7.3 Feature Fields

We begin with the definition of feature fields involving a semi-direct product group $G = \mathbb{R}^d \rtimes H$.

This definition will be generalized later to a G-bundle on a homogenous space X (see Section 6.13).

To help intuition, suppose that $G = \mathbb{R}^2 \rtimes \mathbf{SO}(2)$.

A scalar-valued function $f\colon \mathbb{R}^2 \to \mathbb{R}$ (more generally $f\colon \mathbb{R}^2 \to \mathbb{C}$) can be viewed as a gray-scale image, or temperature field, or pressure field.

The group $G = \mathbb{R}^2 \rtimes \mathbf{SO}(2)$ acts on such an image by moving each pixel at t to the new position $Rt + x$, since $f \mapsto \mathbf{R}_{(x,R)}f$, with $(\mathbf{R}_{(x,R)}f)(t) = f((x,R)^{-1} \cdot t) = f(R^{-1}(t - x))$, where $g = (x, R) \in \mathbb{R}^2 \rtimes \mathbf{SO}(2)$, so

$$(\mathbf{R}_{(x,R)}f)(Rt + x) = f(R^{-1}(Rt + x - x)) = f(t);$$

see Figure 7.1.

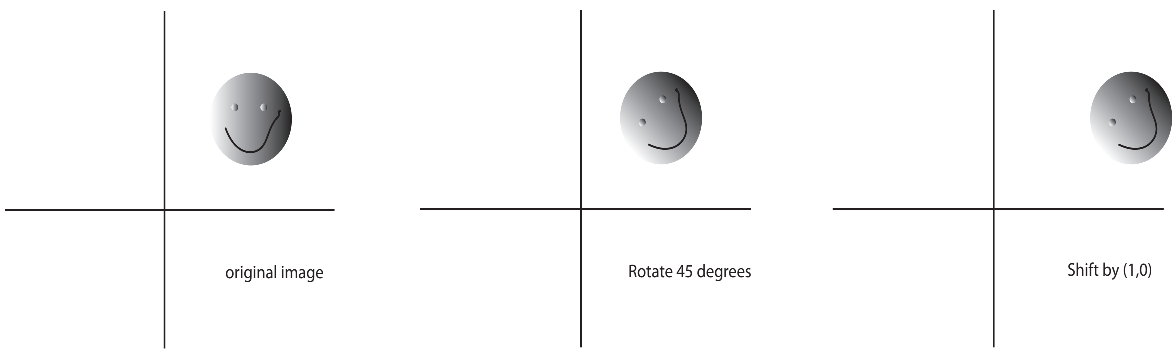

Figure 7.1: The image of $f(t)$ is the gray-scaled smiley face. The action of $G = \mathbb{R}^2 \rtimes \mathbf{SO}(2)$ on this image moves each pixel to $Rt + x$, where R is a rotation by 45 degrees counter-clockwise and x is a translation by $[1\ 0]^T$.

On the other hand, a function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defines a vector field, such as a velocity field, an optical flow, or a gradient image.

This time such a vector field transforms under the action of $G = \mathbb{R}^2 \rtimes \mathbf{SO}(2)$ as follows: the vector $v = f(t)$ originally located at t is moved to the location $Rt + x$, and then *rotated* by R, so that the overall action results in the vector

$$Rv \text{ in location } Rt + x.$$

See Figure 7.2.

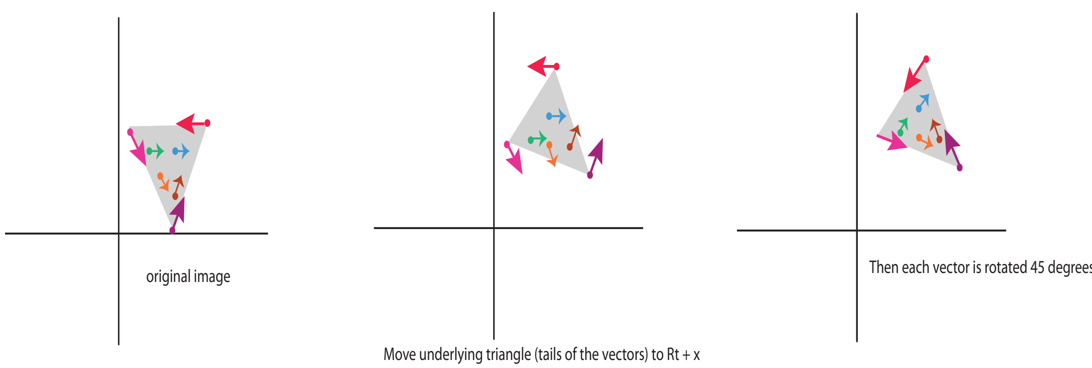

Figure 7.2: The image of $f(t)$ is the vectorized triangular smiley face. The action of $G = \mathbb{R}^2 \rtimes \mathbf{SO}(2)$ on this image moves each pixel to $Rt + x$, (where R is a rotation by 45 degrees counter-clockwise and x is a translation by $[1\ 0]^T$), and then rotates the vector by 45 degrees counter-clockwise.

Given a more general vector field $f\colon \mathbb{R}^2 \to E$, where E is some finite-dimensional hermitian vector space, it is useful to generalize the action on a vector $v = f(t)$ so that it is specified by a representation $\sigma\colon \mathbf{SO}(2) \to \mathbf{U}(E)$ as

$$\sigma(R)(v) \text{ in location } Rt + x.$$

The preceding discussion suggests the following definition.

Definition 7.3. Let $G = \mathbb{R}^d \rtimes H$ be a semi-direct product with H a compact group and let $\sigma\colon H \to \mathbf{GL}(\mathcal{H})$ be a representation, where $\mathcal{H}$ is any complex vector space (possibly infinite dimensional). If $\mathcal{H}$ is finite dimensional or a separable Hilbert space we assume that
$\sigma\colon H \to \mathbf{U}(\mathcal{H})$ is a unitary representation.
A *feature field* is any function $f\colon \mathbb{R}^d \to \mathcal{H}$.

The space of such feature fields is denoted by
$\mathbf{FF}(\mathbb{R}^d, H, \sigma\colon H \to \mathbf{GL}(\mathcal{H}))$.

The representation σ is called the *type* of the feature field.

The group G acts on feature fields *via* the induced representation $\mathrm{Ind}_H^G \sigma$, namely

$$[(\mathrm{Ind}_H^G \sigma)_{(x,h)} f](t) = \sigma(h)(f(h^{-1} \cdot (t - x))),$$
$$(x, h) \in \mathbb{R}^d \rtimes H,\ t \in \mathbb{R}^d. \quad (\dagger_2)$$

Note that $(\dagger_2)$ is the immediate generalization of the formula obtained in Example 6.1. for the induced representation $[(\mathrm{Ind}_H^G \sigma)_{(x,Q)}](f) = \Pi_{(x,Q)}(f)$.

Most authors use ρ instead of σ. This clashes with our notation used for indexing the irreducible representations of the group H so we use σ instead.

A scalar field, namely a function $f\colon \mathbb{R}^d \to \mathbb{C}$ in $\mathrm{L}^2(\mathbb{R}^d)$, is the special case corresponding to $\mathcal{H} = \mathbb{C}$ and representation $\sigma\colon H \to \mathbf{U}(1)$ given by $\sigma(h) = \mathrm{id}_{\mathbb{C}}$ for all $h \in H$.

