Chapter 6

Induced Representations

If G is a locally compact group and if H is a closed subgroup of G, under certain conditions, it is possible to construct a Hilbert space $\mathcal{H}$ and a unitary representation $\Pi\colon G \to \mathbf{U}(\mathcal{H})$ of G in $\mathcal{H}$ from a unitary representation $U\colon H \to \mathbf{U}(E)$ of H in a (separable) Hilbert space E.

The representation Π is called an *induced representation* and it is often denoted by $\mathrm{Ind}_H^G U$.

There are two approaches for the construction of the Hilbert space $\mathcal{H}$:

1. The Hilbert space $\mathcal{H}$ is a set of functions from $X = G/H$ to E.
2. The Hilbert space $\mathcal{H}$ is a set of functions from G to E.

In the first approach we will construct unitary representations of G in $\mathcal{H}$ using certain functions $\alpha\colon G \times (G/H) \to \mathbf{GL}(E)$ called *cocycles*.

In the second approach the construction of the Hilbert space $\mathcal{H}$ is more complicated, but the definition of the operator Π_s is simpler.

The general construction (in the first approach) consists of seven steps, where the first four are purely algebraic and do not deal with continuous unitary representations, but instead linear representations (group homomorphisms $U: G \to \mathbf{GL}(E)$, where G is a group not equipped with any topology and E is just a vector space with no additional structure):

(1) Let G be a group acting (on the left) on a set X, say $(s,x) \mapsto s \cdot x$ ($s \in G, x \in X$), and let E be a vector space. In Section 6.1 we define the notion of *equilinear action* of G on $X \times E$, which is an action of the form

$$s \cdot (x,z) = (s \cdot x, \alpha(s,x)(z)), \quad s \in G, x \in X, z \in E,$$

where $\alpha(s,x)$ is a linear automorphism of E satisfying the conditions

(a) For all $x \in X$

$$\alpha(e,x) = \mathrm{id}_E.$$

(b) For all $x \in X$ and all $s,t \in G$,

$$\alpha(st,x) = \alpha(s,t \cdot x) \circ \alpha(t,x).$$

A map $\alpha\colon G \times X \to \mathbf{GL}(E)$ satisfying Conditions (a) and (b) is called a *cocycle of G with values in $\mathbf{GL}(E)$*.

Conversely, an action of G on X and a cocycle $\alpha\colon G \times X \to \mathbf{GL}(E)$ determines an equilinear action of G on $X \times E$.

Then we show that an equilinear action of G on $X \times E$ induces a homomorphism $\Pi\colon G \to \mathbf{GL}(E^X)$, where E^X is the vector space of all functions from X to E.

More precisely, for every function $f\colon X \to E$, for every $s \in G$, $\Pi_s(f)\colon X \to E$ is function given by

$$(\Pi_s(f))(x) = \alpha(s, s^{-1} \cdot x)(f(s^{-1} \cdot x)), \quad \text{for every } x \in X.$$

(2) In Section 6.2 we specialize the construction to the homogeneous space $X = G/H$ of left cosets.

Then G acts on G/H on the left by

$$s \cdot (gH) = sgH.$$

By choosing a *set of representatives $(r_x)_{x \in G/H}$ in the cosets of $X = G/H$* (with $x_0 = H$ and $r_{x_0} = e$), a cocycle $\alpha \colon G \times X \to \mathbf{GL}(E)$ determines a homomorphism $\sigma \colon H \to \mathbf{GL}(E)$ given by $\sigma(h) = \alpha(h, x_0)$ and a map $\beta \colon X \to \mathbf{GL}(E)$ given by $\beta(x) = \alpha(r_x, x_0)$.

If $\pi \colon G \to G/H$ denotes the projection map then picking a set of coset representatives $(r_x)_{x \in G/H}$ is equivalent to picking a *section of $\pi \colon G \to G/H$*, namely a function $r \colon G/H \to G$ such that

$$\pi \circ r = \mathrm{id}_{G/H}.$$

Conversely, a homomorphism $\sigma\colon H \to \mathbf{GL}(E)$ and a map $\beta\colon X \to \mathbf{GL}(E)$ determine a cocycle $\alpha\colon G \times X \to \mathbf{GL}(E)$.

In fact, we may restrict ourselves to the map β given by $\beta(x) = \mathrm{id}_E$, and if we define $u\colon G \times X \to H$ by

$$u(s, x) = r_{s\cdot x}^{-1} s r_x,$$

the map $\alpha\colon G \times X \to \mathbf{GL}(E)$ given by

$$\alpha(s, x) = \sigma(u(s, x))$$

is a cocycle.

The *induced representation* associated with $\sigma\colon H \to \mathbf{GL}(E)$ is given by

$$(\Pi_s(f))(x) = \sigma(u(s, s^{-1} \cdot x))(f(s^{-1} \cdot x)),$$
$$f \in E^X,\ x \in X.$$

The induced representation $\Pi\colon G \to \mathbf{GL}(E^X)$ is usually denoted by

$$\mathrm{Ind}_H^G\, \sigma\colon G \to \mathbf{GL}(E^X).$$

This step is the most important application of Step 1, and E is an arbitrary vector space.

(3) For a given homomorphism $\sigma\colon H \to \mathbf{GL}(E)$, the homomorphisms $\Pi\colon G \to \mathbf{GL}(E^X)$ corresponding to cocycles associated with different maps β are equivalent.

(4) In Section 6.3 we show that a cocycle $\alpha\colon G \times X \to \mathbf{GL}(E)$ determines a bijection between E^X and a subspace L^α of the set E^G of maps from G to E defined by

$$L^\alpha = \{f \in E^G \mid f(sh) = \sigma(h^{-1})(f(s)),\\ s \in G,\ h \in H\}.$$

Remark: The functions in L^α may be called *Frobenius functions*, but they are often incorrectly called *Mackey functions*.

As a consequence, the representation $\Pi\colon G \to \mathbf{GL}(E^X)$ corresponding to a cocycle α is equivalent to the representation $\Pi_{L^\alpha}\colon G \to \mathbf{GL}(L^\alpha)$ given by

$$((\Pi_{L^\alpha})_s(g))(t) = g(s^{-1}t) \quad \text{for all } g \in L^\alpha \text{ and all } s, t \in G.$$

Observe that *this is simply the left regular representation of L^α*.

The issue of choosing between representations in the space E^X or representations in the space L^α comes up in Chapter 7.

This completes the purely algebraic construction.

The next steps use topology and analysis to construct *unitary* representations.

(5) In Section 6.4 we assume that G is a locally compact group and H is a closed subgroup of G, in which case G/H is also locally compact.

Let μ be a positive measure on $X = G/H$, and assume that E is a separable Hilbert space. We then define a Hilbert space $\mathcal{L}^2_\mu(X; E)$ consisting of measurable functions from X to E.

(6) In Section 6.5, given a unitary representation U of H in E, we assume that the measure μ on $X = G/H$ is G-invariant and that the cocycle α satisfies the conditions:

(i) The linear automorphisms $\alpha(s, x)$ of E are unitary operators of E for all $s \in G$ and all $x \in G/H$, and $\alpha(h, x_0) = U(h)$ for all $h \in H$ (where x_0 denotes the coset H).

(ii) For every $s \in G$, for every $f \in \mathcal{L}^2_\mu(X; E)$, the map $x \mapsto \alpha(s, x)(f(x))$ from X to E is μ-measurable.

(iii) For every $f \in \mathcal{L}^2_\mu(X; E)$, the map $s \mapsto [\Pi_s(f)]$ from G to $\mathrm{L}^2_\mu(X; E)$ is continuous.

Then the homomorphism $s \mapsto \Pi_s([f]) = [\Pi_s(f)]$ is a unitary representation of G in $\mathrm{L}^2_\mu(X; E) = \mathcal{H}$.

(7) In Sections 6.6 and 6.7 we generalize the previous construction to certain measure called *quasi-invariant*. If the measure μ on G/H is quasi-invariant and another technical condition is satisfied, then the homomorphism $s \mapsto \Pi_s([f]) = [\Pi_s(f)]$ is a unitary representation of G in $\mathrm{L}^2_\mu(X; E)$.

Quasi-invariant measures on G/H always exist and can be constructed using rho-functions.

In Section 6.8 we illustrate the method of Section 6.7 by showing how to construct unitary representations of $\mathbf{SL}(2, \mathbb{R})$ using induced representations. One example involves the action of $\mathbf{SL}(2, \mathbb{R})$ on the projective line $\mathbb{RP}^1$, and the other example involves the action of $\mathbf{SL}(2, \mathbb{R})$ on the upper half plane.

In Section 6.9 we consider a compact (metrizable) group G and a closed subgroup H of G, and our goal is to determine the canonical (unitary) representation of G in $\mathrm{L}^2_\mu(G/H;\mathbb{C})$ induced by the trivial representation of H in $E = \mathbb{C}$ (see Definition 6.11), where μ is the G-invariant measure on G/H induced by a Haar measure λ on G.

For simplicity of notation we write $\mathrm{L}^2_\mu(G/H)$ instead of $\mathrm{L}^2_\mu(G/H;\mathbb{C})$.

To do this it is necessary to understand what is the restriction of the representation $M_\rho\colon G \to \mathbf{U}(\mathbb{C}^{n_\rho})$ to H, with $\rho \in R(G)$.

