}(d)$ and $y \in \mathbb{R}^d$.

Using $(*_{\mathcal{K}_1})$ we have

$$
\begin{aligned}
&\widehat{\Phi}[(\mathrm{Ind}_H^G\,\sigma_{\mathrm{in}})_{(x,h)}f](t) \\
&= \int_{\mathbb{R}^d} \mathcal{K}(t,y)(\sigma_{\mathrm{in}}(h)(f(h^{-1}(y-x))))\,dy,
\end{aligned}
$$

and since $g^{-1} = (x,h)^{-1} = (-h^{-1}x, h^{-1})$, if we make the change of variable $y \mapsto hy + x = g \cdot y$, since the determinant of the Jacobian matrix of this affine map is $+1$, by the change of variable formula, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \mathcal{K}(t,y)(\sigma_{\mathrm{in}}(h)(f(h^{-1}(y-x))))\,dy \\
&= \int_{\mathbb{R}^d} \mathcal{K}(t, g\cdot y)(\sigma_{\mathrm{in}}(h)(f(y)))\,dy.
\end{aligned}
$$

Since $\sigma_{\mathrm{out}}(h)$ is linear, by Vol I, Proposition @@@5.24(7), we also have

$$
\begin{aligned}
&[(\mathrm{Ind}_H^G\,\sigma_{\mathrm{out}})_{(x,h)}\widehat{\Phi}](t)\\
&= \sigma_{\mathrm{out}}(h)(\widehat{\Phi}(h^{-1}(t-x)))\\
&= \sigma_{\mathrm{out}}(h)\left(\int_{\mathbb{R}^d} \mathcal{K}(h^{-1}(t-x), y)(f(y))\,dy\right)\\
&= \int_{\mathbb{R}^d} \sigma_{\mathrm{out}}(h)\left(\mathcal{K}(g^{-1}\cdot t, y)(f(y))\right)dy.
\end{aligned}
$$

Consequently, we must have

$$\mathcal{K}(t, g \cdot y) \circ \sigma_{\mathrm{in}}(h) = \sigma_{\mathrm{out}}(h) \circ \mathcal{K}(g^{-1} \cdot t, y)$$

for all $g \in G = \mathbb{R}^d \rtimes H$ and all $t, y \in \mathbb{R}^d$, which by replacing t by $g \cdot t$ is equivalent to

$$\begin{aligned} \mathcal{K}(g \cdot t, g \cdot y) = \sigma_{\mathrm{out}}(h) \circ \mathcal{K}(t, y) \circ \sigma_{\mathrm{in}}(h)^{-1}, \\ g \in G,\ h \in H,\ t, y \in \mathbb{R}^d. \end{aligned} \tag{$\mathcal{K}_1$}$$

In particular, for $g = -t$ and $h = e$, we get

$$\mathcal{K}(0, y - t) = \mathcal{K}(t, y), \qquad (\mathcal{K}_1')$$

so we define K such that

$$K(y) = \mathcal{K}(0, y),$$

and since $\mathcal{K}(t, y) = \mathcal{K}(0, y-t) = K(y-t)$, $(*_{\mathcal{K}_1})$ becomes

$$\widehat{\Phi}(f)(t) = \int_{\mathbb{R}^d} K(y - t)(f(y))\, dy,$$

as claimed.

By setting $t = 0$ in $(\mathcal{K}_1)$, we see that K satisfies the condition

$$K(g \cdot y) = \sigma_{\mathrm{out}}(h) \circ K(y) \circ \sigma_{\mathrm{in}}(h)^{-1},$$
$$g \in G,\ h \in H,\ y \in \mathbb{R}^d.$$

Since the expression given by $(K1)$ is already translation invariant, it suffices to require the above condition for $g \in H = \mathbf{SO}(d)$, which is (EC_1). $\square$

Observe that a crucial point of the proof is that we are using the Lebesgue measure on $\mathbb{R}^d$ and that the determinant of the Jacobian of the change of variable is $+1$, because we are considering transformations in the affine group of rigid motions $\mathbf{SE}(d)$.

Earlier, Bekkers [1] considered a situation which is *less general in a way, because no representations are involved, but more general in another way, because he is dealing with two homogeneous spaces* $X_{\text{in}} = G/H_{\text{in}}$ and $X_{\text{out}} = G/H_{\text{out}}$, where G is a locally compact group which is not necessarily a semi-direct product.

In this case, we would like to know when a continuous linear map Φ from $\mathrm{L}^2(X_{\text{in}})$ to $\mathrm{L}^2(X_{\text{out}})$ is equivariant with respect to the regular representations $\mathbf{R}^{G\to \mathrm{L}^2(X_{\text{in}})}$ and $\mathbf{R}^{G\to \mathrm{L}^2(X_{\text{out}})}$ induced by G on $\mathrm{L}^2(X_{\text{in}})$ and $\mathrm{L}^2(X_{\text{out}})$.

A new difficulty that now comes up is that X_{in} may not have a G-invariant measure.

Although Bekkers [1] does not make use of quasi-invariant measures, he proves a result in terms of Radon-Nikodym derivatives of measures which can be translated as follows using ϱ-functions. Let x_0^{out} be a chosen point in $X_{\mathrm{out}} = G/H_{\mathrm{out}}$, so that H_{out} is the stabilizer of x_0^{out}.

Suppose that ϱ defines a *quasi-invariant measure* μ on $X_{\mathrm{in}} = G/H_{\mathrm{in}}$.

First we have the fact that every equivariant continuous linear map Φ from $\mathrm{L}^2(X_{\mathrm{in}})$ to $\mathrm{L}^2(X_{\mathrm{out}})$ is given by

$$\Phi(f)(y) = \int_{X_{\mathrm{in}}} \mathcal{K}(x, y) f(x)\, d\mu(x), \qquad y \in X_{\mathrm{out}},\ f \in \mathrm{L}^2(X_{\mathrm{in}}), \tag{$*_{\mathcal{K}_2}$}$$

for some kernel $\mathcal{K} \in \mathrm{L}^1(X_{\mathrm{in}} \times X_{\mathrm{out}})$.