In this case, $\mathrm{Ind}_H^G \sigma = \mathbf{R}^{G \to \mathrm{L}^2(\mathbb{R}^d)}$, the left regular representation of G.

A vector field $f\colon \mathbb{R}^d \to \mathbb{C}^d$ corrresponds to the case where H is a closed subgroup of $\mathbf{GL}(d, \mathbb{C})$ and the representation $\sigma\colon H \to \mathbf{GL}(d, \mathbb{C})$ is the standard representation given by $\sigma(h) = h$, namely $\sigma(h)(x) = hx$ for any $x \in \mathbb{C}^d$, where h is a matrix in H.

Example 7.7. Let us show how G-feature maps $f\colon \mathbb{R}^d \times H \to \mathbb{C}$ in $\mathrm{L}^2(\mathbb{R}^d \rtimes H)$ can be viewed as feature fields $f^H\colon \mathbb{R}^d \to \mathrm{L}^2(H)$ (with $G = \mathbb{R}^d \rtimes H$).

The left regular representation $\mathbf{R}^{G\to \mathrm{L}^2(\mathbb{R}^d \rtimes H)}$ acts on G-feature maps *via*

$$(\mathbf{R}^{G\to \mathrm{L}^2(\mathbb{R}^d \rtimes H)}_{(x,h)} f)(x_1, h_1) = f(h^{-1} \cdot (x_1 - x), h^{-1}h_1),$$

$x, x_1 \in \mathbb{R}^d,\ h, h_1 \in H$

A G-feature map can be converted into a feature field as follows.

Given $f\colon \mathbb{R}^d \times H \to \mathbb{C}$ in $\mathrm{L}^2(\mathbb{R}^d \rtimes H)$, let $f^H\colon \mathbb{R}^d \to \mathrm{L}^2(H)$, where

$$(f^H(x))(h) = f(x,h), \quad x \in \mathbb{R}^d,\ h \in H.$$

From an intuitive point of view, for $h \in H$ fixed, the map $x \mapsto f(x,h)$ can be viewed as a sort of image based on $\mathbb{R}^d$, where the value $f(x,h)$ is the color at the location $x \in \mathbb{R}^d$. see Figure 7.3.

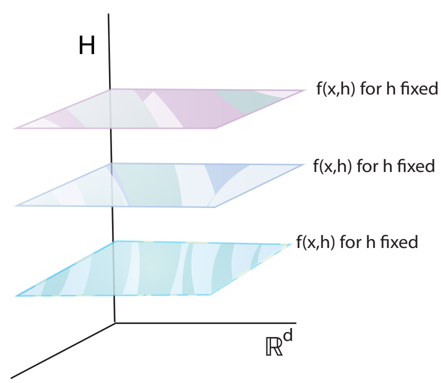

Figure 7.3: A schematic illustration of $f^H(x) = f(x,h)$, where $H = \mathbf{SO}(2)$. For each fixed $h \in H$, the image of $f(x,h)$ is the horizontal colored layer.

These images can be thought of as *parallel layers*, and *for x fixed, as h varies the color $f(x, h)$ moves along a sort of fibre* that passes through each of the layers "above x."

For $d = 2$ and $H = \mathbf{SO}(2)$, it is possible to visualize these fibres.

They are circles, but it is simpler to view them as line segments of height 2π with both endpoints identified. See Figure 7.4.

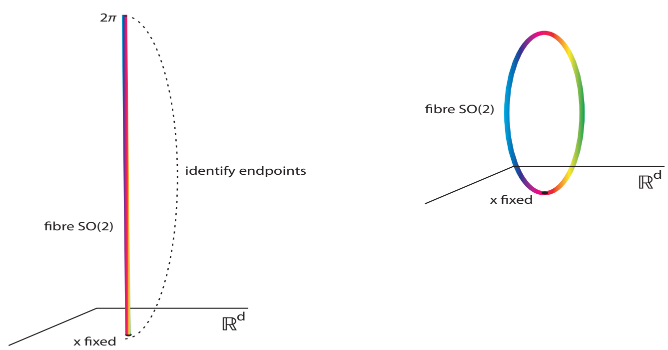

Figure 7.4: Two illustrations of the fibre $\mathbf{SO}(2)$ above a fixed $x \in \mathbb{R}^2$.

The left regular representation $\mathbf{R}^{H\to \mathrm{L}^2(H)}$ acts on $\mathrm{L}^2(H)$ in the usual way, namely

$$(\mathbf{R}_h^{H\to \mathrm{L}^2(H)} g)(h_1) = g(h^{-1}h_1), \quad g \in \mathbb{C}^H,\ h, h_1 \in H.$$

Then the induced representation $\mathrm{Ind}_H^G \mathbf{R}^{H\to \mathrm{L}^2(H)}$ (here $\sigma = \mathbf{R}^{H\to \mathrm{L}^2(H)}$) acts on the feature fields $f^H \colon \mathbb{R}^d \to \mathrm{L}^2(H)$ by

$$\begin{aligned}&[(\mathrm{Ind}_H^G \mathbf{R}^{H\to \mathrm{L}^2(H)})_{(x,h)} f^H](x_1)\\ &= \mathbf{R}_h^{H\to \mathrm{L}^2(H)}(f^H(h^{-1}\cdot(x_1 - x))).\end{aligned}$$

By definition of $\mathbf{R}^{H\to \mathrm{L}^2(H)}$ we get

$$\begin{aligned}&(\mathbf{R}_h^{H\to \mathrm{L}^2(H)}(f^H(h^{-1}\cdot(x_1-x))))(h_1)\\ &=(f^H(h^{-1}\cdot(x_1-x)))(h^{-1}h_1)\\ &=f(h^{-1}\cdot(x_1-x),h^{-1}h_1)=(\mathbf{R}_{(x,h)}^{G\to \mathrm{L}^2(\mathbb{R}^d\rtimes H)}f)(x_1,h_1).\end{aligned}$$

Therefore,

$$(\mathrm{Ind}_H^G\,\mathbf{R}^{H\to \mathrm{L}^2(H)})_{(x,h)}f^H=\mathbf{R}_{(x,h)}^{G\to \mathrm{L}^2(\mathbb{R}^d\rtimes H)}f,$$

which shows that G-feature maps $f\colon \mathbb{R}^d\times H\to\mathbb{C}$ can be viewed as feature fields $f^H\colon \mathbb{R}^d\to \mathrm{L}^2(H)$, using the left regular representations $\mathbf{R}^{H\to \mathrm{L}^2(H)}$.

In this case, $\mathcal{H}=\mathrm{L}^2(H)$ and $\sigma=\mathbf{R}^{H\to \mathrm{L}^2(H)}$.

Definition 7.4. Let $\sigma\colon H \to \mathbf{GL}(F)$ be a representation with F finite-dimensional. Define the function $\mathrm{Hom}(\sigma, \mathrm{id})$ by

$$\mathrm{Hom}(\sigma, \mathrm{id})_h f = f \circ \sigma_{h^{-1}}, \quad f \in \mathrm{Hom}(F, F),\ h \in H.$$

Actually, the representation $\mathrm{Hom}(\sigma, \mathrm{id})$ is a special case of the Hom representation in Definition 4.18 with $\sigma_1\colon H \to \mathbf{GL}(F)$ the representation $\sigma_1 = \sigma$ and σ_2 the trivial representation given by $\sigma_2(h) = \mathrm{id}_F$ for all $h \in H$.