In Proposition 6.10 we show that the space $\mathrm{L}^2_\mu(G/H)$ is the Hilbert sum of subspaces $L_\rho \subseteq \mathfrak{a}_\rho$.

If the trivial representation σ_0 of H is contained $d = (\rho : \sigma_0) \geq 1$ times in the restriction of M_ρ to H, then L_ρ is the direct sum of the first d columns of the matrix $M_\rho^{(H)} = P^* M_\rho P$, where P is a suitable change of basis matrix, namely,

$$L_\rho = \bigoplus_{j=1}^{d} \mathfrak{l}_j^{(\rho,H)} \quad \text{and} \quad \mathfrak{l}_j^{(\rho,H)} = \bigoplus_{k=1}^{n_\rho} \mathbb{C} m_{kj}^{(\rho,H)}.$$

If $d = 0$, then $L_\rho = (0)$.

6.1 Cocycles and Induced Representations

As a warm up and as an example of the second approach, we consider the case where G is compact, H is a closed subgroup of G, and U is a linear representation of H in a *finite-dimensional* vector space E.

This means that U is a homomorphism $U\colon H \to \mathbf{GL}(E)$ and that

Condition (C) of Definition 2.1 is dropped.

Consider the Hilbert space $\mathrm{L}^2(G; E)$ consisting of all functions $f\colon G \to E$ such that for any orthonormal basis $(e_1, \ldots, e_n)$ of E, $f = f_1e_1 + \cdots + f_ne_n$, where the f_i are functions in $\mathrm{L}^2(G)$.

Equivalently, $\mathrm{L}^2(G;E)$ is the finite Hilbert sum $\mathrm{L}^2(G;E) = \bigoplus_{i=1}^n \mathrm{L}^2(G)e_i$.

The inner product of two functions $f = \sum_{i=1} f_i e_i$ and $g = \sum_{i=1} g_i e_i$ is

$$\langle f, g\rangle = \sum_{i=1}^{n} \int_G f_i(s)\overline{g_i(s)}\, d\lambda(s),$$

where λ is a Haar measure on G. This construction will be generalized in Section 6.4 to an infinite-dimensional Hilbert space.

Consider the subspace $\mathcal{H}$ of $\mathrm{L}^2(G; E)$ consisting of all functions f such that

$$f(sh) = U(h^{-1})(f(s)), \text{ for all } s \in G \text{ and all } h \in H. \tag{$*$}$$

It is easy to check that $\mathcal{H}$ is closed in $\mathrm{L}^2(G; E)$, so it is a Hilbert space.

For any $f \in \mathcal{H}$, as before, let $\lambda_s f$ be the function given by

$$(\lambda_s f)(t) = f(s^{-1}t), \qquad s, t \in G.$$

If we define the map $\Pi\colon G \to \mathbf{GL}(\mathcal{H})$ by

$$\Pi_s(f) = \lambda_s f, \qquad s \in G,\ f \in \mathcal{H},$$

equivalently

$$(\Pi_s(f))(t) = f(s^{-1}t), \qquad s, t \in G,\ f \in \mathcal{H},$$

then we see that Π *is a linear representation of* G *in* $\mathcal{H}$ (Condition (C) of Definition 2.1 may fail, but here we are not considering continuous representations).

Since the Haar measure is left and right invariant, the maps $\lambda_t f$ are unitary ($f \in \mathcal{H}$), so $\Pi\colon G \to \mathbf{GL}(\mathcal{H})$ is a unitary representation of G in $\mathcal{H}$, called the representation *induced* by $U\colon H \to \mathbf{GL}(E)$.

Let us now consider a more general situation.

Our first construction is purely algebraic and *does not assume that the group G or the vector space E have any topology*.

As a consequence, until Section 6.4 we consider linear representations of G in E; these are simply homomorphisms $U\colon G \to \mathbf{GL}(E)$, *with no continuity requirement*.

Definition 6.1. Let G be a left action of a group G on a set X, and let E be a vector space. Let $\alpha\colon G \times X \to \mathbf{GL}(E)$ be a function and assume that the following conditions hold:

(a) For all $x \in X$

$$\alpha(e, x) = \mathrm{id}_E.$$

(b) For all $x \in X$ and all $s, t \in G$,

$$\alpha(st, x) = \alpha(s, t \cdot x) \circ \alpha(t, x).$$

A map $\alpha\colon G \times X \to \mathbf{GL}(E)$ satisfying Conditions (a) and (b) is called a *cocycle of G with values in $\mathbf{GL}(E)$*.

The point of cocycles is that they yield homomorphisms $\Pi\colon G \to \mathbf{GL}(E^X)$, that is, *linear representations of G in the vector space $[X \to E] = E^X$*.

Definition 6.2. Let G be a left action of a group G on a set X, and let E be a vector space. For every cocycle $\alpha\colon G \times X \to \mathbf{GL}(E)$, for every function $f\colon X \to E$, for every $s \in G$, let $\Pi^{\alpha}_{s}(f)\colon X \to E$ be the function given by

$$(\Pi^{\alpha}_{s}(f))(x) = \alpha(s, s^{-1}\cdot x)(f(s^{-1}\cdot x)), \quad \text{for every } x \in X. \tag{Π^{α}_{s}}$$

The above equation defines a map $\Pi^{\alpha}_{s}\colon E^{X} \to E^{X}$. The map $\Pi^{\alpha}\colon G \to \mathbf{GL}(E^{X})$ given by $s \mapsto \Pi^{\alpha}_{s}$ is the *(linear) representation of G in E^{X} induced by the cocycle α*.

For simplicity of notation, we write Π instead of Π^{α}.

The following proposition confirms that the map Π is a linear representation of G in the vector space E^{X}.

Proposition 6.1. *Let G be a left action of a group G on a set X, and let E be a vector space. For every cocycle $\alpha\colon G \times X \to \mathbf{GL}(E)$, for every $s \in G$, the map $\Pi_s\colon E^X \to E^X$ is a linear isomorphism, and the map $\Pi\colon G \to \mathbf{GL}(E^X)$ given by $s \mapsto \Pi_s$ is a homomorphism, that is, a linear representation of G in the vector space E^X.*

If we let $t = s^{-1}$ in (b) of Definition 6.1, we obtain

$$\alpha(s^{-1}, x) = (\alpha(s, s^{-1} \cdot x))^{-1},$$

so $\Pi_s(f)$ can also be written as

$$(\Pi_s(f))(x) = (\alpha(s^{-1}, x))^{-1}(f(s^{-1} \cdot x)). \qquad (\Pi_s)$$

6.2 Cocycles on a Homogeneous Space $X = G/H$

We now consider the special case where $X = G/H$ is the homogeneous space of left cosets for some subgroup H of G, and the left action of G acts on G/H given by

$$s \cdot (gH) = sgH.$$

Definition 6.3. Given a group G and a subgroup H of G, a *set of representatives $(r_x)_{x\in G/H}$ for the cosets of G/H* is the choice for every coset $x \in G/H$ of some element $r_x \in G$ so that $x = r_x H$.

Then every element g of $x = r_x H$ is written uniquely as $g = r_x h$, with $h \in H$.

We denote the coset H by x_0 and pick $r_{x_0} = e$.

For any $s \in G$, the representative of $s \cdot x = s \cdot r_x H = sr_x H$ is denoted by $r_{s\cdot x}$.

If we denote the quotient map by $\pi\colon G \to G/H$, then picking a set of representatives $(r_x)_{x\in G/H}$ in the cosets of G/H is *equivalent to picking a section of* π, that is, a map $r\colon G/H \to G$ such that $\pi \circ r = \mathrm{id}_{G/H}$.

Definition 6.4. Given $\alpha\colon G \times X \to \mathbf{GL}(E)$ as in Definition 6.1, for all $s \in G$, all $h \in H$, and all $x \in X$, define $\alpha_0(s), \sigma(h), \beta(x)$ and $u(s, x)$ by

$$\begin{aligned}
\alpha_0(s) &= \alpha(s, x_0)\\
\sigma(h) &= \alpha(h, x_0) = \alpha_0(h)\\
\beta(x) &= \alpha(r_x, x_0) = \alpha_0(r_x)\\
u(s, x) &= r_{s\cdot x}^{-1} s r_x \in H. \qquad \text{(u)}
\end{aligned}$$

Proposition 6.2. *Let G be a group, H be a subgroup of G, and E be a vector space. Choose a set $(r_x)_{x\in G/H}$ of representatives for the cosets of $X = G/H$ as explained above, with $x_0 = H$ and $r_{x_0} = e$. Every cocycle $\alpha\colon G\times X \to \mathbf{GL}(E)$ determines a homomorphism $\sigma\colon H \to \mathbf{GL}(E)$ with $\sigma(h) = \alpha(h, x_0)$ for all $h \in H$, a map $\beta\colon X \to \mathbf{GL}(E)$ given by $\beta(x) = \alpha(r_x, x_0)$ for all $x \in X$, and a map $u\colon G \times G/H \to H$ given by $u(s, x) = r_{s\cdot x}^{-1} s r_x \in H$, such that*

$$\alpha(s, x) = \beta(s \cdot x) \circ \sigma(u(s, x)) \circ (\beta(x))^{-1}.$$

Conversely, given a homomorphism $\sigma\colon H \to \mathbf{GL}(E)$ and a map $\beta\colon X \to \mathbf{GL}(E)$, if we set

$$u(s, x) = r_{s\cdot x}^{-1} s r_x \tag{u}$$

and

$$\alpha(s, x) = \beta(s \cdot x) \circ \sigma(u(s, x)) \circ (\beta(x))^{-1}, \tag{α}$$

then $\alpha\colon G \times X \to \mathbf{GL}(E)$ is a cocycle.