To say that Φ is G-equivariant means that the following diagrams commute

$$\begin{tikzcd}
\mathrm{L}^2(X_{\mathrm{in}}) \arrow[r, "\Phi"] \arrow[d, "\mathbf{R}_g^{G\to \mathrm{L}^2(X_{\mathrm{in}})}"'] & \mathrm{L}^2(X_{\mathrm{out}}) \arrow[d, "\mathbf{R}_g^{G\to \mathrm{L}^2(X_{\mathrm{out}})}"] \\
\mathrm{L}^2(X_{\mathrm{in}}) \arrow[r, "\Phi"'] & \mathrm{L}^2(X_{\mathrm{out}})
\end{tikzcd}$$

for all $g \in G$. For any $f \in \mathrm{L}^2(X_{\mathrm{in}})$ and any $y \in X_{\mathrm{out}}$ we have

$$[(\Phi \circ \mathbf{R}_g^{G\to \mathrm{L}^2(X_{\mathrm{in}})})(f)](y) = \int_{X_{\mathrm{in}}} \mathcal{K}(x,y) f(g^{-1}\cdot x)\, d\mu(x)$$

and

$$\begin{aligned}
[(\mathbf{R}_g^{G\to \mathrm{L}^2(X_{\mathrm{out}})} \circ \Phi)(f)](y) &= \int_{X_{\mathrm{in}}} \mathcal{K}(x, g^{-1}\cdot y) f(x)\, d\mu(x) \\
&= \int_{X_{\mathrm{in}}} \varrho(g^{-1}, x)\mathcal{K}(g^{-1}\cdot x, g^{-1}\cdot y) f(g^{-1}\cdot x)\, d\mu(x).
\end{aligned}$$

The equation

$$[(\Phi \circ \mathbf{R}_g^{G \to \mathrm{L}^2(X_{\mathrm{in}})})(f)](y) = [(\mathbf{R}_g^{G \to \mathrm{L}^2(X_{\mathrm{out}})} \circ \Phi)(f)](y)$$

asserting the commutativity of the above diagram implies that $\mathcal{K}$ satisfies the equation

$$\begin{aligned} \mathcal{K}(x, y) = \varrho(g^{-1}, x)\mathcal{K}(g^{-1} \cdot x, g^{-1} \cdot y), \\ g \in G,\ x \in X_{\mathrm{in}},\ y \in X_{\mathrm{out}}. \end{aligned} \tag{$\mathcal{K}_2$}$$

If we define $K \colon X_{\text{in}} \to \mathbb{C}$ by

$$K(x) = \mathcal{K}(x, x_0^{\text{out}}),$$

then for any $g_y \in G$ such that $y = g_y \cdot x_0^{\text{out}}$,

$$\begin{aligned}
\mathcal{K}(x, y) = \mathcal{K}(x, g_y \cdot x_0^{\text{out}}) &= \varrho(g_y^{-1}, x)\mathcal{K}(g_y^{-1} \cdot x, g_y^{-1} \cdot y) \\
&= \varrho(g_y^{-1}, x)\mathcal{K}(g_y^{-1} \cdot x, x_0^{\text{out}}) \\
&= \varrho(g_y^{-1}, x)K(g_y^{-1} \cdot x).
\end{aligned}$$

Consequently, every equivariant continuous linear map Φ from $\mathrm{L}^2(X_{\mathrm{in}})$ to $\mathrm{L}^2(X_{\mathrm{out}})$ is given by

$$\Phi(f)(y) = \int_{X_{\mathrm{in}}} \varrho(g_y^{-1}, x) K(g_y^{-1} \cdot x) f(x)\, d\mu(x), \qquad y \in X_{\mathrm{out}},\ f \in \mathrm{L}^2(X_{\mathrm{in}}), \tag{K2}$$

where $g_y \in G$ is any element such that $y = g_y \cdot x_0^{\mathrm{out}}$.

Since $h \cdot x_0^{\mathrm{out}} = x_0^{\mathrm{out}}$ for all $h \in H_{\mathrm{out}}$, by setting $g = h \in H_{\mathrm{out}}$ and $y = x_0^{\mathrm{out}}$ in $(\mathcal{K}_2)$, we deduce that the map $K\colon X_{\mathrm{in}} \to \mathbb{C}$ satisfies the condition

$$K(x) = \varrho(h^{-1}, x) K(h^{-1} \cdot x), \quad h \in H_{\mathrm{out}},\ x \in X_{\mathrm{in}}. \tag{$\mathcal{K}_3$}$$

The factor involving ϱ disappears or is replaced by a more tractable term in many practical cases.

This is the case when G is unimodular. If $X_{\mathrm{in}} = \mathbb{R}^d$ and $G = \mathbb{R}^d \rtimes H_{\mathrm{out}}$ with H_{out} a closed subgroup of $\mathbf{GL}(d)$, then if $g = (x, h) \in G$, the condition on K becomes

$$K(x) = \frac{1}{|\det(h)|} K(h^{-1} \cdot x), \quad h \in H_{\mathrm{out}}, x \in X_{\mathrm{in}}, \quad (\mathcal{K}_4)$$

where $\det(h)$ is the determinant of the matrix representing h. For more details, see Bekkers [1].

7.7 Equivariant Correlation G-Kernels; General Case

Until now we have been assuming that we are dealing with feature fields defined on $X = \mathbb{R}^d$ and that the group G is a semi-direct product $G = \mathbb{R}^d \rtimes H$ with $H = \mathbf{SO}(d)$, and more generally a compact group.

It is possible to deal with the more general situation where X is a homogeneous space of the form $X = G/H$ with G locally compact and unimodular and H compact equipped with a unitary representation $\sigma \colon H \to \mathbf{U}(\mathcal{H}_\sigma)$.

The main problem is to define the "right" notion of feature field.

Cohen, Geiger and Weiler [8] propose to use the G-bundle $E = G \times_H \mathcal{H}_\sigma$ introduced in Section 6.13; see Definition 6.12.

But then we might as well use the hermitian G-bundles of finite rank of Definition 6.18 (see Section 6.13) and *the natural choice for the space of feature fields is the subspace* $\mathrm{L}^2(X; E)$ *of the space of sections of the hermitian G-bundle $p\colon E \to X$*, with $X = G/H$ (see Definition 6.20).

Recall that the restriction of the action of G to H on the fibre E_0 is a unitary representation $\sigma\colon H \to \mathbf{U}(E_0)$, and that for every fibre E_x, there is a representation $\sigma_x\colon H \to \mathbf{U}(E_x)$ equivalent to the representation $\sigma\colon H \to \mathbf{U}(E_0)$.

For the time being we will assume that there exists a section $r\colon X \to G$ such that the maps $\mathcal{L}\colon \mathrm{L}^2(X;E) \to L^\sigma$ and $\mathcal{S}\colon L^\sigma \to \mathrm{L}^2(X;E)$ define isomorphisms between $\mathrm{L}^2(X;E)$ and L^σ.

Recall from Equation $(\dagger_4)$ of Definition 6.19 that L^σ is the set consisting of all functions $f \in \mathrm{L}^2(G;E_0)$ such that

$$f(gh) = \sigma(h^{-1})(f(g)) = h^{-1}\cdot f(g), \quad \text{for all } g \in G \text{ and all } h \in H.$$

We will assume that the representations $\sigma\colon H \to \mathbf{U}(E_0)$ are irreducible.

Then the feature fields with values in the fibre E_x transform according to the induced representation
$\mathrm{Ind}_H^G \sigma_x = \Pi$; see Equation ($\dagger_7$) in Section 6.13.