If $F = \mathbb{C}^n$, then $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ is isomorphic to the space $\mathrm{M}_n(\mathbb{C})$ of $n \times n$ matrices, and if H is a closed subgroup of $\mathbf{GL}(n, \mathbb{C})$, then $\mathrm{Hom}(\sigma, \mathrm{id})$ acts on $\mathrm{M}_n(\mathbb{C})$ by multiplication on the right by the matrix σ_h^{-1}, namely

$$\mathrm{Hom}(\sigma, \mathrm{id})_h(A) = A\sigma_h^{-1}, \quad A \in \mathrm{M}_n(\mathbb{C}). \qquad (*_{22})$$

This is the situation that occurs in practice.

If $\mathbb{C}^n$ is equipped with its standard hermitian inner product and if $\sigma\colon H \to \mathbf{U}(n)$ is a unitary representation, so that σ_h is a unitary matrix, if we give $\mathrm{M}_n(\mathbb{C})$ the hermitian inner product $\langle A, B\rangle = \mathrm{tr}(B^* A)$, then the representation $\mathrm{Hom}(\sigma, \mathrm{id})$ is unitary because using the fact that $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ we have

$$\begin{aligned}\langle A\sigma_h^{-1}, B\sigma_h^{-1}\rangle &= \langle A\sigma_h^*, B\sigma_h^*\rangle \\ &= \mathrm{tr}((B\sigma_h^*)^*(A\sigma_h^*)) \\ &= \mathrm{tr}(\sigma_h B^* A\sigma_h^*) \\ &= \mathrm{tr}(\sigma_h^*\sigma_h B^* A) = \mathrm{tr}(B^* A) = \langle A, B\rangle.\end{aligned}$$

In the next section we show how to construct a Fourier transform on a semi-direct product $G = \mathbb{R}^d \rtimes H$ where H is compact in terms of the Fourier transform $\mathcal{F}$ on H.

7.4 Promoting the Fourier Transform from H to $\mathbb{R}^d \rtimes H$

If we view a function defined on $G = \mathbb{R}^d \rtimes H$ as a function $f\colon \mathbb{R}^d \rtimes H \to \mathbb{C}$, the new twist is that the Fourier coefficients of f are now tuples $(\widehat{f}_\rho)_{\rho \in R(H)}$ of *functions* $\widehat{f}_\rho\colon \mathbb{R}^d \to \mathrm{M}_{n_\rho}(\mathbb{C})$.

This causes new problems to reconstruct a function from its Fourier coefficients because even if the functions $\widehat{f}_\rho$ belong to $\mathrm{L}^2(\mathbb{R}^d, \mathrm{M}_{n_\rho}(\mathbb{C}))$, *there is no guarantee that the function obtained from the inverse Fourier transform belongs to* $\mathrm{L}^2(G)$.

Some additional condition is required on the functions $\widehat{f}_\rho$.

We provide a solution to this problem below by constructing a Hilbert space $\mathrm{L}^2(\mathbb{R}^d, \widehat{H})$ *such that the new Fourier transform* $\mathcal{F}^\tau \colon \mathrm{L}^2(G) \to \mathrm{L}^2(\mathbb{R}^d, \widehat{H})$ *and the Fourier cotransform* $\overline{\mathcal{F}^\tau} \colon \mathrm{L}^2(\mathbb{R}^d, \widehat{H}) \to \mathrm{L}^2(G)$ *are mutual inverses.*

We found the key idea in a paper by Mensah and Awussi [33] who investigate the situation of a semi-direct product $H \rtimes \mathbb{R}^d$, where $\mathbb{R}^d$ acts on H by automorphisms.

The first crucial observation is that for any function $f \in \mathrm{L}^2(\mathbb{R}^d \rtimes H)$, by Fubini, for any fixed $x \in \mathbb{R}^d$ we have $f^H(x) \in \mathrm{L}^2(H)$, where f^H is the function defined in Example 7.7.

Since H is a compact group, the Fourier transform $\mathcal{F}(f^H(x))$ is well-defined.

For every $\rho \in R(H)$ and every *fixed* $x \in \mathbb{R}^d$, recall that $\mathcal{F}(f^H(x))(\rho)$ is the $n_\rho \times n_\rho$ matrix given by

$$\begin{aligned}\mathcal{F}(f^H(x))(\rho) &= \int_H (f^H(x))(h) M_\rho(h)^* \, d\lambda(h) \\ &= \int_H f(x,h) M_\rho(h)^* \, d\lambda(h),\end{aligned}$$

where M_ρ is an irreducible representation of H in $\mathbb{C}^{n_\rho}$.

To reduce the amount of superscripts we also denote $f^H(x)$ as $f(x,-)$.

Technically $\mathcal{F}\colon \mathrm{L}^2(H) \to \mathrm{L}^2(\widehat{H})$ is defined for functions with domain H, with

$$\mathrm{L}^2(\widehat{H}) = \left\{ F \in \prod_{\rho \in R(H)} \mathrm{M}_{n_\rho}(\mathbb{C}) \mid \|F\|_{\mathrm{L}^2(\widehat{H})} < \infty \right\},$$

and

$$\begin{aligned} \|F\|_{\mathrm{L}^2(\widehat{H})} &= \left(\sum_{\rho \in R(H)} n_\rho \, \|F(\rho)\|_{\mathrm{HS}}^2 \right)^{1/2} \\ &= \left(\sum_{\rho \in R(H)} n_\rho \operatorname{tr}\Big(F(\rho)^* F(\rho) \Big) \right)^{1/2}; \end{aligned}$$

see Definition 4.22 and Definition 4.23.

The vector space $\mathrm{L}^2(\widehat{H})$ is a Hilbert space under the inner product

$$\begin{aligned}\langle F_1, F_2\rangle_{\mathrm{L}^2(\widehat{H})} &= \sum_{\rho\in R(H)} n_\rho \langle F_1(\rho), F_2(\rho)\rangle_{\mathrm{HS}} \\ &= \sum_{\rho\in R(H)} n_\rho \operatorname{tr}\Big(F_2(\rho)^* F_1(\rho)\Big);\end{aligned}$$

see Theorem 4.19.

We would like to define a notion of Fourier transform *on functions in* $\mathrm{L}^2(\mathbb{R}^d \rtimes H)$ that makes use of the Fourier transform $\mathcal{F}$ defined on H, so to avoid confusion we will denote this new Fourier transform by $\mathcal{F}^\tau$.

The motivation is that $\tau\colon H \to \mathbf{GL}(n)$ is the action of H on $\mathbb{R}^d$, with $\tau(h)(x) = hxh^{-1}$.

Definition 7.5. For any *fixed* $x \in \mathbb{R}^d$ and any G-feature map $f \in \mathrm{L}^2(\mathbb{R}^d \rtimes H)$, we define

$$\mathcal{F}(f(x,-)) = (\mathcal{F}(f(x,-))_\rho)_{\rho \in R(H)} \in \mathrm{L}^2(\widehat{H}),$$

also denoted $\widehat{f}(x)$, by

$$\mathcal{F}(f(x,-))_\rho = \widehat{f}(x)_\rho = \int_H f(x,h) M_\rho(h)^* \, d\lambda(h), \tag{$\widehat{f}(x)$}$$

with $\rho \in R(H)$.