Remark: Kirillov [29] (Appendix V, Section 2.1) calls (u) the *Master equation*. See also Proposition 5, Lemma 2, and Lemma 3.

This material is also discussed in Kririllov [28] (Sections 13.1 and 13.2).

In view of Proposition 6.2 we make the following definition.

Definition 6.5. Given a homomorphism $\sigma\colon H \to \mathbf{GL}(E)$ and a map $\beta\colon X \to \mathbf{GL}(E)$, if α is the cocycle associated with σ and β, we say that the representation Π^α of G in E^X defined by α is the *representation induced by σ and β*.

Remarkably, for a given homomorphism $\sigma\colon H \to \mathbf{GL}(E)$, the representations $\Pi_1\colon G \to \mathbf{GL}(E^X)$ and $\Pi_2\colon G \to \mathbf{GL}(E^X)$ corresponding to the cocycles α_1 and α_2 associated with two maps β_1 and β_2 are equivalent, in the sense that there is an automorphism γ of E^X such that

$$\Pi_2 = \gamma \circ \Pi_1 \circ \gamma^{-1}.$$

This is proven as follows.

Proposition 6.3. *Let G be a group, H be a subgroup of G, and E be a vector space. Choose a set $(r_x)_{x\in G/H}$ of representatives for the cosets of $X = G/H$ as explained above, with $x_0 = H$ and $r_{x_0} = e$. Let $\sigma\colon H \to \mathbf{GL}(E)$ be a homomorphism, let $\beta\colon X \to \mathbf{GL}(E)$ be a map, and let α be the cocycle determined by σ and β as in Proposition 6.2, and let $\Pi\colon G \to \mathbf{GL}(E^X)$ be the corresponding representation. If $c(x) = \beta(x)^{-1}$ for all $x \in X$, then define the automorphism γ of E^X by*

$$(\gamma(f))(x) = c(x)(f(x)), \qquad f \in E^X,\ x \in X.$$

Then the representation

$$\Pi' = \gamma \circ \Pi \circ \gamma^{-1}$$

is associated with the cocycle α' given by

$$\alpha'(s, x) = \sigma(u(s, x)), \tag{α'}$$

with

$$u(s, x) = r_{s\cdot x}^{-1} s r_x. \tag{u}$$

Thus, the representation Π *induced by* σ *and* β *is equivalent to the representation induced by* σ *and* β'*, with* $\beta'(x) = \mathrm{id}_E$ *for all* $x \in X$.

The induced representation Π' *associated with* α' *is given by*

$$\begin{aligned}(\Pi'_s(f))(x) &= \sigma(u(s^{-1}, x)^{-1})(f(s^{-1} \cdot x))\\ &= \sigma(u(s, s^{-1} \cdot x))(f(s^{-1} \cdot x)),\\ &\quad f \in E^X,\ x \in X. \qquad (\Pi')\end{aligned}$$

It is also easy to check that if σ is replaced by an equivalent representation σ' of H in E^X, then the corresponding representations Π and Π' of G in E^X are equivalent.

Therefore, the process for making a representation Π of G in E^X from a representation σ of H in E and a function $\beta\colon X \to \mathbf{GL}(E)$ defines a class of representations of G in E^X.

Furthermore, *there is a special representation associated with σ and the constant function β given by* $\beta(x) = \mathrm{id}_E$, for all $x \in X$.

In summary, the method is: find a set $(r_x)_{x\in G/H}$ of representatives for the cosets of G/H, then to construct u given by $u(s, x) = r_{s\cdot x}^{-1} s r_x$ as in Equation (u), and then to define α by $\alpha(s, x) = \sigma(u(s, x))$.

The induced representation is given by

$$(\Pi_s(f))(x) = \sigma(u(s, s^{-1}\cdot x))(f(s^{-1}\cdot x)), \\ f \in E^X,\ x \in X. \qquad (*)$$

Vilenkin [39] (Chapter 1, Section 7) calls such a representation *representation with operator factor*.

From a theoretical point of view, a cocycle α is equivalent to a pair (σ, β) as in Proposition 6.2, but from a practical point of view, it may be very hard (if not impossible) to find constructively a set $(r_x)_{x \in G/H}$ of representatives for the cosets of G/H.

Thus we use cocycles α that agree with a given representation $\sigma\colon H \to \mathbf{GL}(E)$, in the sense that $\alpha(h, x_0) = \sigma(h)$ for all $h \in H$.

A case of practical case interest in equivariant machine learning is the case where $G = \mathbf{SE}(3)$ and $H = \mathbf{SO}(3)$.

Example 6.1. Let $G = \mathbf{SE}(3)$ and $H = \mathbf{SO}(3)$. The group $\mathbf{SE}(3)$ is the group of affine rigid motions of $\mathbb{R}^3$ consisting of rotations and translations.

Here we view $\mathbf{SE}(3)$ as the group of matrices

$$s = \begin{pmatrix} Q & a \\ 0 & 1 \end{pmatrix}, \quad Q \in \mathbf{SO}(3),\ a \in \mathbb{R}^3$$

under multiplication.

For short we denote the above matrix by (a, Q).

The group $\mathbf{SE}(3)$ acts on $\mathbb{R}^3$ by

$$(a, Q) \cdot x = Qx + a, \quad x \in \mathbb{R}^3.$$

Multiplication in $\mathbf{SE}(n)$ is given by

$$(a, Q)(b, R) = (a + Qb, QR),$$

and the inverse of (a, Q) is

$$(a, Q)^{-1} = (-Q^\top a, Q^\top).$$

It is easy to see that the homogeneous space $\mathbf{SE}(3)/\mathbf{SO}(3)$ is $\mathbb{R}^3$.

Indeed $\mathbf{SE}(3)$ acts on $\mathbb{R}^3$, and the stabilizer of the origin 0_3 is $\mathbf{SO}(3)$ viewed as the set of matrices

$$\begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}, \quad Q \in \mathbf{SO}(3).$$

We now use the method based on Proposition 6.2 and Proposition 6.3 to construct an induced representation of $\mathbf{SE}(3)$ from a representation $\sigma\colon \mathbf{SO}(3) \to \mathbf{GL}(E)$ of $\mathbf{SO}(3)$.

For this we need to find a set of representative for the cosets of $\mathbb{R}^3 = \mathbf{SE}(3)/\mathbf{SO}(3)$ in order to define u, and then $\alpha(s, x)$ is given by $\alpha(s, x) = \sigma(u(s, x))$ and the induced representation Π is given by $(*)$.

This is a case where it is easy to pick a set of coset representatives, namely for each $x \in \mathbb{R}^3$, $r_x \in \mathbf{SE}(3)$ is the matrix

$$r_x = \begin{pmatrix} I_3 & x \\ 0 & 1 \end{pmatrix},$$

the translation by x.

The coset $x\mathbf{SO}(3)$ consists of the matrices

$$\begin{pmatrix} Q & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_3 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}$$

with x fixed.

Let us compute $u(s, x) = r_{s\cdot x}^{-1} s r_x$.

First $s \cdot x = (a, Q) \cdot x = Qx + a$, so

$$r_{s\cdot x} = \begin{pmatrix} I_3 & Qx + a, \\ 0 & 1 \end{pmatrix}, \quad r_{s\cdot x}^{-1} = \begin{pmatrix} I_3 & -Qx - a, \\ 0 & 1 \end{pmatrix},$$

and finally

$$\begin{aligned} u(s, x) = r_{s\cdot x}^{-1} s r_x &= \begin{pmatrix} I_3 & -Qx - a, \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Q & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_3 & x \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} Q & -Qx \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_3 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Consequently, if $\sigma\colon \mathbf{SO}(3) \to \mathbf{GL}(E)$ is any representation of $\mathbf{SO}(3)$ is a finite-dimensional (nontrivial) vector space E, the above shows that $u(s, x)$ is independent of x and given by

$$u(s, x) = u((a, Q), x) = Q$$

and so $\alpha((a, Q), x)$ is given by

$$\alpha((a, Q), x) = \sigma(u((a, Q), x)) = \sigma(Q).$$

Then by $(*)$ we obtain the representation $\Pi\colon \mathbf{SE}(3) \to \mathbf{GL}(E^{\mathbb{R}^3})$ of $\mathbf{SE}(3)$ in $E^{\mathbb{R}^3}$ given by

$$\begin{aligned}(\Pi_{(a,Q)}(f))(x) &= \sigma(u(s, s^{-1} \cdot x))(f(s^{-1} \cdot x)) \\ &= \sigma(Q)(f((a, Q)^{-1} \cdot x)) \\ &= \sigma(Q)(f(Q^{\top}(x - a))),\end{aligned}$$

that is,

$$(\Pi_{(a,Q)}(f))(x) = \sigma(Q)(f(Q^{\top}(x - a))),\ f \in E^{\mathbb{R}^3},\ x \in \mathbb{R}^3.$$

This representation is reducible because the subspace of constant functions from $\mathbb{R}^3$ to E is invariant.

6.3 Converting Induced Representations of G From E^X to E^G

We can also show that a cocycle $\alpha\colon G \times X \to \mathbf{GL}(E)$ defines an isomorphism τ between the space E^X and a subspace L^α of the space E^G.