In view of the isomorphism between $\mathrm{L}^2(X; E)$ and L^σ given by the map $\mathcal{L}\colon \mathrm{L}^2(X; E) \to L^\sigma$ (see Definition 6.17, Equation ($\mathcal{L}_3$)), with

$$\mathcal{L}(s)(g) = g^{-1} \cdot s(g \cdot x_0), \quad s \in \mathrm{L}^2(X; E),\ g \in G,$$

the induced representation $\mathrm{Ind}_H^G \sigma_x = \Pi$ is equivalent to the left regular representation of G in L^σ.

We also assume that the section $r\colon X \to G$ makes the representation Π continuous.

Inspired by Cohen, Geiger and Weiler [8] we consider the more general situation in which we have two hermitian G-bundles of finite rank $p_{\mathrm{in}}\colon E_{\mathrm{in}} \to X_{\mathrm{in}}$ and $p_{\mathrm{out}}\colon E_{\mathrm{out}} \to X_{\mathrm{out}}$, where $X_{\mathrm{in}} = G/H_{\mathrm{in}}$ and $X_{\mathrm{out}} = G/H_{\mathrm{out}}$ for the *same* group G, input and output representations σ_{in} and σ_{out}, and determine what are the linear maps $\Phi\colon L^{\sigma_{\mathrm{in}}} \to L^{\sigma_{\mathrm{out}}}$ that are equivariant with respect to the representations $\mathrm{Ind}_{H_{\mathrm{in}}}^{G} \sigma_{\mathrm{in}}$ and $\mathrm{Ind}_{H_{\mathrm{out}}}^{G} \sigma_{\mathrm{out}}$, which means that the following diagram commutes

$$\begin{tikzcd}
L^{\sigma_{\mathrm{in}}} \arrow[r, "\Phi"] \arrow[d, "(\mathrm{Ind}_{H_{\mathrm{in}}}^{G} \sigma_{\mathrm{in}})(g)"'] & L^{\sigma_{\mathrm{out}}} \arrow[d, "(\mathrm{Ind}_{H_{\mathrm{out}}}^{G} \sigma_{\mathrm{out}})(g)"] \\
L^{\sigma_{\mathrm{in}}} \arrow[r, "\Phi"'] & L^{\sigma_{\mathrm{out}}}
\end{tikzcd}$$

for all $g \in G$, where

$$[(\mathrm{Ind}_{H_{\mathrm{in}}}^{G}\sigma_{\mathrm{in}})(g)](f_{\mathrm{in}})(g_1) = f_{\mathrm{in}}(g^{-1}g_1), \qquad g, g_1 \in G,\ f_{\mathrm{in}} \in L^{\sigma_{\mathrm{in}}}$$
$$[(\mathrm{Ind}_{H_{\mathrm{out}}}^{G}\sigma_{\mathrm{out}})(g)](f_{\mathrm{out}})(g_1) = f_{\mathrm{out}}(g^{-1}g_1), \qquad g, g_1 \in G,\ f_{\mathrm{out}} \in L^{\sigma_{\mathrm{out}}}.$$

To reduce the amount of subscripts we will denote the fibre $(E_{\mathrm{in}})_0$ above $x_0^{\mathrm{in}} = H_{\mathrm{in}}$ by E_0^{in} and the fibre $(E_{\mathrm{out}})_0$ above $x_0^{\mathrm{out}} = H_{\mathrm{out}}$ by E_0^{out}.

Then our representations σ_{in} and σ_{out} are
$\sigma_{\mathrm{in}} \colon H_{\mathrm{in}} \to \mathbf{U}(E_0^{\mathrm{in}})$ and $\sigma_{\mathrm{out}} \colon H_{\mathrm{out}} \to \mathbf{U}(E_0^{\mathrm{out}})$.

The following proposition generalizes results proven in Cohen, Geiger and Weiler [8] (see Theorem 3.1 and Theorem 3.2).

In the sequel we assume that *all hermitian G-bundles have finite rank*.

Proposition 7.6. *Let $p_{\mathrm{in}} \colon E_{\mathrm{in}} \to X_{\mathrm{in}}$ and $p_{\mathrm{out}} \colon E_{\mathrm{out}} \to X_{\mathrm{out}}$ be two hermitian G-bundles where $X_{\mathrm{in}} = G/H_{\mathrm{in}}$ and $X_{\mathrm{out}} = G/H_{\mathrm{out}}$ for the same locally compact and unimodular group G. If the space of equivariant G-kernels is defined as*

$$\begin{aligned}
&\mathrm{Hom}_{H_{\mathrm{in}},H_{\mathrm{out}}}(G, \mathrm{Hom}(E_0^{\mathrm{in}}, E_0^{\mathrm{out}})) \\
&= \{K \colon G \to \mathrm{Hom}(E_0^{\mathrm{in}}, E_0^{\mathrm{out}}) \mid \\
&\qquad K(h_2 g h_1) = \sigma_{\mathrm{out}}(h_2) \circ K(g) \circ \sigma_{\mathrm{in}}(h_1), \\
&\qquad g \in G,\ h_1 \in H_{\mathrm{in}},\ h_2 \in H_{\mathrm{out}}\},
\end{aligned} \tag{EC$_2$}$$

then every equivariant linear map $\Phi \in \mathrm{Hom}_{H_{\mathrm{in}},H_{\mathrm{out}}}(\mathrm{Ind}_{H_{\mathrm{in}}}^{G} \sigma_{\mathrm{in}}, \mathrm{Ind}_{H_{\mathrm{out}}}^{G} \sigma_{\mathrm{out}})$ is of the form

$$\begin{aligned}
(\Phi(f_{\mathrm{in}}))(g) = \int_G K(g^{-1}t)(f_{\mathrm{in}}(t))\, d\lambda_G(t) = (K \star f_{\mathrm{in}})(g),& \\
f_{\mathrm{in}} \in L^{\sigma_{\mathrm{in}}},\ g \in G&
\end{aligned} \tag{Φ}$$

for a unique $K \in \mathrm{Hom}_{H_{\mathrm{in}},H_{\mathrm{out}}}(G, \mathrm{Hom}(E_0^{\mathrm{in}}, E_0^{\mathrm{out}}))$.

Observe that if G is not unimodular, in which case the Haar measure λ_G is only left-invariant, the modular term $\Delta(h_1)$ needs to be added, namely we have the equation

$$\begin{aligned} K(gh_1) = \Delta(h_1)K(g) \circ \sigma_{\mathrm{in}}(h_1), \\ g \in G,\ h_1 \in H_{\mathrm{in}}. \end{aligned} \tag{$\mathcal{K}'_6$}$$

Observe that $\Phi(f_{\mathrm{in}})$ is a generalization of group correlation as defined in Definition **??** to vector valued-functions.

Since we are dealing with finite-dimensional vector spaces, we don't need the notion of weak integral and in (Φ) we use component-wise integration.