Then if we *let x vary in $\mathbb{R}^d$*, for any *fixed* ρ we obtain a function $\widehat{f}_\rho \colon \mathbb{R}^d \to \mathrm{M}_{n_\rho}(\mathbb{C})$ given by

$$\widehat{f}_\rho(x) = \widehat{f}(x)_\rho = \int_H f(x,h) M_\rho(h)^* \, d\lambda(h), \quad x \in \mathbb{R}^d. \tag{$\widehat{f}_\rho$}$$

By Fubini, since $f \in \mathrm{L}^2(\mathbb{R}^d \rtimes H)$, we have $\widehat{f}_\rho \in \mathrm{L}^2(\mathbb{R}^d, \mathrm{M}_{n_\rho}(\mathbb{C}))$. This step requires a justification that we postpone for now.

The function $\widehat{f}_\rho$ is called a *Fourier coefficients feature field of type ρ* or *steerable feature field of type ρ*.

The $R(H)$-indexed family $(\widehat{f}_\rho)_{\rho \in R(H)}$ is denoted by $\widehat{f}$ and is called the *family of Fourier coefficients feature fields of f* or *family of steerable feature fields of f*.

Observe that

$$\widehat{f}(x) = (\widehat{f}_\rho(x))_{\rho \in R(H)} \in \mathrm{L}^2(\widehat{H}) \quad \text{for every } x \in \mathbb{R}^d,$$

and consequently $(\widehat{f}_\rho)_{\rho \in R(H)}$ belongs to the space $\mathfrak{E}^\tau(\widehat{H})$ defined next.

Definition 7.6. The vector space $\mathfrak{E}^\tau(\widehat{H})$ is defined by

$$\mathfrak{E}^\tau(\widehat{H}) = \Bigg\{ F \in \prod_{\rho \in R(H)} \mathrm{L}^2(\mathbb{R}^d, \mathrm{M}_{n_\rho}(\mathbb{C})) \\ \mid (F_\rho(x))_{\rho \in R(H)} \in \mathrm{L}^2(\widehat{H}),\ x \in \mathbb{R}^d \Bigg\}. \qquad (\mathfrak{E}^\tau(\widehat{H}))$$

Note the analogy with the space $\mathfrak{E}(\widehat{H})$ of Definition 4.23.

Definition 7.7. We define the map $\mathcal{F}^\tau$ from $\mathrm{L}^2(\mathbb{R}^d \rtimes H)$ to $\mathfrak{E}^\tau(\widehat{H})$ by setting

$$\begin{aligned} \mathcal{F}^\tau(f) &= (\mathcal{F}^\tau_\rho(f))_{\rho \in R(H)}, \quad f \in \mathrm{L}^2(G), \text{ with} \\ \mathcal{F}^\tau_\rho(f)(x) &= \widehat{f}_\rho(x) = \mathcal{F}(f(x,-))_\rho \\ &= \int_H f(x,h) M_\rho(h)^* \, d\lambda(h), \quad x \in \mathbb{R}^d, \ \rho \in R(H). \end{aligned} \tag{$\mathcal{F}^\tau$}$$

Observe that by Line $(\widehat{f}(x))$, for every fixed $x \in \mathbb{R}^d$, we have

$$\mathcal{F}^\tau(f)(x) = (\mathcal{F}^\tau_\rho(f)(x))_{\rho \in R(H)} = \mathcal{F}(f(x,-)). \tag{$\mathcal{F}^\tau(f)(x)$}$$

We will see shortly that steerable feature fields of type ρ transform under the representation $\mathrm{Hom}(M_\rho, \mathrm{id})$.

For this reason the space of steerable feature fields of type ρ is denoted by $\mathbf{FF}(\mathbb{R}^d, H, \mathrm{Hom}(M_\rho, \mathrm{id}))$.

These are *matrix-valued functions* $\widehat{f}_\rho \colon \mathbb{R}^d \to \mathrm{M}_{n_\rho}(\mathbb{C})$ that belong to $\mathrm{L}^2(\mathbb{R}^d, \mathrm{M}_{n_\rho}(\mathbb{C}))$.

Actually, we will see below (see Definition 7.8) that there is some extra condition on the family $(\widehat{f}_\rho)_{\rho \in R(H)}$ that ensures that Fourier inversion yields a function in $\mathrm{L}^2(G)$.

For every *fixed* $x \in \mathbb{R}^d$, the function $f^H(x) \in \mathrm{L}^2(H)$ can be recovered by Fourier inversion using the Fourier cotransform $\overline{\mathcal{F}}$ from $\mathrm{L}^2(\widehat{H})$ to $\mathrm{L}^2(H)$ from the family of Fourier coefficients feature fields $\widehat{f} = (\widehat{f}_\rho)_{\rho \in R(H)} \in \mathfrak{E}^\tau(\widehat{H})$ evaluated at x, namely the $R(H)$-indexed family $\widehat{f}(x) = (\widehat{f}_\rho(x))_{\rho \in R(H)} \in \mathrm{L}^2(\widehat{H})$, using the formula

$$
\begin{aligned}
(f^H(x))(h) &= [\overline{\mathcal{F}}(\widehat{f}(x))](h) \\
&= \sum_{\rho \in R(H)} n_\rho \operatorname{tr}\left(\widehat{f}_\rho(x) M_\rho(h)\right), \quad h \in H.
\end{aligned}
$$

Thus the G-feature map $f\colon \mathbb{R}^d \times H \to \mathbb{C}$ can also be recovered *pointwise, via*

$$f(x,h) = [\overline{\mathcal{F}}(\widehat{f}(x))](h) = \sum_{\rho \in R(H)} n_\rho \,\mathrm{tr}\Big(\widehat{f}_\rho(x) M_\rho(h)\Big). \qquad (\overline{\mathcal{F}}(\widehat{f}(x)))$$

The definition of a map $\overline{\mathcal{F}^\tau}$ from $\mathfrak{E}^\tau(\widehat{H})$ to $\mathrm{L}^2(\mathbb{R}^d \rtimes H)$ is more delicate.

The space $\mathfrak{E}^\tau(\widehat{H})$ is actually too big to ensure that the resulting functions belong to $\mathrm{L}^2(\mathbb{R}^d \rtimes H)$.

Inspired by Mensah and Awussi [33] we define the following space.