Definition 6.6. Let G be a group, H be a subgroup of G, E be a vector space, and write $X = G/H$. Given any cocycle $\alpha\colon G \times X \to \mathbf{GL}(E)$, for any function $f\colon X \to E$, the function $f^\alpha\colon G \to E$ is given by

$$\begin{aligned} f^\alpha(s) &= \alpha(s^{-1}, s \cdot x_0)(f(s \cdot x_0)) \\ &= (\alpha(s, x_0))^{-1}(f(s \cdot x_0)), \quad \text{for all } s \in G, \quad (*_{\alpha_1}) \end{aligned}$$

with $x_0 = H$.

Recall from Definition 6.4 that $\sigma(h) = \alpha(h, x_0)$ for all $h \in H$.

Proposition 6.4. *With the hypotheses of Definition 6.6, the function f^α satisfies the equation*

$$f^\alpha(sh) = \sigma(h^{-1})(f^\alpha(s)), \quad \text{for all } h \in H \text{ and all } s \in G. \qquad (*_{\alpha_2})$$

Definition 6.7. Let G be a group, H be a subgroup of G, E be a vector space, and write $X = G/H$. Given any cocycle $\alpha\colon G \times X \to \mathbf{GL}(E)$, let L^α be the subspace of E^G consisting of all functions $g\colon G \to E$ such that

$$g(sh) = \sigma(h^{-1})(g(s)), \quad \text{for all } s \in G \text{ and all } h \in H, \qquad (*_{\alpha_3})$$

where $\sigma(h) = \alpha(h, x_0)$, for all $h \in H$ (with $x_0 = H$).

Proposition 6.5. *With the hypotheses of Definition 6.6, for every $g \in L^\alpha$, there is a unique function $f\colon E \to X$ such that $g = f^\alpha$. Therefore, the map $\tau\colon E^X \to L^\alpha$ given by $\tau(f) = f^\alpha$ is an isomorphism.*

Observe that in the proof of Proposition 6.5, Equation $(*_f)$ and the fact that $\tau(f) = f^\alpha = g$ show that if $g \in L^\alpha$, then

$$(\tau^{-1}(g))(s \cdot x_0) = \alpha(s, x_0)(g(s)). \qquad (*_{\tau^{-1}(g)})$$

For any cocycle $\alpha\colon G \times X \to \mathbf{GL}(E)$, we can use the isomorphism $\tau\colon E^X \to L^\alpha$ to convert the representation $\Pi\colon G \to \mathbf{GL}(E^X)$ defined by α into the equivalent representation Π_{L^α} given by $\Pi_{L^\alpha}(s) = \tau \circ \Pi(s) \circ \tau^{-1}$.

Proposition 6.6. *For every cocycle $\alpha\colon G \times X \to \mathbf{GL}(E)$, if $\Pi\colon G \to \mathbf{GL}(E^X)$ is the representation defined by α, then the equivalent representation $\Pi_{L^\alpha}\colon G \to \mathbf{GL}(L^\alpha)$ defined by $\Pi_{L^\alpha}(s) = \tau \circ \Pi(s) \circ \tau^{-1}$ is given by*

$$((\Pi_{L^\alpha})_s(g))(t) = g(s^{-1}t) \quad \text{for all } g \in L^\alpha \text{ and all } s, t \in G. \tag{Π_{L^α}}$$

Remark: Observe that L^α only depends on σ, so we may write L^σ instead of L^α, and Π_{L^α} depends only on σ, so we may also write Π_{L^σ} instead of Π_{L^α}.

The representation Π_{L^σ}, *which is simply the left regular representation of G on L^α, is more intrinsic than the representations Π^α acting on the space of functions in E^X*.

The representations Π^α acting on the space of functions in E^X *require for their construction the choice of a set of coset representatives* $(r_x)_{x\in G/H}$ in addition to the representation $\sigma\colon H \to \mathbf{GL}(E)$ in order to define a cocycle α.

However, if $X = G/H$ is a lot "smaller" than G, then the space of functions in L^α (a space of functions from G to E) is very redundant and from a practical point of view, it might be better to use the representations defined on the smaller space of functions from X to E.

This issue will come up in Chapter 7.

We have concluded our discussion of algebraic methods for constructing representations of G from representations of a subgroup H of G.

6.4 Construction of the Hilbert Space $L^2_\mu(X; E)$

We now assume that G is a locally compact group and that H is a closed subgroup of G.

By Vol I, Proposition @@@8.6(1), the space $X = G/H$ is also locally compact. If G is separable, then so is G/H, and if G is metrizable, then so G/H; see Dieudonné [13] (Chapter XII, Sections 10 and 11).

Given a unitary representation $U\colon H \to \mathbf{U}(E)$ of H we would like to construct a unitary representation $\Pi\colon G \to \mathbf{U}(\mathcal{H})$ of G.

This is possible under certain conditions on H and G and on measures on $X = G/H$.

Note that unlike in the previous sections we are now considering *continuous* unitary representations.

The first step is to construct a Hilbert space $\mathcal{H}$ that will be the representation space of a unitary representation of G. There are two approaches:

1. The Hilbert space $\mathcal{H}$ is a set of functions from $X = G/H$ to E.
2. The Hilbert space $\mathcal{H}$ is a set of functions from G to E, analogous to the space L^α of Section 6.3.

The second step is to define the operators Π_s (for $s \in G$) so that they are unitary operators of $\mathcal{H}$.

This involves defining an inner product in $\mathcal{H}$ that makes the operators Π_s unitary.

In the first approach that makes use of cocycles, the definition of the inner product on $\mathcal{H}$ is straightforward.

To ensure that the operators Π_s are unitary, a Borel measure μ on $X = G/H$ is needed, and the cocycles must satisfy some additional conditions with respect to the measure μ.

The case where the measure μ is G-invariant is simpler than the case where μ is only quasi-invariant.

In the second approach, the definition of the Hilbert space $\mathcal{H}$ is more complicated and requires a completion.

A good candidate for the first approach is a subspace $\mathrm{L}^2_\mu(X;E)$ of the vector space E^X, where μ is positive Borel measure on G/H.

In the special case where H is compact, given a cocycle α on $G \times X$ satisfying some suitable conditions, the space L^α will be a subspace of $\mathrm{L}^2_\lambda(G;E) \subseteq E^G$, where λ is a left-invariant Haar measure on G.

Whether μ is G-invariant is an issue that will come up later, but for the time being we can ignore it.

Let E be a separable Hilbert space, and let (a_n) be a Hilbert basis of E.

Every function $f\colon X \to E$ can be written uniquely as $f = \sum_n f_n a_n$, where $f_n\colon X \to \mathbb{C}$, and such that the series $\sum_n |f_n(x)|^2$ converges for all $x \in X$.

By definition, we let

$$\|f(x)\|_E^2 = \sum_n |f_n(x)|^2.$$

It can be shown that a function $f\colon X \to E$ is μ-measurable iff all the f_n are μ-measurable.

Definition 6.8. Let G be a locally compact group, let H be a closed subgroup of G, let μ be a positive Borel measure on $X = G/H$, and let E be a separable Hilbert space. For any Hilbert basis (a_n) of E, let $\mathcal{L}^2_\mu(X; E)$ be the space of all μ-measurable functions $f\colon X \to E$ with $f = \sum_n f_n a_n$, such that the function $x \mapsto \sum_n |f_n(x)|^2 = \|f(x)\|_E^2$ is μ-integrable.

It is easy to see that if $f = \sum_n f_n a_n$, then $f_n \in \mathcal{L}^2_\mu(X;\mathbb{C})$, and

$$\int_{G/H} \|f\|_E^2 \, d\mu = \sum_n \int_{G/H} |f_n|^2 \, d\mu = \sum_n \|f_n\|_2^2 ;$$

see Dieudonné [13] (Chapter XIII, Sections 8 and 9).

As a consequence, given two functions $f = \sum_n f_n a_n$ and $g = \sum_n g_n a_n$ in $\mathcal{L}^2_\mu(X;E)$, by Vol I, Proposition @@@5.41, the function $x \mapsto \langle f(x), g(x)\rangle$ is integrable and

$$\int_{G/H} \langle f(x), g(x)\rangle \, d\mu(x) = \sum_n \int_{G/H} f_n(x)\overline{g_n(x)} \, d\mu(x).$$

Definition 6.9. We say that a function $f \in \mathcal{L}^2_\mu(X; E)$ is *negligeable* if the function $x \mapsto \|f(x)\|_E^2$ is zero almost everywhere.

The quotient of the space $\mathcal{L}^2_\mu(X; E)$ by the subspace of negligeable functions is denoted by $\mathrm{L}^2_\mu(X; E)$.

It is a hermitian space under the inner product

$$\langle [f], [g] \rangle = \int_{G/H} \langle f(x), g(x) \rangle \, d\mu(x),$$

and we have the norm N_2[1] given by

$$N_2([f]) = \sqrt{\langle [f], [f] \rangle}.$$

[1]We are using the notation N_2 for the norm on $\mathrm{L}^2_\mu(X; E)$ to avoid a confusion with the norm $\| \, \|_2$ on $\mathrm{L}^2_\mu(X; \mathbb{C})$.