Recall that $\mathrm{Ind}_{H_{\mathrm{in}}}^{G} \sigma_{\mathrm{in}} \colon G \to \mathbf{U}(L^{\sigma_{\mathrm{in}}})$ and $\mathrm{Ind}_{H_{\mathrm{out}}}^{G} \sigma_{\mathrm{out}} \colon G \to \mathbf{U}(L^{\sigma_{\mathrm{out}}})$ and that $L^{\sigma_{\mathrm{in}}}$ is a space of functions from G to E_0^{in} and that $L^{\sigma_{\mathrm{out}}}$ is a space of functions from G to E_0^{out}.

The equivariance Condition (EC_2) is a bit awkward since it involves the two-sided term $h_2 g h_1$, with $h_1 \in H_{\mathrm{in}}$ and $h_2 \in H_{\mathrm{out}}$.

Lang and Weiler [31] showed that by considering the group $H = H_{\mathrm{out}} \times H_{\mathrm{in}}$, Condition ($\mathrm{EC}_2$) can be reduced to the familiar condition

$$K(h \cdot g) = \sigma_{\mathrm{out}}(h) \circ K(g) \circ \sigma_{\mathrm{in}}(h)^{-1}, \quad h \in H,\ g \in G. \tag{EC}$$

The key observation is that as subgroups of G, H_{in} and H_{out} act on G, but we can consider the more general situation where a compact group H acts on G and seek kernels $K\colon G \to \mathrm{Hom}(E_0^{\mathrm{in}}, E_0^{\mathrm{out}})$ satisfying the condition (EC).

Indeed if we let $H = H_{\mathrm{out}} \times H_{\mathrm{in}}$ and define the left action of $H = H_{\mathrm{out}} \times H_{\mathrm{in}}$ on G by

$$(h_2, h_1) \cdot g = h_2 g h_1^{-1}, \quad h_1 \in H_{\mathrm{in}},\ h_2 \in H_{\mathrm{out}}$$

and the representations $\sigma_{\mathrm{in}}^H\colon H \to \mathbf{U}(E_0^{\mathrm{in}})$ and $\sigma_{\mathrm{out}}^H\colon H \to \mathbf{U}(E_0^{\mathrm{out}})$ by

$$\begin{aligned}\sigma_{\mathrm{in}}^H(h_2, h_1) &= \sigma_{\mathrm{in}}(h_1)\\ \sigma_{\mathrm{out}}^H(h_2, h_1) &= \sigma_{\mathrm{out}}(h_2),\end{aligned}$$

then the condition (EC), namely

$$K(h \cdot g) = \sigma^H_{\mathrm{out}}(h) \circ K(g) \circ (\sigma^H_{\mathrm{in}}(h))^{-1},\\ h = (h_2, h_1) \in H,\ g \in G$$

is equivalent to

$$K(h_2 g h_1^{-1}) = \sigma_{\mathrm{out}}(h_2) \circ K(g) \circ \sigma_{\mathrm{in}}(h_1)^{-1},\\ h_1 \in H_{\mathrm{in}},\ h_2 \in H_{\mathrm{out}},\ g \in H,$$

which is equivalent to

$$K(h_2 g h_1) = \sigma_{\mathrm{out}}(h_2) \circ K(g) \circ \sigma_{\mathrm{in}}(h_1),\\ h_1 \in H_{\mathrm{in}},\ h_2 \in H_{\mathrm{out}},\ g \in H \qquad (\mathrm{EC}_2)$$

since in the quantification over $h_1 \in H_{\mathrm{in}}$ we can replace h_1^{-1} by h_1.

Unlike the previous cases, the kernels K are defined on the group G and the formula (Φ) expressing $K \star f_{\mathrm{in}}$ as an integral requires integration over G.

This is more expensive that the previous cases that only required integration over $\mathbb{R}^d$ or more generally over X_{in}.

The technical reason is that the definition of the induced representations $\mathrm{Ind}_{H_{\mathrm{in}}}^{G} \sigma_{\mathrm{in}}$ and $\mathrm{Ind}_{H_{\mathrm{out}}}^{G} \sigma_{\mathrm{out}}$ is a lot simpler when they are acting on the spaces $L^{\sigma_{in}}$ and $L^{\sigma_{out}}$, since they are simply the regular representations.

The representations σ_{in} and σ_{out} are hidden in the definition of the spaces $L^{\sigma_{in}}$ and $L^{\sigma_{out}}$.

To define these representations on functions defined on X_{in} or X_{out} is more complicated because this requires picking some sets of coset representatives $(r_x^{\mathrm{in}})_{x\in G/H_{\mathrm{in}}}$ and $(r_x^{\mathrm{out}})_{x\in G/H_{\mathrm{out}}}$ but then, there is no guarantee that the corresponding sections are continuous.

We will assume in the sequel that the maps $\mathcal{L}_{\mathrm{in}}$ and $\mathcal{S}_{\mathrm{in}}$ are continuous, and similarly for the maps $\mathcal{L}_{\mathrm{out}}$ and $\mathcal{S}_{\mathrm{out}}$

7.8 Equivariant Correlation X_{in}-Kernels

Cohen, Geiger and Weiler [8] give other characterizations of the space $\mathrm{Hom}_{H_{\mathrm{in}},H_{\mathrm{out}}}(\mathrm{Ind}_{H_{\mathrm{in}}}^{G}\sigma_{\mathrm{in}}, \mathrm{Ind}_{H_{\mathrm{out}}}^{G}\sigma_{\mathrm{out}})$; one in terms of kernels defined on $X_{\mathrm{in}} = G/H_{\mathrm{in}}$, and the other in terms of kernels on the space $H_{\mathrm{out}}\backslash G/H_{\mathrm{in}}$ of double cosets.

We discuss the solution in terms of kernels on $X_{\mathrm{in}} = G/H_{\mathrm{in}}$ and refer the reader to Cohen, Geiger and Weiler [8] for the third solution (see Theorem 3.4).

The key is to pick a set of coset representatives $(r_x^{\mathrm{in}})_{x\in G/H_{\mathrm{in}}}$. Here $x_0^{\mathrm{in}} = H_{\mathrm{in}}$, and as usual $r_{x_0^{\mathrm{in}}}^{\mathrm{in}} = e$.

As we said earlier we assume that maps $\mathcal{L}_{\text{in}}$ and $\mathcal{S}_{\text{in}}$ are continuous.