Definition 7.8. Define the vector space $\mathrm{L}^2(\mathbb{R}^d, \widehat{H})$ by

$$\mathrm{L}^2(\mathbb{R}^d, \widehat{H}) = \left\{ F \in \mathfrak{E}^\tau(\widehat{H}) \;\middle|\; \|F(-)\|_{\mathrm{L}^2(\widehat{H})} \in \mathrm{L}^2(\mathbb{R}^d) \right\}, \tag{$\mathrm{L}^2(\mathbb{R}^d, \widehat{H})$}$$

where $\|F(-)\|_{\mathrm{L}^2(\widehat{H})}$ is the function defined such that if $F = (F_\rho)_{\rho \in R(H)}$, then

$$\|F(x)\|_{\mathrm{L}^2(\widehat{H})} = \left(\sum_{\rho \in R(H)} n_\rho \, \|F_\rho(x)\|_{\mathrm{HS}}^2 \right)^{1/2}. \tag{$\|F(-)\|_{\mathrm{L}^2(\widehat{H})}$}$$

Note that $\|F(-)\|_{\mathrm{L}^2(\widehat{H})} \in \mathrm{L}^2(\mathbb{R}^d)$ implies that

$$\int_{\mathbb{R}^d} \|F(x)\|_{\mathrm{L}^2(\widehat{H})}^2 \, dx < \infty.$$

The vector space $\mathrm{L}^2(\mathbb{R}^d, \widehat{H})$ is equipped with the norm $\| \, \|_{\mathrm{L}^2(\mathbb{R}^d, \widehat{H})}$ given by

$$\begin{aligned} \|F\|^2_{\mathrm{L}^2(\mathbb{R}^d, \widehat{H})} &= \int_{\mathbb{R}^d} \|F(x)\|^2_{\mathrm{L}^2(\widehat{H})} \, dx \\ &= (\|F(-)\|^2_{\mathrm{L}^2(\widehat{H})})^2_{\mathrm{L}^2(\mathbb{R}^d)}. \qquad (\|F\|_{\mathrm{L}^2(\mathbb{R}^d, \widehat{H})}) \end{aligned}$$

Note the analogy with the definition of the space $\mathrm{L}^2(\widehat{H})$ in Definition 4.23.

We also define an inner product on $\mathrm{L}^2(\mathbb{R}^d, \widehat{H})$ as follows.

Definition 7.9. For any two sequences of functions $F_1, F_2 \in \mathrm{L}^2(\mathbb{R}^d, \widehat{H})$, let $\langle F_1, F_2 \rangle_{\mathrm{L}^2(\mathbb{R}^d, \widehat{H})}$ be given by

$$\langle F_1, F_2 \rangle_{\mathrm{L}^2(\mathbb{R}^d, \widehat{H})} = \int_{\mathbb{R}^d} \sum_{\rho \in R(H)} n_\rho \operatorname{tr}\Big(((F_2)_\rho(x))^* (F_1)_\rho(x) \Big)\, dx. \quad (\langle -, - \rangle)$$

Observe that

$$\|F\|^2_{\mathrm{L}^2(\mathbb{R}^d, \widehat{H})} = \langle F, F \rangle_{\mathrm{L}^2(\mathbb{R}^d, \widehat{H})},$$

but we still need to prove that the integral in $(\langle -, - \rangle)$ is well defined.

We will use the Cauchy-Schwarz inequality both in $\mathrm{L}^2(\widehat{H})$ and $\mathrm{L}^2(\mathbb{R}^d)$.

We have

$$
\begin{aligned}
&|\langle F_1, F_2\rangle_{\mathrm{L}^2(\mathbb{R}^d,\widehat{H})}| \\
&= \left|\int_{\mathbb{R}^d} \sum_{\rho\in R(H)} n_\rho \, \langle (F_1)_\rho(x), (F_2)_\rho(x)\rangle_{\mathrm{HS}} \, dx\right| && (1)\\
&\le \int_{\mathbb{R}^d} \left| \sum_{\rho\in R(H)} n_\rho \, \langle (F_1)_\rho(x), (F_2)_\rho(x)\rangle_{\mathrm{HS}} \right| dx && (2)\\
&= \int_{\mathbb{R}^d} \left| \langle F_1(x), F_2(x)\rangle_{\mathrm{L}^2(\widehat{H})} \right| dx && (3)\\
&\le \int_{\mathbb{R}^d} \|F_1(x)\|_{\mathrm{L}^2(\widehat{H})} \, \|F_2(x)\|_{\mathrm{L}^2(\widehat{H})} \, dx && (4)\\
&\le \left(\int_{\mathbb{R}^d} \|F_1(x)\|^2_{\mathrm{L}^2(\widehat{H})} \, dx\right)^{1/2} \left(\int_{\mathbb{R}^d} \|F_2(x)\|^2_{\mathrm{L}^2(\widehat{H})} \, dx\right)^{1/2} && (5)\\
&= \|F_1\|_{\mathrm{L}^2(\mathbb{R}^d,\widehat{H})} \, \|F_2\|_{\mathrm{L}^2(\mathbb{R}^d,\widehat{H})} \, , && (6)
\end{aligned}
$$

where (1) holds by definition, (2) by a standard property of the integral, (3) by definition of the inner product in $\mathrm{L}^2(\widehat{H})$, (4) by the Cauchy-Schwarz inequality in $\mathrm{L}^2(\widehat{H})$, (5) by the Cauchy-Schwarz inequality in $\mathrm{L}^2(\mathbb{R}^d)$, and (6) by definition (see ($\|F\|_{\mathrm{L}^2(\mathbb{R}^d,\widehat{H})}$)).

We will also need the projection $\mathrm{L}^2(\mathbb{R}^d, \widehat{H})_\rho$ of $\mathrm{L}^2(\mathbb{R}^d, \widehat{H})$ on the ρ-th factor, that is,

$$\mathrm{L}^2(\mathbb{R}^d, \widehat{H})_\rho = \{F_\rho \mid (F_\rho)_{\rho \in R(H)} \in \mathrm{L}^2(\mathbb{R}^d, \widehat{H})\}. \tag{$\mathrm{L}^2(\mathbb{R}^d, \widehat{H})_\rho$}$$

We have the following important version of Plancherel theorem for our Fourier transform $\mathcal{F}^\tau \colon \mathrm{L}^2(G) \to \mathrm{L}^2(\mathbb{R}^d, \widehat{H})$.

Theorem 7.2. *(Generalized Plancherel) The map* $\mathcal{F}^\tau \colon \mathrm{L}^2(G) \to \mathrm{L}^2(\mathbb{R}^d, \widehat{H})$ *(with* $G = \mathbb{R}^d \rtimes H$*) is an isometric isomorphism of Hilbert spaces. That is, it is bijective and*

$$\langle \mathcal{F}^\tau(f), \mathcal{F}^\tau(g) \rangle_{\mathrm{L}^2(\mathbb{R}^d, \widehat{H})} = \langle f, g \rangle_{\mathrm{L}^2(G)}, \quad f, g \in \mathrm{L}^2(\mathbb{R}^d \rtimes H).$$

In particular, it is continuous.

Proof. First we prove that the map $\mathcal{F}^\tau$ is an isometry.

Since $\mathrm{L}^2(G)$ is a Hilbert space, this proves that $\mathrm{L}^2(\mathbb{R}^d, \widehat{H})$ is also a Hilbert space.

Since the norm on $\mathrm{L}^2(\mathbb{R}^d, \widehat{H})$ is induced by the inner product on $\mathrm{L}^2(\mathbb{R}^d, \widehat{H})$, it suffices to prove that the norm is preserved.

This is a standard result of linear algebra; for example, see Gallier and Quaintance [24] (Chapter 13, Proposition 13.1).