If $[f]$ is represented by $f = \sum_n f_n a_n$, then

$$N_2([f])^2 = \int_{G/H} \|f\|_E^2 \, d\mu = \sum_n \|f_n\|_2^2 .$$

Actually, it turns out that the hermitian space $\mathrm{L}^2_\mu(X;E)$ is complete, that is, it is a Hilbert space.

In fact, it is a separable Hilbert space.

Proposition 6.7. *Let G be a locally compact group, let H be a closed subgroup of G, let μ be a positive Borel measure on $X = G/H$, and let E be a separable Hilbert space. The space $\mathrm{L}^2_\mu(X;E)$ is a separable Hilbert space.*

6.5 Induced Representations, I; G/H has a G-Invariant Measure

In the rest of this chapter, by unitary representation, we mean *continuous* unitary representation.

We will now assume that the positive Borel measure μ on $X = G/H$ is G-invariant.

Recall from Voll I, Section @@@8.10 (Definition @@@8.18) that

$$(\lambda_s(\mu))(A) = \mu(s^{-1} \cdot A),$$

for every Borel subset A of X, so μ is G-invariant if for every Borel subset A of X,

$$\mu(s^{-1} \cdot A) = \mu(A) \qquad \text{for all } s \in G.$$

In this case,

$$\int_{G/H} f(s\cdot x)\, d\mu(x) = \int_{G/H} f(x)\, d\mu(x), \qquad \text{for all } s \in G.$$

Let E be a separable Hilbert space, and let $U\colon H \to \mathbf{U}(E)$ be a unitary representation of H.

Theorem 6.8. *Let G be a locally compact group, H be a closed subgroup of G, E be a separable Hilbert space, and $U\colon H \to \mathbf{U}(E)$ be a unitary representation of H. If $X = G/H$ admits a G-invariant σ-Radon measure μ, and for any cocycle $\alpha\colon G \times X \to \mathbf{U}(E)$, if the following conditions hold*

(1) We have $\alpha(h, x_0) = U(h)$ for all $h \in H$;

(2) For every $s \in G$, for every $f \in \mathrm{L}^2_\mu(X; E)$, the map $x \mapsto \alpha(s, x)(f(x))$ from X to E is μ-measurable;

(3) For every $f \in \mathrm{L}^2_\mu(X; E)$, the map $s \mapsto \Pi_s(f)$ is a continuous map from G to $\mathrm{L}^2_\mu(X; E)$, where Π is the homomorphism $\Pi\colon G \to \mathbf{GL}(E^X)$ induced by the cocycle α;

then the homomorphism $\Pi\colon G \to \mathbf{U}(\mathrm{L}^2_\mu(X; E))$ induced by the cocycle α given by

$$(\Pi_s(f))(x) = (\alpha(s^{-1}, x))^{-1}(f(s^{-1} \cdot x)), \quad f \in \mathrm{L}^2_\mu(X; E),\ x \in X,$$

(see Definition 6.2) is a unitary representation of G.

Definition 6.10. The unitary representation $\Pi\colon G \to \mathbf{U}(\mathrm{L}^2_\mu(X;E))$ induced by the cocycle α (and the unitary representation $U\colon H \to \mathbf{U}(E)$) is denoted $\mathrm{Ind}_H^G\,\alpha$, or by abuse of notation even $\mathrm{Ind}_H^G\,U$.

Remark: To be very precise, the representing space $\mathrm{L}^2_\mu(X;E)$ of this representation should be specified, for example as in $\mathrm{Ind}^G_{H,\mathrm{L}^2_\mu(X;E)}\,\alpha$, because there are variants of this construction that use a different representation space.

If U is the trivial representation of H in E, and if we choose $\alpha(s,x) = \mathrm{id}_E$ for all $(s,x) \in G \times (G/H)$, then it can be verified that the hypotheses of Theorem 6.8 are satisfied.

In this case, the subspace L^α corresponding to $\mathcal{L}^2_\mu(X;E)$ consists of all functions of the form $f \circ \pi$ with $f \in \mathcal{L}^2_\mu(X;E)$, where $\pi\colon G \to G/H$ is the projection map.

If H is a (closed) compact subgroup of G, then by Vo I, Proposition @@@8.43, the space G/H has G-invariant measures (unique up to a scalar).

This is a special case of particular interest. A good illustration of this situation is provided by Example 6.1 that we now revisit.

Example 6.2. As in Example 6.1 consider the groups $G = \mathbf{SE}(3)$ and $H \approx \mathbf{SO}(3)$, where G is locally compact and H is compact and closed in G.

Consequently $X = G/H \approx \mathbb{R}^3$ has an $\mathbf{SE}(3)$-invariant Radon measure μ.

Consider any unitary representation $\sigma\colon \mathbf{SO}(3) \to \mathbf{U}(E)$ of $\mathbf{SO}(3)$ in a separable Hilbert space E.

We showed in Example 6.1 that we have a cocycle $\alpha\colon \mathbf{SE}(3)\times\mathbb{R}^3 \to \mathbf{U}(E)$ given by

$$\alpha((a,Q),x) = \sigma(Q), \quad a,x\in\mathbb{R}^3,\ Q\in\mathbf{SO}(3),$$

and the homomorphism $\Pi\colon \mathbf{SE}(3)\to\mathbf{GL}(E^{\mathbb{R}^3})$ induced by α is given by

$$(\Pi_{(a,Q)}(f))(x) = \sigma(Q)f(Q^\top(x-a)), \quad f\in E^{\mathbb{R}^3}, x\in\mathbb{R}^3.$$

We leave it as an exercise to check that Conditions (1)-(3) of Theorem 6.8 are satisfied, and so Π is a unitary representation $\Pi\colon \mathbf{SE}(3)\to\mathbf{U}(\mathrm{L}^2_\mu(\mathbb{R}^3;E))$ of $\mathbf{SE}(3)$ in the Hilbert space $\mathrm{L}^2_\mu(\mathbb{R}^3;E)$.

If E is finite-dimensional, say of dimension $n \geq 1$, then the Hilbert space $\mathrm{L}^2_\mu(\mathbb{R}^3; E)$ is isomorphic to the direct sum of n copies of $\mathrm{L}^2_\mu(\mathbb{R}^3; \mathbb{C})$.

Then every function $f \in \mathrm{L}^2_\mu(\mathbb{R}^3; E)$ is identified with the n-tuple $f = (f_1, \ldots, f_n)$ where $f_i \in \mathrm{L}^2_\mu(\mathbb{R}^3; \mathbb{C})$, with the inner product of
$f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_n)$ given by

$$\langle f, g \rangle = \sum_{i=1}^{n} \int_{\mathbb{R}^3} f_n(x) \overline{g(x)} \, d\mu(x).$$

Another example of induced representations of $G = \mathbf{SE}(n)$ arises from the normal abelian subgroup $H = \mathbb{R}^n$.

In this case we make use of the characters of $\mathbb{R}^n$. Such repesentations of $\mathbf{SE}(n)$ are discussed in Vilenkin [39] (Chapter XI, Section 2).

Under some mild additional conditions, induced unitary representations of G in $\mathrm{L}^2_\mu(X; E)$ can be converted to unitary representations of G in a closed subspace of $\mathrm{L}^2_\lambda(G; E)$ (where λ is a left Haar measure on G).

Suppose that the unitary cocycle α has the property that the map

$$s \mapsto f^\alpha(s) = \alpha(s^{-1}, s \cdot x_0)(f(s \cdot x_0))$$

from G to E is λ-measurable for every $f \in \mathcal{L}^2_\mu(X; E)$.

If so, it can be shown that $f^\alpha \in \mathcal{L}^2_\lambda(G; E)$.

Conversely, if $g \in \mathcal{L}^2_\lambda(G; E)$ satisfies the property

$$g(sh) = U(h^{-1})(g(s)) \quad \text{for all } s \in G \text{ and all } h \in H, \tag{$*_U$}$$

and if the map $s \mapsto \alpha(s, x_0)(g(s))$ from G to E is λ-measurable, then as in Proposition 6.5 we can write this map as $f \circ \pi$ for some $f \in \mathrm{L}^2_\mu(X; E)$, and we have $g = f^\alpha$.

In this case, up to equivalence, we can consider the unitary representation $\mathrm{Ind}_{H,F}^G\, \alpha$ induced by α as a unitary representation of G in the closed subspace F of $\mathrm{L}^2_\lambda(G; E)$ spanned by the functions $g \in \mathcal{L}^2_\lambda(G; E)$ satisfying property $(*_U)$.

Then for all $s \in G$,

$$(\mathrm{Ind}_{H,F}^{G}\,\alpha)_s(g) = \lambda_s g, \qquad \text{for all } g \in F, \qquad (\mathrm{Ind}_G)$$

equivalently, for all $s, t \in G$,

$$((\mathrm{Ind}_{H,F}^{G}\,\alpha)_s(g))(t) = g(s^{-1}t), \qquad \text{for all } g \in F.$$

Notice the analogy with Proposition 6.6.

Note that $\mathrm{Ind}_{H,F}^{G}\,\alpha$ depends only on U, so we usually write $\mathrm{Ind}_{H,F}^{G}\,U$ instead of $\mathrm{Ind}_{H,F}^{G}\,\alpha$.

If $E = \mathbb{C}$, then $\mathrm{Ind}_{H,F}^{G}\,U$ is a subrepresentation of the regular representation of G in $\mathrm{L}^2(G)$.