Then recall from Definition 6.4 that for every coset $x \in X_{\text{in}} = G/H_{\text{in}}$ and every $g \in G$ we set

$$u^{\text{in}}(g, x) = (r^{\text{in}}_{g\cdot x})^{-1} g r^{\text{in}}_{x} \in H_{\text{in}}, \tag{u}$$

and that by Equation (s), if $x = gH_{\text{in}} = g \cdot x^{\text{in}}_0$, we have

$$g = r^{\text{in}}_x u^{\text{in}}(g, x^{\text{in}}_0).$$

Then for any $g \in G$, if $x = gH_{\mathrm{in}} = g \cdot x_0^{\mathrm{in}}$, by setting $h_2 = e$ in (EC_2), we have $K(g_1h_1) = K(g_1) \circ \sigma_{\mathrm{in}}(h_1)$ for all $g_1 \in G$ and all $h_1 \in H_{\mathrm{in}}$, so we can write

$$K(g) = K(r_x^{\mathrm{in}} u^{\mathrm{in}}(g, x_0^{\mathrm{in}})) = K(r_x^{\mathrm{in}}) \circ \sigma_{\mathrm{in}}(u^{\mathrm{in}}(g, x_0^{\mathrm{in}})).$$

This suggests defining $\kappa\colon X_{\mathrm{in}} \to \mathrm{Hom}(E_0^{\mathrm{in}}, E_0^{\mathrm{out}})$ by

$$\kappa(x) = K(r_x^{\mathrm{in}}), \quad x \in X_{\mathrm{in}} = G/H_{\mathrm{in}}. \qquad (\kappa)$$

The following proposition which generalizes a result originally proven in Cohen, Geiger and Weiler [8] (Theorem 3.3) is obtained.

Proposition 7.7. *Let $p_{\mathrm{in}}\colon E_0^{\mathrm{in}} \to X_{\mathrm{in}}$ and $p_{\mathrm{out}}\colon E_0^{\mathrm{out}} \to X_{\mathrm{out}}$ be two hermitian G-bundles where $X_{\mathrm{in}} = G/H_{\mathrm{in}}$ and $X_{\mathrm{out}} = G/H_{\mathrm{out}}$ for the same locally compact and unimodular group G. If the space of equivariant G-kernels is defined as*

$$
\begin{aligned}
&\mathrm{Hom}_{H_{\mathrm{in}},H_{\mathrm{out}}}(G, \mathrm{Hom}(E_0^{\mathrm{in}}, E_0^{\mathrm{out}}))\\
&= \{K\colon G \to \mathrm{Hom}(E_0^{\mathrm{in}}, E_0^{\mathrm{out}}) \mid\\
&\qquad K(h_2 g h_1) = \sigma_{\mathrm{out}}(h_2) \circ K(g) \circ \sigma_{\mathrm{in}}(h_1),\\
&\qquad g \in G,\ h_1 \in H_{\mathrm{in}},\ h_2 \in H_{\mathrm{out}}\}
\end{aligned}
$$

and the space of equivariant X_{in}-kernels is defined as

$$
\begin{aligned}
&\mathrm{Hom}_{H_{\mathrm{out}}}(X_{\mathrm{in}}, \mathrm{Hom}(E_0^{\mathrm{in}}, E_0^{\mathrm{out}}))\\
&= \{\kappa\colon X_{\mathrm{in}} \to \mathrm{Hom}(E_0^{\mathrm{in}}, E_0^{\mathrm{out}}) \mid\\
&\qquad \kappa(h_2 \cdot x) = \sigma_{\mathrm{out}}(h_2) \circ \kappa(x) \circ \sigma_{\mathrm{in}}(u^{\mathrm{in}}(h_2, x)^{-1}),\\
&\qquad x \in X_{\mathrm{in}},\ h_2 \in H_{\mathrm{out}}\},
\end{aligned}
\tag{EC$_3$}
$$

then the map that assigns to every X_{in}-kernel κ the G-kernel K defined such that for every $g \in G$, if $x = gH_{\mathrm{in}}$ then

$$K(g) = \kappa(x) \circ \sigma_{\mathrm{in}}(u^{\mathrm{in}}(g, x_0^{\mathrm{in}})),$$

is a bijection.

The dependency on x of the term $\sigma_{\mathrm{in}}(u^{\mathrm{in}}(h_2, x)^{-1})$ is a problem.

It would be nice if H_{in} had the property that we could find a section (a set of coset representatives) $r^{\mathrm{in}} \colon G/H_{\mathrm{in}} \to G$ satisfying the property

$$r^{\mathrm{in}}_{h_2 \cdot x} = h_2 r^{\mathrm{in}}_x h_2^{-1}, \quad x \in X_{\mathrm{in}} = G/H_{\mathrm{in}},\ h_2 \in H_{\mathrm{out}}. \quad (\dagger_3)$$

Indeed, in this case, from (u) rewritten as

$$r^{\mathrm{in}}_{h_2\cdot x}u^{\mathrm{in}}(h_2,x) = h_2 r^{\mathrm{in}}_x,$$

we get

$$h_2 r^{\mathrm{in}}_x h_2^{-1} u^{\mathrm{in}}(h_2,x) = h_2 r^{\mathrm{in}}_x,$$

that is,

$$u^{\mathrm{in}}(h_2,x) = h_2. \tag{$\dagger_4$}$$

It follows that

$$\sigma_{\mathrm{in}}(u^{\mathrm{in}}(h_2, x)^{-1}) = \sigma_{\mathrm{in}}(h_2)^{-1},$$

and (EC_3) is then the more friendly condition

$$\begin{aligned} \kappa(h_2 \cdot x) = \sigma_{\mathrm{out}}(h_2) \circ \kappa(x) \circ \sigma_{\mathrm{in}}(h_2)^{-1}, \\ h_2 \in H_{\mathrm{out}},\ x \in X_{\mathrm{in}}. \end{aligned} \tag{EC$_4$}$$

It is not hard to show that Equation $(\dagger_3)$ holds in the case where $H = H_{\mathrm{in}} = H_{\mathrm{out}}$ and G is a semi-direct product.

In general there does not appear to be a simple way to find conditions for which the term $\sigma^{\mathrm{in}}(u^{\mathrm{in}}(h_2, x)^{-1})$ goes away.

Cohen, Geiger and Weiler [8] (Theorem 3.4) show that by considering kernels defined on the double coset space $H_{\mathrm{out}}\backslash G/H_{\mathrm{in}}$, Condition ($\mathrm{EC}_3$) almost becomes Condition (EC_5), but the analog of the representation σ_{in} *depends on x*, so this is not a reduction to (EC_5).

7.9 Passing from $L^{\sigma_{\rm in}}$ and $L^{\sigma_{\rm out}}$ to $\mathrm{L}^2(X_{\rm in}, E_{\rm in})$ and $\mathrm{L}^2(X_{\rm out}, E_{\rm out})$

The G-equivariant maps in $\mathrm{Hom}_{H_{\rm in}, H_{\rm out}}(\mathrm{Ind}_{H_{\rm in}}^G \sigma_{\rm in}, \mathrm{Ind}_{H_{\rm out}}^G \sigma_{\rm out})$ are functions from $L^{\sigma_{\rm in}}$ to $L^{\sigma_{\rm out}}$ and still require integration over G to be computed using equivariant kernels in the space $\mathrm{Hom}_{H_{\rm in}, H_{\rm out}}(G, \mathrm{Hom}(E_0^{\rm in}, E_0^{\rm out}))$.