For any $f \in \mathrm{L}^2(\mathbb{R}^d \rtimes H)$, we have

$$\begin{aligned} \|\mathcal{F}^\tau(f)\|^2_{\mathrm{L}^2(\mathbb{R}^d,\widehat{H})} &= \int_{\mathbb{R}^d} \|\mathcal{F}^\tau(f)(x)\|^2_{\mathrm{L}^2(\widehat{H})}\, dx \quad \text{by definition} \\ &= \int_{\mathbb{R}^d} \|\mathcal{F}(f(x,-))\|^2_{\mathrm{L}^2(\widehat{H})}\, dx \;\; \text{by } (\mathcal{F}^\tau(f)(x)). \end{aligned}$$

However, for fixed x, $\mathcal{F}(f(x,-))$ is the Fourier transform of the function $f(x,-) \in \mathrm{L}^2(H)$.

By Plancherel Theorem (Theorem 4.23), we have

$$\|\mathcal{F}(f(x,-))\|^2_{\mathrm{L}^2(\widehat{H})} = \|f(x,-)\|^2_{\mathrm{L}^2(H)}\,.$$

Since $f \in \mathrm{L}^2(\mathbb{R}^d \rtimes H)$, by Fubini

$$\begin{aligned}\|f\|^2_{\mathrm{L}^2(G)} &= \int_G |f(x,h)|^2 \, d\lambda_G(x,h) \\ &= \int_{\mathbb{R}^d}\int_H |f(x,h)|^2 \, d\lambda_H(h) \, dx < \infty,\end{aligned}$$

but

$$\int_{\mathbb{R}^d}\int_H |f(x,h)|^2 \, d\lambda_H(h) \, dx = \int_{\mathbb{R}^d} \|f(x,-)\|^2_{\mathrm{L}^2(H)} \, dx,$$

which shows that the function $x \mapsto \|\mathcal{F}(f(x,-))\|_{\mathrm{L}^2(\widehat{H})}$ is in $\mathrm{L}^2(\mathbb{R}^d)$.

Consequently, we have

$$
\begin{aligned}
&\|\mathcal{F}^\tau(f)\|^2_{\mathrm{L}^2(\mathbb{R}^d,\widehat{H})} \\
&= \int_{\mathbb{R}^d} \|\mathcal{F}(f(x,-))\|^2_{\mathrm{L}^2(\widehat{H})}\, dx \\
&= \int_{\mathbb{R}^d} \|f(x,-)\|^2_{\mathrm{L}^2(H)}\, dx && \text{by Plancherel} \\
&= \int_{\mathbb{R}^d}\int_H |f(x,h)|^2\, d\lambda_H(h)\, dx && \text{by definition of the } \mathrm{L}^2(H)\text{-norm} \\
&= \|f\|^2_{\mathrm{L}^2(G)}\,. && \text{by Fubini}
\end{aligned}
$$

Since $\mathcal{F}^\tau$ is an isometry, it is injective. It remains to prove that it is surjective.

For any $F = (F_\rho)_{\rho \in R(H)} \in \mathrm{L}^2(\mathbb{R}^d, \widehat{H})$ and for every fixed $x \in \mathbb{R}^d$, we have

$$F(x) = (F_\rho(x))_{\rho \in R(H)} \in \mathrm{L}^2(\widehat{H}).$$

By Plancherel applied to the Fourier transform $\mathcal{F}$ between $\mathrm{L}^2(H)$ and $\mathrm{L}^2(\widehat{H})$, there is a unique function $f_x \in \mathrm{L}^2(H)$ such that

$$\mathcal{F}(f_x) = F(x) \quad \text{and} \quad \|f_x\|_{\mathrm{L}^2(H)} = \|F(x)\|_{\mathrm{L}^2(\widehat{H})}. \qquad (*_{23})$$

Define the function $f\colon \mathbb{R}^d \rtimes H \to \mathbb{C}$ by

$$f(x, h) = f_x(h) \quad x \in \mathbb{R}^d,\, h \in H. \qquad (*_{24})$$

Observe that

$$f(x, -) = f_x, \qquad (*_{25})$$

so we get

$$\begin{aligned}
&\|f\|_{\mathrm{L}^2(G)} \\
&= \int_{\mathbb{R}^d}\int_H |f(x,h)|^2\, d\lambda_H(h)\, dx && \text{by definition of } \|f\|^2_{\mathrm{L}^2(\mathrm{G})} \\
&= \int_{\mathbb{R}^d}\int_H |f_x(h)|^2\, d\lambda_H(h)\, dx && \text{by } (*_{24}) \\
&= \int_{\mathbb{R}^d} \|f_x\|^2_{\mathrm{L}^2(\mathrm{H})}\, dx && \text{by definition of } \|f_x\|^2_{\mathrm{L}^2(\mathrm{H})} \\
&= \int_{\mathbb{R}^d} \|F(x)\|^2_{\mathrm{L}^2(\widehat{\mathrm{H}})}\, dx && \text{by } (*_{23}) \\
&= \|F\|^2_{\mathrm{L}^2(\mathbb{R}^d, \widehat{H})} < \infty, && \text{by definition of } \|F\|^2_{\mathrm{L}^2(\mathbb{R}^d, \widehat{H})}
\end{aligned}$$

and the last step because $F \in \mathrm{L}^2(\mathbb{R}^d, \widehat{H})$.

Therefore $f \in \mathrm{L}^2(G)$. Then by $(\mathcal{F}^\tau(f)(x))$, $(*_{25})$ and $(*_{23})$, we have

$$\mathcal{F}^\tau(f)(x) = \mathcal{F}(f(x,-)) = \mathcal{F}(f_x) = F(x), \quad x \in \mathbb{R}^d,$$

which means that $\mathcal{F}^\tau(f) = F$, and thus $\mathcal{F}^\tau$ is indeed surjective. □

Since we already know that functions in $\mathrm{L}^2(G)$ can be recovered pointwise using the Fourier transform on H, we can exhibit the inverse $\overline{\mathcal{F}^\tau}$ of the Fourier transform $\mathcal{F}^\tau$.

Definition 7.10. Define the map

$\overline{\mathcal{F}^\tau}_\rho \colon \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_\rho \to \mathrm{L}^2(G)$

for every $\rho \in R(H)$ by

$$\begin{aligned} \overline{\mathcal{F}^\tau}_\rho(\widehat{f}_\rho)(x,h) &= n_\rho \operatorname{tr}\left(\widehat{f}_\rho(x) M_\rho(h)\right), \\ & x \in \mathbb{R}^d,\ h \in H,\ \widehat{f}_\rho \in \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_\rho, \end{aligned} \tag{$\overline{\mathcal{F}^\tau}_\rho$}$$

and the map $\overline{\mathcal{F}^\tau} \colon \mathrm{L}^2(\mathbb{R}^d, \widehat{H}) \to \mathrm{L}^2(G)$ by

$$\begin{aligned} \overline{\mathcal{F}^\tau}((\widehat{f}_\rho)_{\rho\in R(H)})(x,h) &= \sum_{\rho \in R(H)} \overline{\mathcal{F}^\tau}_\rho(\widehat{f}_\rho)(x,h), \\ & x \in \mathbb{R}^d,\ h \in H,\ (\widehat{f}_\rho)_{\rho\in R(H)} \in \mathrm{L}^2(\mathbb{R}^2, \widehat{H}). \end{aligned} \tag{$\overline{\mathcal{F}^\tau}$}$$

Then $\mathcal{F}^\tau\colon \mathrm{L}^2(G) \to \mathrm{L}^2(\mathbb{R}^d, \widehat{H})$ and $\overline{\mathcal{F}^\tau}\colon \mathrm{L}^2(\mathbb{R}^d, \widehat{H}) \to \mathrm{L}^2(G)$ are mutual inverses.