Definition 6.11. If we choose U to be the trivial representation of H in E, then the functions $g \in \mathrm{L}^2_\lambda(G; E)$ satisfying Condition $(*_U)$ are constant on the classes sH, so we can identify F with $\mathrm{L}^2_\mu(X; E)$. In this case we say that the induced representation $\mathrm{Ind}_H^G U$ of G in $\mathrm{L}^2_\mu(X; E)$ is the *canonical representation* of G corresponding to the compact subgroup H and to its trivial representation in E.

If $H = (e)$ and $E = \mathbb{C}$, then the induced representation is the regular representation of G in $\mathrm{L}^2(G)$.

Going back to the case where H is an arbitrary closed subgroup of G, and where there is a G-invariant measure on G/H, there is another method, not using cocyles, for defining a unitary induced representation of G from a unitary representation $U\colon H \to \mathbf{U}(E)$.

We can define a Hilbert space $\mathcal{H}$ such that formula (Ind_G) defines a unitary induced representation $\mathrm{Ind}_{H,\mathcal{H}}^{G} U$ of G in $\mathcal{H}$.

This method is described in Folland [19] (Chapter 6, Section 1), and we briefly describe it.

Given a unitary representation $U\colon H \to \mathbf{U}(E)$, let $\mathcal{H}_0$ be the following set of functions:

$$\begin{aligned}\mathcal{H}_0 = \{f \in \mathcal{C}(G, E) \mid\ & \pi(\mathrm{supp}(f)) \text{ is compact and} \\ & f(sh) = U(h^{-1})(f(s)) \\ & \text{for all } s \in G \text{ and all } h \in H\}.\end{aligned}$$

The problem is that it is not obvious that $\mathcal{H}_0$ is nonempty!

However, the following result proven in Folland [19] (Chapter 6, Proposition 6.1) shows that this is not the case.

Proposition 6.9. *If $\varphi\colon G \to E$ is a continuous function with compact support, then the function f_φ from G to E given by*

$$f_\varphi(s) = \int_H U(h)(\varphi(hs))\, d\lambda_H(h)$$

belongs to $\mathcal{H}_0$ and is uniformly continuous on G. Moreover, every element of $\mathcal{H}_0$ is of the form f_φ for some $\varphi \in \mathcal{K}(G, E)$.

The group G acts on the left on $\mathcal{H}_0$ by $f \mapsto \lambda_s f$.

In order to act by unitary maps, we need to define an inner product on $\mathcal{H}_0$ with respect to which these left translations are isometries.

Since G/H has G-invariant measures, this is easy to achieve.

If $f, g \in \mathcal{H}_0$, then the map $s \mapsto \langle f(s), g(s)\rangle_E$ depends only on the coset sH, so we can define the inner product $\langle f, g\rangle$ by

$$\langle f, g\rangle = \int_{G/H} \langle f(s), g(s)\rangle_E \, d\mu(sH).$$

This is clearly a positive hermitian form, and it is positive definite because $\mu(A) > 0$ for every nonempty open set A.

This inner product is is nvariant under the left translations λ_s because μ is G-invariant.

Therefore, with respect to this inner product, the maps $f \mapsto \lambda_s f$ are unitary.

If $\mathcal{H}$ is the Hilbert space which is the completion of $\mathcal{H}_0$, then the maps $f \mapsto \lambda_s f$ extend to unitary operators on $\mathcal{H}$.

It follows from Proposition 6.9 that the map $s \mapsto \lambda_s f$ from G to $\mathcal{H}$ are continuous for every $f \in \mathcal{H}_0$.

Therefore, they define a unitary representation of G in $\mathcal{H}$ given by

$$(\mathrm{Ind}_{H,\mathcal{H}}^{G} U)_s(f) = \lambda_s(f), \qquad f \in \mathcal{H}.$$

This unitary representation has the advantage that it depends only on U, but one should not neglect the fact that the construction involving cocycles allows more flexibility.

The Hilbert space $\mathcal{H}$ is also more complicated than the Hilbert space $\mathrm{L}^2_\mu(X; E)$.

When G/H admits no G-invariant measure, then we need to use a weaker notion of invariance. It turns out that the notion of (strong) quasi-invariance does the job.

6.6 Quasi-Invariant Measures on G/H

6.7 Induced Representations, II; G/H has a Quasi-Invariant Measure

6.8 Examples of Induced Representations Via Method II

6.9 Induced Representations of Compact Groups

In this section we consider a compact (metrizable) group G and a closed subgroup H of G, and our goal is to determine the canonical (unitary) representation of G in $\mathrm{L}^2_\mu(G/H;\mathbb{C})$ induced by the trivial representation of H in $E=\mathbb{C}$ (see Definition 6.11), where μ is the G-invariant measure on G/H induced by a Haar measure λ on G.

For simplicity of notation we write $\mathrm{L}^2_\mu(G/H)$ instead of $\mathrm{L}^2_\mu(G/H;\mathbb{C})$.

To do this it is *necessary to understand what is the restriction of the representation* $M_\rho\colon G\to\mathbf{U}(\mathbb{C}^{n_\rho})$ *to* H, with $\rho\in R(G)$.

We will denote the complete set of the irreducible representations of G given by the Peter-Weyl theorem I (Theorem 4.3) by $\rho \in R(G)$, the corresponding representations by $M_\rho \colon G \to \mathbf{U}(\mathbb{C}^{n_\rho})$, and the identity element of $\mathfrak{a}_\rho$ by $u_\rho = \frac{1}{n_\rho}\chi_\rho$, where χ_ρ is the character associated with ρ.

Similarly, we will denote the complete set of irreducible representations of H given by the Peter-Weyl theorem I by $\sigma \in R(H)$, the corresponding representations by $M_\sigma \colon H \to \mathbf{U}(\mathbb{C}^{n_\sigma})$, and the identity element of $\mathfrak{a}_\sigma$ by $u_\sigma = \frac{1}{n_\sigma}\chi_\sigma$, where χ_σ is the character associated with σ.

The Haar measure on G is denoted by λ_G, and the Haar measure on H is denoted by λ_H.

Consider the restriction $V\colon H \to \mathbf{U}(\mathbb{C}^{n_\rho})$ of the representation $M_\rho\colon G \to \mathbf{U}(\mathbb{C}^{n_\rho})$ to H.

Recall that for any function $f \in \mathrm{L}^2(H)$ and any $x \in \mathbb{C}^{n_\rho}$, $V_{\mathrm{ext}}(f)(x)$ is the weak integral of the function $t \mapsto V(t)(x)$ with respect to $f d\lambda_H$ $(t \in H)$.

We will write $M_\rho(f)$ for $V_{\mathrm{ext}}(f)$.

By the Peter–Weyl theorem II (Theorem 4.8), for every $\sigma \in R(H)$, the map $\pi_\sigma^{M_\rho} = M_\rho(u_{\overline{\sigma}})$ given by

$$\pi_\sigma^{M_\rho}(x) = \frac{1}{n_\sigma}\int_H \overline{\chi_\sigma(t)} M_\rho(t)(x)\, d\lambda_H(t), \quad x \in \mathbb{C}^{n_\rho},$$

is the orthogonal projection of $\mathbb{C}^{n_\rho}$ onto a closed subspace E_σ of $\mathbb{C}^{n_\rho}$, and we have a Hilbert sum

$$\mathbb{C}^{n_\rho} = \bigoplus_{\sigma \in R(H)} E_\sigma.$$

Recall from Section 4.7 that the integral defining $\pi_\sigma^{M_\rho}(x)$ can be computed by integrating the matrix $\overline{\chi_\sigma(t)}M_\rho(t)(x)$ term by term.

Furthermore, for each subspace $E_\sigma \neq (0)$, each irreducible representation M_σ of H is contained a certain number of times in the restriction of M_ρ to H, which we denote $d_\sigma = (\rho : \sigma)$, so E_σ is a finite Hilbert sum

$$E_\sigma = \bigoplus_{k=1}^{d_\sigma} F_k^\sigma,$$

of subspaces $F_1^\sigma, F_2^\sigma, \ldots, F_{d_\sigma}^\sigma$ of dimension n_σ, invariant under $M_\rho(t)$ for every $t \in H$, and such that the restriction of M_ρ to H and to each F_k^σ is equivalent to the irreducible representation M_σ.

Thus E_σ has dimension $p_\sigma = d_\sigma n_\sigma$.

We can pick an orthonormal basis of $\mathbb{C}^{n_\rho}$ consisting of the union of orthonormal bases of each of the F_j^σ and of a basis of the orthogonal complement F' of E_σ in $\mathbb{C}^{n_\rho}$.

Let P be the change of basis matrix, which is unitary.

For the basis of E_σ consisting of the first $p_\sigma = d_\sigma n_\sigma$ vectors of this basis, the matrix $M_{\rho,\sigma}(t)$ of the restriction of $P^* M_\rho(t) P$ to E_σ is a block diagonal matrix (consisting of d_σ blocks) of the form

$$M_{\rho,\sigma}(t) = \begin{pmatrix} M_\sigma(t) & 0 & \cdots & 0 \\ 0 & M_\sigma(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_\sigma(t) \end{pmatrix}$$

for every $t \in H$.

Since G and H are compact, G/H has a G-invariant measure μ induced by a Haar measure on G.

We now try to understand what the canonical unitary representation of G in $\mathrm{L}^2_\mu(G/H)$ induced by the trivial representation of H in $E = \mathbb{C}$ looks like.

With the notations as above, we have $n_{\sigma_0} = 1$, and $p_{\sigma_0} = d$.