It would be nice if we could transform the integration over G to a more practically computable integration over $X_{\rm in}$.

This can be achieved by using the maps $\mathcal{S}_{\rm out}\colon L^{\sigma_{\rm out}} \to \mathrm{L}^2(X_{\rm out}, E_{\rm out})$ and $\mathcal{L}_{\rm in}\colon \mathrm{L}^2(X_{\rm in}, E_{\rm in}) \to L^{\sigma_{\rm in}}$ given by ($\mathcal{S}_3''$) and ($\mathcal{L}_3'$) of Section 6.13.

When these maps are well-defined, which is our assumption, they can be used to define maps from $\mathrm{L}^2(X, E_{\mathrm{in}})$ to $\mathrm{L}^2(X, E_{\mathrm{out}})$ from functions from $L^{\sigma_{\mathrm{in}}}$ to $L^{\sigma_{\mathrm{out}}}$.

Recall that $(\mathcal{L}_3')$ is given by

$$\begin{aligned}\mathcal{L}(s)(g) = \sigma(u(g, x_0)^{-1})(r_x^{-1} \cdot s(x)),\\ x = gH = g \cdot x_0,\ g \in G,\ s \in \mathrm{L}^2(X, E),\end{aligned}$$

and $(\mathcal{S}_3'')$ is given by

$$\begin{aligned}\mathcal{S}(f)(gH) = \mathcal{S}(f)(g \cdot x_0) = g \cdot f(g),\\ g \in G,\ f \in L^{\sigma}.\end{aligned}$$

Pick a set of coset representatives $(r_x^{\mathrm{in}})_{x\in G/H_{\mathrm{in}}}$ for $X_{\mathrm{in}} = G/H_{\mathrm{in}}$ and a set of coset representatives $(r_x^{\mathrm{out}})_{x\in G/H_{\mathrm{out}}}$ for $X_{\mathrm{out}} = G/H_{\mathrm{out}}$.

Then for every section $s \in \mathrm{L}^2(X_{\mathrm{in}}, E_{\mathrm{in}})$, for every $x \in X_{\mathrm{out}}$, observe that for every equivariant kernel $K \in \mathrm{Hom}_{H_{\mathrm{in}},H_{\mathrm{out}}}(G, \mathrm{Hom}(E_0^{\mathrm{in}}, E_0^{\mathrm{out}}))$, the function $\widetilde{\Phi}$ given by

$$\widetilde{\Phi}(s) = \mathcal{S}_{\mathrm{out}}(K \star (\mathcal{L}_{\mathrm{in}}(s))$$

maps $\mathrm{L}^2(X_{\mathrm{in}}, E_{\mathrm{in}})$ to $\mathrm{L}^2(X_{\mathrm{out}}, E_{\mathrm{out}})$, because $\mathcal{L}_{\mathrm{in}}(s) \in L^{\sigma_{\mathrm{in}}}$, $K \star (\mathcal{L}_{\mathrm{in}}(s)) \in L^{\sigma_{\mathrm{out}}}$, and $\mathcal{S}_{\mathrm{out}}(K \star (\mathcal{L}_{\mathrm{in}}(s))) \in \mathrm{L}^2(X_{\mathrm{out}}, E_{\mathrm{out}})$, as illustrated in the following diagram.

$$\begin{tikzcd}
L^{\sigma_{\mathrm{in}}} \arrow[r, "\Phi = K\star -"] & L^{\sigma_{\mathrm{out}}} \arrow[d, "\mathcal{S}_{\mathrm{out}}"] \\
\mathrm{L}^2(X_{\mathrm{in}}, E_{\mathrm{in}}) \arrow[u, "\mathcal{L}_{\mathrm{in}}"] \arrow[r, "\widetilde{\Phi}"'] & \mathrm{L}^2(X_{\mathrm{out}}, E_{\mathrm{out}}).
\end{tikzcd}$$

We now work out several explicit formulae for $\mathcal{S}_{\mathrm{out}}(K \star (\mathcal{L}_{\mathrm{in}}(s))$, the most general ones being $(\dagger_8)$ and $(\dagger_9)$.

For any $s \in \mathrm{L}^2(X_{\mathrm{in}}, E_{\mathrm{in}})$ we obtain

$$
\begin{aligned}
&[\mathcal{S}_{\mathrm{out}}(K \star (\mathcal{L}_{\mathrm{in}}(s))](x) \\
&= r_x^{\mathrm{out}} \cdot \int_G \kappa((r_x^{\mathrm{out}})^{-1} y) \\
&\qquad\qquad [\sigma_{\mathrm{in}}(u^{\mathrm{in}}((r_x^{\mathrm{out}})^{-1}, y))((r_y^{\mathrm{in}})^{-1} \cdot s(y))]\, d\lambda_G(t),
\end{aligned}
\tag{$\dagger_7$}
$$

with $y = t \cdot x_0^{\mathrm{in}}$, $t \in G$, and $x = r_x^{\mathrm{out}} \cdot x_0^{\mathrm{out}} \in X_{\mathrm{out}}$.

By Vol I, Proposition @@@8.43, since G is a locally compact group and $H_{\rm in}$ is a compact subgroup of G, the space $X_{\rm in} = G/H_{\rm in}$ admits a G-invariant σ-Radon measure γ so that for any $s \in \mathrm{L}^2(X_{\rm in}, E_{\rm in})$ and any
$x = r^{\rm out}_x \cdot x^{\rm out}_0 \in X_{\rm out}$,

$$\begin{aligned}
&[\mathcal{S}_{\rm out}(K \star (\mathcal{L}_{\rm in}(s))](x) \\
&= r^{\rm out}_x \cdot \int_{X_{\rm in}} \kappa((r^{\rm out}_x)^{-1} y) \\
&\qquad [\sigma_{\rm in}(u^{\rm in}((r^{\rm out}_x)^{-1}, y))((r^{\rm in}_y)^{-1} \cdot s(y))] \, d\gamma(y).
\end{aligned} \tag{$\dagger_8$}$$

This is the main formula of this section.

It uses a cheaper integration over $X_{\rm in}$ and the simpler kernel κ.

This formula holds in the general framework of hermitian G-bundles of finite rank.

A similar formula is given in Cohen, Geiger and Weiler [8] (Formula (14)), but with the term $u^{\rm in}((r_x^{\rm out})^{-1} r_y^{\rm in}, x_0^{\rm in})$ instead of the term $u^{\rm in}((r_x^{\rm out})^{-1}, y)$. In fact these terms are equal.