We claim that the map $\widehat{f_\rho} \in \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_\rho$ is indeed a feature field, with $\mathcal{H} = \mathrm{M}_{n_\rho}(\mathbb{C})$ and $\sigma = \mathrm{Hom}(M_\rho, \mathrm{id})$.

For this we need to see how the function $\widehat{f_\rho}$ changes when $G = \mathbb{R}^d \rtimes H$ acts on f *via* the left regular action $\mathbf{R}^{G \to \mathrm{L}^2(G)}$ given by

$$\mathbf{R}^{G \to \mathrm{L}^2(G)}_{(x,h)}(f)(x_1, h_1) = f(h^{-1} \cdot (x_1 - x), h^{-1} h_1).$$

Proposition 7.3. *For every $\rho \in R(H)$, let $\sigma_\rho \colon H \to \mathbf{U}(\mathrm{M}_{n_\rho}(\mathbb{C}))$ be the representation*

$$\sigma_\rho = \mathrm{Hom}(M_\rho, \mathrm{id})$$

associated with the representation $M_\rho \colon H \to \mathbf{U}(\mathbb{C}^{n_\rho})$ as in Definition 7.4. For every function $\widehat{f}_\rho \in \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_\rho$, we have

$$\begin{aligned} \mathcal{F}^\tau_\rho[\mathbf{R}^{G \to \mathrm{L}^2(G)}_{(x,h)}(f)](x_1) &= [(\mathrm{Ind}_H^G\,(\sigma_\rho)_{(x,h)}\,\widehat{f}_\rho](x_1) \\ &= \widehat{f}_\rho(h^{-1} \cdot (x_1 - x))\, M_\rho(h)^*. \end{aligned} \qquad (*_{26})$$

Proof. Using the fact that the Haar measure λ is left (and right) invariant and the fact that M_ρ is a representation, we have

$$\begin{aligned}
&\mathcal{F}^\tau_\rho[\mathbf{R}^{G\to \mathrm{L}^2(G)}_{(x,h)}(f)](x_1)\\
&= \int_H \mathbf{R}^{G\to \mathrm{L}^2(G)}_{(x,h)}(f)(x_1, h_1) M_\rho(h_1)^*\, d\lambda(h_1)\\
&= \int_H f(h^{-1}\cdot (x_1 - x), h^{-1}h_1) M_\rho(h_1)^*\, d\lambda(h_1)\\
&= \int_H f(h^{-1}\cdot (x_1 - x), h_2) M_\rho(hh_2)^*\, d\lambda(h_2) && h_1 = hh_2\\
&= \left(\int_H f(h^{-1}\cdot (x_1 - x), h_2) M_\rho(h_2)^*\, d\lambda(h_2)\right) M_\rho(h)^*\\
&= \widehat{f}_\rho(h^{-1}\cdot (x_1 - x))\, M_\rho(h)^*.
\end{aligned}$$

The above computation shows that

$$\mathcal{F}^\tau_\rho[\mathbf{R}^{G\to \mathrm{L}^2(G)}_{(x,h)}(f)](x_1) = \widehat{f}_\rho(h^{-1}\cdot (x_1 - x))\, M_\rho(h)^*,$$

as claimed. □

Equation $(*_{26})$ shows that the group $G = \mathbb{R}^d \rtimes H$ acts on the feature fields of type ρ *via*

$$[(\mathrm{Ind}_H^G\,(\sigma_\rho)_{(x,h)}\,\widehat{f}_\rho](x_1) = \widehat{f}_\rho(h^{-1}\cdot(x_1 - x))\,M_\rho(h)^*, \tag{σ_ρ}$$

for all $(x, h) \in \mathbb{R}^d \rtimes H$ and all $x_1 \in \mathbb{R}^d$, and $(*_{26})$ is equivalent to the commutativity of the following diagram

$$\begin{CD}
\mathrm{L}^2(G) @>{\mathcal{F}_\rho^\tau}>> \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_\rho \\
@V{\mathbf{R}_{(x,h)}^{G\to \mathrm{L}^2(G)}}VV @VV{(\mathrm{Ind}_H^G\,\sigma_\rho)_{(x,h)}}V \\
\mathrm{L}^2(G) @>>{\mathcal{F}_\rho^\tau}> \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_\rho
\end{CD}$$

for all $(x, h) \in G = \mathbb{R}^d \rtimes H$.

We also package the representations $\operatorname{Ind}_H^G \sigma_\rho \colon G \to \mathbf{U}(\mathrm{L}^2(\mathbb{R}^d, \widehat{H})_\rho)$ in the map

$$\operatorname{Ind}_H^G \sigma \colon G \times \mathrm{L}^2(\mathbb{R}^d, \widehat{H}) \to \mathrm{L}^2(\mathbb{R}^d, \widehat{H})$$

defined such that for any $\widehat{f} = (\widehat{f}_\rho)_{\rho \in R(H)}$,

$$[(\operatorname{Ind}_H^G \sigma)_{(x,h)} \widehat{f}]_\rho(x_1) = [(\operatorname{Ind}_H^G \sigma_\rho)_{(x,h} \widehat{f}_\rho](x_1), \quad x_1 \in \mathbb{R}^d,\ \rho \in R(H). \qquad (\sigma)$$

The following result should not be too surprising.

Proposition 7.4. *The following diagram commutes*

$$\begin{array}{ccc} \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_\rho & \xrightarrow{\ \overline{\mathcal{F}^\tau}_\rho\ } & \mathrm{L}^2(G) \\ {\scriptstyle (\mathrm{Ind}_H^G \sigma_\rho)_{(x,h)}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{R}^{G\to \mathrm{L}^2(G)}_{(x,h)}} \\ \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_\rho & \xrightarrow[\ \overline{\mathcal{F}^\tau}_\rho\ ]{} & \mathrm{L}^2(G) \end{array}$$

for all $(x, h) \in G = \mathbb{R}^d \rtimes H$*.*

Proof. For any $\widehat{f}_\rho \in \mathrm{L}^2(\mathbb{R}^d, \widehat{H})_\rho$ we have

$$
\begin{aligned}
&\overline{\mathcal{F}^\tau}_\rho((\mathrm{Ind}_H^G \sigma_\rho)_{(x,h)} \widehat{f}_\rho)(x_1, h_1) \\
&= n_\rho \operatorname{tr}\Big(((\mathrm{Ind}_H^G \sigma_\rho)_{(x,h)} \widehat{f}_\rho)(x_1) M_\rho(h_1) \Big) && \text{by } (\overline{\mathcal{F}^\tau}_\rho) \\
&= n_\rho \operatorname{tr}\Big(\widehat{f}_\rho(h^{-1} \cdot (x_1 - x)) M_\rho(h)^* M_\rho(h_1) \Big) && \text{by } (*_{26}) \\
&= n_\rho \operatorname{tr}\Big(\widehat{f}_\rho(h^{-1} \cdot (x_1 - x)) M_\rho(h^{-1} h_1) \Big).
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\mathbf{R}_{(x,h)}(\overline{\mathcal{F}^\tau}_\rho(\widehat{f}_\rho))(x_1, h_1) \\
&= \overline{\mathcal{F}^\tau}_\rho(\widehat{f}_\rho)(h^{-1} \cdot (x_1 - x)), h^{-1} h_1) && \text{by definition of } \mathbf{R}_{(x,h)} \\
&= n_\rho \operatorname{tr}\Big(\widehat{f}_\rho(h^{-1} \cdot (x_1 - x)) M_\rho(h^{-1} h_1) \Big). && \text{by } (\overline{\mathcal{F}^\tau}_\rho)
\end{aligned}
$$