First, let us observe that a function $g \in \mathcal{L}^2_\mu(G/H)$ can be viewed as a function $g \in \mathcal{L}^2(G)$ such that

$$g(st) = g(s) \qquad \text{for all } t \in H \text{ and all } s \in G. \quad (*_{G/H})$$

Since $(g*\delta_t)(s) = g(st)$, the above condition is equivalent to

$$g * \delta_t = g \qquad \text{for all } t \in H, \qquad (*'_{G/H})$$

and thus for any measure $\nu \in \mathcal{M}^1(G)$, the function $\nu * g \in \mathcal{L}^2_\mu(G/H)$ also satisfies the equation

$$(\nu * g) * \delta_t = \nu * g,$$

so we deduce that $\mathrm{L}^2_\mu(G/H)$ is a closed left ideal in $\mathcal{M}^1(G)$, which implies that $\mathrm{L}^2_\mu(G/H)$ is a closed left ideal in $\mathrm{L}^2(G)$.

In particular, for every $\rho \in R(G)$, the projection $g \mapsto u_\rho*g$ of $\mathrm{L}^2(G)$ onto the ideal $\mathfrak{a}_\rho$ maps $\mathrm{L}^2_\mu(G/H)$ onto itself, *so $\mathrm{L}^2_\mu(G/H)$ is the Hilbert sum of the subspaces*

$$L_\rho = \mathrm{L}^2_\mu(G/H) \cap \mathfrak{a}_\rho.$$

It remains to determine what the L_ρ are.

We explained that by applying Peter–Weyl II (Theorem 4.8) to the restriction of the representation $M_\rho\colon G \to \mathbf{U}(C^{n_\rho})$ to H we obtain a decomposition of $\mathbb{C}^{n_\rho}$ as a finite Hilbert sum

$$\mathbb{C}^{n_\rho} = E_{\sigma_1} \oplus \cdots \oplus E_{\sigma_q},$$

with each E_{σ_i} a direct sum

$$E_{\sigma_i} = \bigoplus_{k=1}^{d_{\sigma_i}} F_k^{\sigma_i}$$

of subspaces $F_1^{\sigma_i}, F_2^{\sigma_i}, \ldots, F_{d_{\sigma_i}}^{\sigma_i}$ of dimension n_{σ_i}, invariant under $M_\rho(t)$ for every $t \in H$, and such that the restriction of M_ρ to each $F_k^{\sigma_i}$ is equivalent to the irreducible representation M_{σ_i}.

Let us pick for an orthonormal basis of $\mathbb{C}^{n_\rho}$ the union of orthonormal bases of the $F_k^{\sigma_i}$, and let P be the change of basis matrix, which is unitary.

Then for any $t \in H$ we have

$$P^* M_\rho(t) P = \begin{pmatrix} M_{\rho,\sigma_1}(t) & 0 & \cdots & 0 \\ 0 & M_{\rho,\sigma_2}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{\rho,\sigma_q}(t) \end{pmatrix},$$

where $M_{\rho,\sigma_i}(t)$ is the block matrix

$$M_{\rho,\sigma_i}(t) = \begin{pmatrix} M_{\sigma_i}(t) & 0 & \cdots & 0 \\ 0 & M_{\sigma_i}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{\sigma_i}(t) \end{pmatrix}$$

(consisting of d_{σ_i} blocks) defined earlier.

Thus the matrix $M_\rho^{(H)}(t) = P^* M_\rho(t) P$ (with $t \in H$) is the block matrix consisting of the blocks $M_{\sigma_i}(t)$, each one repeated d_{σ_i} times.

We also define the matrices $M_\rho^{(H)}(s) = (m_{ij}^{(\rho,H)}(s))$ for *all $s \in G$* by

$$M_\rho^{(H)}(s) = P^* M_\rho(s) P, \quad s \in G.$$

Beware that if $s \in G$ but $s \notin H$, then the matrix $M_\rho^{(H)}(s)$ does *not* have the nice block structure enjoyed by the matrices $M_\rho^{(H)}(t)$ when $t \in H$.

The representations of G in $\mathbb{C}^{n_\rho}$ defined by the matrices $M_\rho(s)$ and $M_\rho^{(H)}(s)$ ($s \in G$) are equivalent.

The matrix $M_\rho^{(H)}$ denotes the matrix of n_ρ^2 functions $m_{ij}^{(\rho,H)}$ given by $s \mapsto m_{ij}^{(\rho,H)}(s)$ and we also write $M_\rho^{(H)} = P^* M_\rho P$.

By Proposition 4.5, the matrix $M_\rho^{(H)}$ defines n_ρ^2 functions $m_{ij}^{(\rho,H)}$ that form an orthonormal basis of $\mathfrak{a}_\rho$ and satisfy the same properties as the functions $m_{i,j}^{(\rho)}$ defined by the matrix M_ρ.

Proposition 6.10. *The space* $\mathrm{L}^2_\mu(G/H)$ *is the Hilbert sum of subspaces* $L_\rho \subseteq \mathfrak{a}_\rho$*. If the trivial representation* σ_0 *of* H *is contained* $d = (\rho : \sigma_0) \geq 1$ *times in the restriction of* M_ρ *to* H*, then* L_ρ *is the direct sum of the first* d *columns of* $M_\rho^{(H)} = P^* M_\rho P$*,*

$$L_\rho = \bigoplus_{j=1}^{d} \mathfrak{l}_j^{(\rho,H)} \quad \text{and} \quad \mathfrak{l}_j^{(\rho,H)} = \bigoplus_{k=1}^{n_\rho} \mathbb{C} m_{kj}^{(\rho,H)}.$$

If $d = 0$*, then* $L_\rho = (0)$*. The subrepresentation* $\Pi\colon G \to \mathbf{U}(L_\rho)$ *in* L_ρ *of the canonical representation* $\Pi\colon G \to \mathbf{U}(\mathrm{L}^2_\mu(G/H))$ *of* G *in* $\mathrm{L}^2_\mu(G/H)$ *induced by the trivial representation of* H *in* $\mathbb{C}$ *is the Hilbert sum of* $d = (\rho : \sigma_0)$ *irreducible representations equivalent to* $M_{\overline{\rho}}$*.*

We can also consider the space $H\backslash G$ of right cosets Hs of G ($s \in G$). If $\pi\colon G \to H\backslash G$ is the quotient map $\pi(s) = Hs$, the fact that the Haar measure λ on a compact group is left and right invariant implies immediately that there is a G-invariant measure μ' on $H\backslash G$ such that

$$\int_{G/H} g(x)\, d\mu'(x) = \int_G (g \circ \pi)\, d\lambda,$$

and

$$\int_{G/H} g(x{\cdot}s)\, d\mu'(x) = \int_{G/H} g(x)\, d\mu'(x) \qquad \text{for all } s \in G,$$

with

$$(Ht) \cdot s = Hts, \qquad s, t \in G.$$

Every function $g \in \mathcal{L}^2_{\mu'}(H\backslash G)$ can be viewed as a function $g \in \mathcal{L}^2(G)$ such that

$$g(ts) = g(s) \qquad \text{for all } t \in H \text{ and all } s \in G. \quad (*_{H\backslash G})$$

Since $(\delta_t * g)(s) = g(t^{-1}s)$, the above condition is equivalent to

$$\delta_t * g = g \qquad \text{for all } t \in H. \qquad (*'_{H\backslash G})$$

The space $\mathrm{L}^2_{\mu'}(H\backslash G)$ is the image of the space $\mathrm{L}^2_{\mu}(G/H)$ under the isomorphism $g \mapsto \check{\overline{g}}$ (here we use the fact that G is unimodular).

Therefore $\mathrm{L}^2_{\mu'}(H\backslash G)$ is a closed right ideal in $\mathrm{L}^2(G)$, and it is the Hilbert sum of the images $\check{\overline{L}}_\rho$ of the L_ρ;

since by Theorem 4.4(2) we have $m_{ji} = \check{\overline{m}}_{ij}$, we deduce that $\check{\overline{L}}_\rho$ is the direct sum of the first d rows of $M_\rho^{(H)}$ (with $d = (\rho : \sigma_0)$).

Let us record this fact.

Proposition 6.11. *The space* $\mathrm{L}^2_{\mu'}(H\backslash G)$ *is the Hilbert sum of subspaces* $\check{\bar{L}}_\rho \subseteq \mathfrak{a}_\rho$*. If the trivial representation* σ_0 *of* H *is contained* $d = (\rho : \sigma_0) \geq 1$ *times in the restriction of* M_ρ *to* H*, then* $\check{\bar{L}}_\rho$ *is the direct sum of the first* d *rows of* $M_\rho^{(H)}$*; that is,*

$$\check{\bar{L}}_\rho = \bigoplus_{i=1}^{d} \bigoplus_{j=1}^{n_\rho} \mathbb{C} m_{ij}^{(\rho,H)}.$$

Let us now consider the intersection
$\mathrm{L}^2_\mu(G/H) \cap \mathrm{L}^2_{\mu'}(H\backslash G)$.

This is a closed involutive subalgebra of $\mathrm{L}^2(G)$, thus a complete Hilbert algebra.