This is because by $(*_h)$,

$$\begin{aligned} u^{\rm in}((r_x^{\rm out})^{-1} r_y^{\rm in}, x_0^{\rm in}) &= u^{\rm in}((r_x^{\rm out})^{-1}, r_y^{\rm in} \cdot x_0^{\rm in}) u^{\rm in}(r_y^{\rm in}, x_0^{\rm in}) \\ &= u^{\rm in}((r_x^{\rm out})^{-1}, y), \end{aligned}$$

since $u^{\rm in}(r_y^{\rm in}, x_0^{\rm in}) = e$, which follows from Equation (u) since

$$u^{\rm in}(r_y^{\rm in}, x_0^{\rm in}) = (r^{\rm in}_{r_y^{\rm in} \cdot x_0^{\rm in}})^{-1} r_y^{\rm in} r^{\rm in}_{x_0^{\rm in}} = (r_y^{\rm in})^{-1} r_y^{\rm in} e = e.$$

We finish this section by considering two special cases of the main formula.

Example 7.9. If the hermitian G-bundles E_{in} and E_{out} arise from the Borel construction (see Section 6.12) from the representations $\sigma_{\mathrm{in}}\colon H_{\mathrm{in}} \to \mathbf{U}(\mathcal{H}_{\mathrm{in}})$ and $\sigma_{\mathrm{out}}\colon H_{\mathrm{out}} \to \mathbf{U}(\mathcal{H}_{\mathrm{out}})$, then the fibres $E_{x_{\mathrm{in}}}$ (with $x_{\mathrm{in}} \in X_{\mathrm{in}}$) consists of equivalence classes $\{[(r^{\mathrm{in}}_{x_{\mathrm{in}}}, u_{\mathrm{in}})] \mid u_{\mathrm{in}} \in \mathcal{H}_{\mathrm{in}}\}$, and the fibres $E_{x_{\mathrm{out}}}$ (with $x_{\mathrm{out}} \in X_{\mathrm{out}}$) consists of equivalence classes $\{[(r^{\mathrm{out}}_{x_{\mathrm{out}}}, u_{\mathrm{out}})] \mid u_{\mathrm{out}} \in \mathcal{H}_{\mathrm{out}}\}$.

The fibre E^{in}_0 above $x^{\mathrm{in}}_0 = H_{\mathrm{in}}$ consists of equivalence classes of the form $[(e, u_{\mathrm{in}})]$, and the fibre E^{out}_0 above $x^{\mathrm{out}}_0 = H_{\mathrm{out}}$ consists of equivalence classes of the form $[(e, u_{\mathrm{out}})]$.

The fibre E^{in}_0 is isomorphic to $\mathcal{H}_{\mathrm{in}}$, and the fibre E^{out}_0 is isomorphic to $\mathcal{H}_{\mathrm{out}}$; see the discussion just after Definition 6.16.

We also explained in Section 6.13 that the definition of the action of G on these hermitian G-bundles implies that

$$(r^{\mathrm{in}}_{x_{\mathrm{in}}})^{-1} \cdot [(r^{\mathrm{in}}_{x_{\mathrm{in}}}, u_{\mathrm{in}})] = [(e, u_{\mathrm{in}})]$$

and

$$r^{\mathrm{out}}_{x_{\mathrm{out}}} \cdot [(e, u_{\mathrm{out}})] = [(r^{\mathrm{out}}_{x_{\mathrm{out}}}, u_{\mathrm{out}})],$$

so the above maps provide isomorphisms from $E_{x_{\mathrm{in}}}$ to E^{in}_0 and from E^{out}_0 to $E_{x_{\mathrm{out}}}$.

Since the sections in $\Gamma(E_{\rm in})$ are of the form

$$s_{\rm in}(x_{\rm in}) = [(r^{\rm in}_{x_{\rm in}}, u_{\rm in})]$$

and the sections in $\Gamma(E_{\rm out})$ are of the form

$$s_{\rm out}(x_{\rm out}) = [(r^{\rm out}_{x_{\rm out}}, u_{\rm out})],$$

and since $\kappa(x_{\rm in})$ maps the fibre $E^{\rm in}_0$ to the the fibre $E^{\rm out}_0$, we see that if we identify *all* the fibres $E_{x_{\rm in}}$ with $E^{\rm in}_0$ and *all* the fibres $E_{x_{\rm out}}$ with $E^{\rm out}_0$, then we can view sections in $\Gamma(E_{\rm in})$ as functions from $X_{\rm in}$ to $E^{\rm in}_0 \approx \mathcal{H}_{\rm in}$ and sections in $\Gamma(E_{\rm out})$ as functions from $X_{\rm out}$ to $E^{\rm out}_0 \approx \mathcal{H}_{\rm out}$, so we can drop the terms $r^{\rm out}_x$ and $(r^{\rm in}_y)^{-1}$ and we get the formula

$$
\begin{aligned}
&[\mathcal{S}_{\mathrm{out}}(K \star (\mathcal{L}_{\mathrm{in}}(s))](x) \\
&= \int_{X_{\mathrm{in}}} \kappa((r_x^{\mathrm{out}})^{-1}y)[\sigma_{\mathrm{in}}(u^{\mathrm{in}}((r_x^{\mathrm{out}})^{-1}, y))(s(y))]\, d\gamma(y),
\end{aligned}
\tag{$\dagger_9$}
$$

for all $s \in \mathrm{L}^2(X_{\mathrm{in}}, E_{\mathrm{in}})$, with $y \in X_{\mathrm{in}}$ and $x \in X_{\mathrm{out}}$.

The second special case deals with semi-direct products.

Example 7.10. If $H = H_{\mathrm{in}} = H_{\mathrm{out}}$ and G is a semi-direct product $G = N \rtimes H$, then $X = G/H \approx N$.

By $(\dagger_6)$, $r_{n\cdot y} = nr_y$ when $n \in N$, and from

$$r_{n\cdot y}u(n, y) = nr_y,$$

we get $nr_y u(n, y) = nr_y$, that is

$$u(n, y) = e. \tag{$\dagger_{10}$}$$

Consequently, by setting $n = (r_x)^{-1} \in N$ we have $u(r_x^{-1}, y) = e$, and since $r_x = x$ and $r_y = y$, by ($\dagger_{10}$) and ($\dagger_8$) we obtain

$$[\mathcal{S}(K \star (\mathcal{L}(s))](x) = x \cdot \int_N \kappa(x^{-1}y)(y^{-1} \cdot s(y))\, d\gamma(y), \quad x, y \in N, \tag{$\dagger_{11}$}$$

for all $s \in \mathrm{L}^2(X, E)$.

If the hermitian G-bundles are constructed from representations $\sigma_{\rm in}\colon H \to \mathbf{U}(\mathcal{H}_{\rm in})$ and $\sigma_{\rm out}\colon H \to \mathbf{U}(\mathcal{H}_{\rm out})$, the above formula becomes

$$[\mathcal{S}(K \star (\mathcal{L}(s))](x) = \int_N \kappa(x^{-1}y)(s(y))\, d\gamma(y),\ x, y \in N, \tag{$\dagger_{12}$}$$

for all $s \in \mathrm{L}^2(X, E)$. Note the analogy of $(\dagger_{12})$ and (Φ) from Proposition 7.6.