Consequently

$$\overline{\mathcal{F}^{\tau}}_{\rho}((\mathrm{Ind}_H^G\,\sigma_\rho)_{(x,h)}\widehat{f}_\rho)(x_1,h_1) = \mathbf{R}_{(x,h)}(\overline{\mathcal{F}^{\tau}}_{\rho}(\widehat{f}_\rho))(x_1,h_1),$$

as claimed. $\square$

Remark: We also have the representation $\mathrm{Hom}(\mathrm{id}, M_\rho)$ which acts on $\mathrm{M}_{n_\rho}(\mathbb{C})$ by multiplication on the left by $M_\rho(h)$ for every $h \in H$.

The induced representation $\mathrm{Hom}(\mathrm{id}, M_\rho)$ of $\mathbb{R}^d \rtimes H$ on the feature fields of type ρ is then given by

$$\begin{aligned}
&[(\mathrm{Ind}_H^G\,\mathrm{Hom}(\mathrm{id}, M_\rho))_{(x,h)}\,\widehat{f}_\rho](x_1)\\
&= \mathrm{Hom}(\mathrm{id}, M_\rho)(h)(\widehat{f}_\rho(h^{-1}\cdot(x_1 - x)))\\
&= M_\rho(h)\widehat{f}_\rho(h^{-1}\cdot(x_1 - x)),
\end{aligned}$$

for all $(x,h) \in \mathbb{R}^d \rtimes H$ and all $x_1 \in \mathbb{R}^d$. It is a bit more natural than the representation induced by $\mathrm{Hom}(M_\rho, \mathrm{id})$.[1]

[1] Which representation arises naturally depends on the definition of the Fourier transform. The literature is not consistent on this matter. For exampe, Bekkers uses M_ρ instead of M_ρ^*.

Example 7.8. Let $H = \mathbf{SO}(2)$ so that
$G = \mathbb{R}^2 \rtimes \mathbf{SO}(2) = \mathbf{SE}(2)$.

In this case, $R(\mathbf{SO}(2)) = \mathbb{Z}$ and $n_\rho = 1$. We will denote ρ as ℓ.

For any $f \in \mathrm{L}^2(\mathbf{SE}(2))$, for every $x \in \mathbb{R}^2$, the Fourier transform $\mathcal{F}^\tau(f)$ of f is the $\mathbb{Z}$-indexed sequence $(\widehat{f}_\ell)_{\ell\in\mathbb{Z}}$ of functions given by

$$\begin{aligned}\widehat{f}_\ell(x) &= \mathcal{F}^\tau(f(x,-))_\ell \\ &= \int_{S^1} e^{-i\ell\theta} f(x,\theta)\, d\theta, \quad x \in \mathbb{R}^2,\, \ell \in \mathbb{Z}.\end{aligned}$$

The functions $\widehat{f}_\ell$ are the feature fields associated with ℓ. Observe that this is an example of $(\widehat{f}_\rho)$.

Given a family $\widehat{f} = (\widehat{f}_m)_{m\in\mathbb{Z}}$ of function $\widehat{f}_m \in \mathrm{L}^2(\mathbb{R}^2, \mathbb{Z})_m$ such that $\widehat{f}(x) = (\widehat{f}_m(x))_{m\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ for all $x \in \mathbb{R}^2$ and

$$\left(\sum_{m=-\infty}^{\infty} |\widehat{f}_m(-)|^2\right)^{1/2} \in \mathrm{L}^2(\mathbb{R}^2),$$

the Fourier cotransform $\overline{\mathcal{F}^\tau}(\widehat{f})(x, \theta)$ is given by

$$\overline{\mathcal{F}^\tau}(\widehat{f})(x, \theta) = \sum_{m=-\infty}^{\infty} \widehat{f}_m(x) e^{im\theta}.$$

It is instructive to see in this more concrete case how the function $\widehat{f}_\ell$ changes when $\mathbf{SE}(2) = \mathbb{R}^2 \rtimes \mathbf{SO}(2)$ acts on f *via* the left regular action $\mathbf{R}^{\mathbf{SE}(2)\to \mathrm{L}^2(\mathbf{SE}(2))}$ given by

$$\mathbf{R}^{\mathbf{SE}(2)\to \mathrm{L}^2(\mathbf{SE}(2))}_{(x,\theta)}(f)(x_1,\theta_1) = f(R_{-\theta}(x_1 - x), \theta_1 - \theta).$$

Using the fact that the Haar measure on $\mathbf{SO}(2)$ is left (and right) invariant, we have

$$\begin{aligned}
&\mathcal{F}^\tau[\mathbf{R}^{\mathbf{SE}(2)\to \mathrm{L}^2(\mathbf{SE}(2))}_{(x,\theta)}(f)(x_1, -)]_\ell \\
&= \int_{\mathbf{SO}(2)} \mathbf{R}^{\mathbf{SE}(2)\to \mathrm{L}^2(\mathbf{SE}(2))}_{(x,\theta)}(f)(x_1,\theta_1) e^{-i\ell\theta_1}\, d\theta_1 \\
&= \int_{\mathbf{SO}(2)} f(R_{-\theta}(x_1 - x), \theta_1 - \theta) e^{-i\ell\theta_1}\, d\theta_1 \\
&= \int_{\mathbf{SO}(2)} f(R_{-\theta}(x_1 - x), \theta_2) e^{-i\ell(\theta+\theta_2)}\, d\theta_2 \qquad \theta_1 = \theta + \theta_2 \\
&= e^{-i\ell\theta} \int_{\mathbf{SO}(2)} f(R_{-\theta}(x_1 - x), \theta_2) e^{-i\ell\theta_2}\, d\theta_2 \\
&= e^{-i\ell\theta} \widehat{f}_\ell(R_{-\theta}(x_1 - x)).
\end{aligned}$$

Thus we have

$$\widehat{\mathbf{R}_{(x,\theta)}(f)}_{\ell}(x_1) = e^{-i\ell\theta}\widehat{f}_{\ell}(R_{-\theta}(x_1 - x)),$$

so the representation that needs to be associated with the feature fields corresponding to ℓ is $e^{-i\ell\theta}$, and not $e^{i\ell\theta}$.

Since multiplication in $\mathbb{C}$ is commutative, given a character $\chi_\ell(\theta) = e^{i\ell\theta}$, the representation $\mathrm{Hom}(\chi_\ell, \mathrm{id})$ is just multiplication by $e^{-i\ell\theta}$ and the representation $\mathrm{Hom}(\mathrm{id}, \chi_\ell)$ is just multiplication by $e^{i\ell\theta}$.