The issue of finding G-equivariant kernels still remains .

Bibliography

[1] Erik J. Bekkers. B-Spline CNNs on Lie Groups. In *International Conference on Learning Representations (ICLR)*, pages 1–25, 2020.

[2] Marcel Berger. *Géométrie 1*. Nathan, 1990. English edition: Geometry 1, Universitext, Springer Verlag.

[3] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology*. GTM No. 82. Springer Verlag, first edition, 1986.

[4] Nicolas Bourbaki. *Algèbre, Chapitre 9*. Eléments de Mathématiques. Hermann, 1968.

[5] T. Bröcker and T. tom Dieck. *Representations of Compact Lie Groups*. GTM, Vol. 98. Springer Verlag, first edition, 1985.

[6] Roger Carter, Graeme Segal, and Ian Macdonald. *Lectures on Lie Groups and Lie Algebras*. Cambridge University Press, first edition, 1995.

[7] Gabriele Cesa, Leon Lang, and Maurice Weiler. A program to build E(n)-equivariant steerable CNN's. In *International Conference on Learning Representations (ICLR)*, pages 359–368, 2022. URL:https://openreview:net/pdf?id=WE4qe9x1nQw.

[8] Taco S. Cohen, Mario Geiger, and Maurice Weiler. A General Theory of equivariant CNN's on homogeneous spaces. In *Conference on Neural Information Processing Systems (NeurIPS)*, 2019.

[9] Jean Dieudonné. *Special Functions and Linear Representations of Lie Groups*. Regional Conference Series in Mathematics, No. 42. AMS, first edition, 1980.

[10] Jean Dieudonné. *Éléments d'Analyse, Tome V. Groupes de Lie Compacts, Groupes de Lie Semi-Simples*. Edition Jacques Gabay, first edition, 2003.

[11] Jean Dieudonné. *Éléments d'Analyse, Tome VI. Analyse Harmonique*. Edition Jacques Gabay, first edition, 2003.

[12] Jean Dieudonné. *Éléments d'Analyse, Tome VII. Équations Fonctionnelles Linéaires. Première partie, Opérateurs Pseudo-Différentiels*. Edition Jacques Gabay, first edition, 2003.

[13] Jean Dieudonné. *Éléments d'Analyse, Tome II. Chapitres XII à XV*. Edition Jacques Gabay, first edition, 2005.

[14] Jean Dieudonné. *Éléments d'Analyse, Tome III. Chapitres XVI et XVII*. Edition Jacques Gabay, first edition, 2006.

[15] Jean Dieudonné. *Éléments d'Analyse, Tome IV. Chapitres XVIII à XX*. Edition Jacques Gabay, first edition, 2007.

[16] B.A. Dubrovin, A.T. Fomenko, and S.P. Novikov. *Modern Geometry–Methods and Applications. Part I*. GTM No. 93. Springer Verlag, second edition, 1985.

[17] J.J. Duistermaat and J.A.C. Kolk. *Lie Groups*. Universitext. Springer Verlag, first edition, 2000.

[18] Gerald B. Folland. *Fourier Analysis and its Applications*. Brooks/Cole Publishing Company, first edition, 1992.

[19] Gerald B. Folland. *A Course in Abstract Harmonic Analysis*. CRC Press, first edition, 1995.

[20] William T. Freeman and Adelson Edward H. The Design and Use of Steerable Filters. *IEEE Trans-*

actions on Pattern Analysis and Machine Intelligence, 13:891–906, 1991.

[21] William Fulton and Joe Harris. *Representation Theory, A first course.* GTM No. 129. Springer Verlag, first edition, 1991.

[22] Jean H. Gallier. *Geometric Methods and Applications, For Computer Science and Engineering.* TAM, Vol. 38. Springer, second edition, 2011.

[23] Jean H. Gallier and Jocelyn Quaintance. *Differential Geometry and Lie Groups. A Second Course.* Geometry and Computing 13. Springer, first edition, 2020.

[24] Jean H. Gallier and Jocelyn Quaintance. *Linear Algebra and Optimization with Applications to Machine Learning, Volume I.* World Scientific, first edition, 2020.

[25] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian Geometry.* Universitext. Springer Verlag, second edition, 1993.

[26] Marshall Hall. *Theory of Groups.* AMS Chelsea, second edition, 1976.

[27] Edwin Hewitt and Kenneth A. Ross. *Abstract Harmonic Analysis II*. Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, first edition, 1970.

[28] A.A. Kirillov. *Elements of the Theory of Representations*. Springer-Verlag, first edition, 1976.

[29] A.A. Kirillov. *Lectures on the Orbit Method*. GSM, Vol. 64. AMS, first edition, 2004.

[30] Yvette Kosmann-Schwarzbach. *Groups and Symmetries. From Finite Groups to Lie Groups*. Universitext. Springer Verlag, first edition, 2010.

[31] Leon Lang and Maurice Weiler. A Wigner-Eckart Theorem for Group-Equivariant Convolution Kernels. In *Conference on Neural Information Processing Systems (NeurIPS)*, 2019. URL:https://arxiv.org/abs/2010.10952.

[32] Serge Lang. *Real and Functional Analysis*. GTM 142. Springer Verlag, third edition, 1996.

[33] Yaogan Mensah and Yaovi M. Awussi. On the fourier transform on semi-direct products with compact factor. *International Journal of Mathematics and Computation*, 21:65–71, 2013.

[34] Shigeyuki Morita. *Geometry of Differential Forms.* Translations of Mathematical Monographs No 201. AMS, first edition, 2001.

[35] J.J Sakurai and Jim Napolitano. *Modern Quantum Mechanics.* Cambridge University Press, third edition, 2021.

[36] Giovanni Sansone. *Orthogonal Functions.* Dover, first edition, 1991.

[37] Jean-Pierre Serre. *Linear Representations of Finite Groups.* GTM No. 42. Springer, first edition, 1977.

[38] Barry Simon. *Representations of Finite and Compact Groups.* GSM, Vol. 10. AMS, first edition, 1996.

[39] N.J. Vilenkin. *Special Functions and the Theory of Group Representations.* Translations of Mathematical Monographs No 22. AMS, first edition, 1968.

[40] Maurice Weiler and Gabriele Cesa. General E(2)-equivariant steerable CNNs'. In *International Conference on Learning Representations (ICLR)*, 2019. URL:https://arxiv.org/abs/1911.08251.

[41] Maurice Weiler, Mario Geiger, Max Welling, Wouter Boomsma, and Taco Cohen. 3D steerable CNN's:

Learning rotationally equivariant features in volumetric data. In *Conference on Neural Information Processing Systems (NeurIPS)*, 2018. URL:https://arxiv.org/abs/1807.02547.

[42] Eugene P. Wigner. *Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra.* Academic Press, first edition, 1959.