We can view a function $g \in \mathrm{L}^2_{\mu}(G/H) \cap \mathrm{L}^2_{\mu'}(H\backslash G)$ as a function $g \in \mathrm{L}^2(G)$ such that

$$g(tst') = g(s) \quad \text{for all } t, t' \in H \text{ and all } s \in G, \tag{$*_{H\backslash G/H}$}$$

or equivalently

$$\delta_t * g * \delta_{t'} = g \quad \text{for all } t, t' \in H. \tag{$*'_{H\backslash G/H}$}$$

We can also think of the functions
$g \in \mathrm{L}^2_{\mu}(G/H) \cap \mathrm{L}^2_{\mu'}(H\backslash G)$ as functions defined on the *double classes (or double cosets)* HsH of G with respect to H.

In this case, if $\pi\colon G \to H\backslash G/H$ is the quotient map $\pi(s) = HsH$, the fact that the Haar measure λ on a compact group is left and right invariant implies that there is a G-invariant measure μ on $H\backslash G/H$ such that

$$\int_{H\backslash G/H} g(x)\, d\mu(x) = \int_G (g \circ \pi)\, d\lambda.$$

We denote the algebra of functions in $\mathrm{L}^2(G)$ satisfying $(*_{H\backslash G/H})$ as $\mathrm{L}^2_\mu(H\backslash G/H)$, or simply as $\mathrm{L}^2(H\backslash G/H)$.

The following proposition follows immediately from the previous two propositions.

Proposition 6.12. *The algebra* $\mathrm{L}^2(H\backslash G/H)$ *is the Hilbert sum of the minimal two-sided ideals*

$$\mathfrak{a}_{\rho,\sigma_0} = L_\rho \cap \check{\overline{L}}_\rho = \bigoplus_{i=1}^{d} \bigoplus_{j=1}^{d} \mathbb{C} m_{ij}^{(\rho,H)}.$$

Each $\mathfrak{a}_{\rho,\sigma_0}$ *is a matrix algebra of dimension* d^2 *having the family* $(m_{ij}^{(\rho,H)})_{1\leq i,j\leq d}$ *as a basis. The center of* $\mathfrak{a}_{\rho,\sigma_0}$ *is the one-dimensional subspace*

$$\mathbb{C}(m_{11}^{(\rho,H)} + \cdots + m_{dd}^{(\rho,H)}) = \mathbb{C} n_\rho \theta_{\rho,\sigma_0},$$

and $u_{\rho,\sigma_0} = m_{11}^{(\rho,H)} + \cdots + m_{dd}^{(\rho,H)}$ *is the unit of* $\mathfrak{a}_{\rho,\sigma_0}$. *The map* $g \mapsto u_{\rho,\sigma_0} * g = g * u_{\rho,\sigma_0}$ *is the orthogonal projection of* $\mathrm{L}^2(H\backslash G/H)$ *onto* $\mathfrak{a}_{\rho,\sigma_0}$.

Proposition 6.13. *The algebra* $\mathrm{L}^2(H\backslash G/H)$ *is commutative if and only if* $(\rho : \sigma_0) \leq 1$ *for all* $\rho \in R(G)$. *If so, then for every* $\rho \in R(G)$ *such that* $(\rho : \sigma_0) = 1$, *the ideal* $\mathfrak{a}_{\rho,\sigma_0}$ *is one-dimensional and is spanned by the function*

$$\omega_\rho(s) = \theta_{\rho,\sigma_0} = \frac{1}{n_\rho} m_{11}^{(\rho,H)}(s),$$

which is continuous and of positive type. Thus

$$\mathrm{L}^2(H\backslash G/H) = \bigoplus_{\rho|(\rho:\sigma_0)=1} \mathbb{C}\omega_\rho.$$

The orthogonal projection of $\mathrm{L}^2(H\backslash G/H)$ *onto* $\mathbb{C}\omega_\rho$ *is given by*

$$g \mapsto \omega_\rho * g = g * \omega_\rho.$$

The function ω_ρ also satisfies the following equations:

$$\begin{aligned} \omega_\rho(tst') &= \omega_\rho(s), \qquad \textit{for all } s \in G \textit{ and all } t, t' \in H \\ \omega_\rho(e) &= 1. \end{aligned}$$

The function ω_ρ is called a *(zonal) spherical function*.

Since $(\rho : \sigma_0) = 1$, the left ideal L_ρ is equal to the ideal $\mathfrak{l}_1^{(\rho,H)}$, which by Proposition 4.5(5) is a minimal ideal in $\mathfrak{a}_\rho$, and by Proposition **??**, it is spanned by the elements of the form $\lambda_s\omega_\rho = \delta_s * \omega_\rho$, for all $s \in G$.

6.10 Spherical Harmonics on S^n and $\mathrm{L}^2(S^n)$

A nice example of the above situation arises if $G = \mathbf{SO}(n+1)$ and $H = \mathbf{SO}(n)$.

In this case, $G/H = \mathbf{SO}(n+1)/\mathbf{SO}(n) \simeq S^n$.

By Proposition 6.10, the space $\mathrm{L}^2(\mathbf{SO}(n+1)/\mathbf{SO}(n)) \simeq \mathrm{L}^2(S^n)$ is the Hilbert sum of the subspaces $L_\rho \subseteq \mathfrak{a}_\rho$ for which $(\rho : \sigma_0) \geq 1$, where

$$\mathrm{L}^2(\mathbf{SO}(n+1)) = \bigoplus_{\rho} \mathfrak{a}_\rho$$

is the Hilbert sum given by Peter–Weyl I and where $d = (\rho : \sigma_0) \geq 1$ is the number of times that the trivial representation σ_0 of $\mathbf{SO}(n)$ is contained in the restriction of M_ρ to $\mathbf{SO}(n)$.

Then L_ρ is the direct sum of the first d columns of $M_\rho^{(H)}$,

$$L_\rho = \bigoplus_{j=1}^{d} \mathfrak{l}_j^{(\rho,H)} \quad \text{and} \quad \mathfrak{l}_j^{(\rho,H)} = \bigoplus_{k=1}^{n_\rho} \mathbb{C} m_{kj}^{(\rho,H)}.$$

The subrepresentation $\Pi\colon \mathbf{SO}(n+1) \to \mathbf{U}(L_\rho)$ of the canonical representation (see Definition 6.11) $\Pi\colon \mathbf{SO}(n+1) \to \mathbf{U}(\mathrm{L}^2(S^n))$ of $\mathbf{SO}(n+1)$ in $\mathrm{L}^2(S^n)$ induced by the trivial representation of $\mathbf{SO}(n)$ in $\mathbb{C}$ is the Hilbert sum of $d = (\rho : \sigma_0)$ irreducible representations equivalent to $M_{\overline{\rho}}$.

Recall (see (Ind_G) before Definition 6.11) that

$$(\Pi_Q(f))(x) = f(Q^{-1}x) = f(Q^\top x), \qquad Q \in \mathbf{SO}(n+1),\ f \in \mathrm{L}^2(S^n),\ x \in S^n.$$

However, $(\mathbf{SO}(n+1), \mathbf{SO}(n))$ is one of examples of a *Gelfand pair* given in Section **??**, Case 1, so $\mathrm{L}^2(H\backslash G/H)$ is commutative.

We need to exhibit $\mathbf{SO}(n)$ as a subgroup of the fixed points of an involution σ of $\mathbf{SO}(n+1)$.

To do this, let $s\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the reflection about the hyperplane $x_1 = 0$, which is given by

$$s(x_1, x_2, \ldots, x_{n+1}) = (-x_1, x_2, \ldots, x_{n+1}).$$

Obviously $s^{-1} = s$. Then let $\sigma\colon \mathbf{SO}(n+1) \to \mathbf{SO}(n+1)$ be the automorphism given by

$$\sigma(Q) = sQs, \quad Q \in \mathbf{SO}(n+1).$$

Since $s^2 = I$, we also have $\sigma^2 = \mathrm{id}$.

In matrix form

$$\sigma(Q) = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} Q \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}.$$

The groups $\mathbf{SO}(n+1)^\sigma$ of fixed points of σ are the rotations $Q \in \mathbf{SO}(n+1)$ such that

$$Q = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} Q \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix},$$

and if we write

$$Q = \begin{pmatrix} q_{11} & u \\ v & Q_1 \end{pmatrix},$$

we must have

$$\begin{aligned} \begin{pmatrix} q_{11} & u \\ v & Q_1 \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} q_{11} & u \\ v & Q_1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \\ &= \begin{pmatrix} q_{11} & -u \\ -v & Q_1 \end{pmatrix}, \end{aligned}$$

and so $u = v = 0$.

Consequently, $\mathbf{SO}(n+1)^\sigma = S(\mathbf{O}(1) \times \mathbf{O}(n))$, with

$$S(\mathbf{O}(1) \times \mathbf{O}(n)) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & Q_1 \end{pmatrix} \;\middle|\; \lambda = \pm 1, \right. \\ \left. Q_1 \in \mathbf{O}(n),\ \lambda \det(Q_1) = 1 \right\}.$$

The stabilizer of $e_1 = (1, 0, \ldots, 0)$ corresponds to $\lambda = +1$, and it is indeed isomorphic to $\mathbf{SO}(n)$.

Since $(\mathbf{SO}(n+1), \mathbf{SO}(n))$ is a Gelfand pair, $\mathrm{L}^2(H \backslash G / H)$ is commutative (with $G = \mathbf{SO}(n+1)$, $H = \mathbf{SO}(n)$), so by Proposition 6.13, *we have $d = (\rho : \sigma_0) \leq 1$ for all ρ*.

It can be shown that the L_ρ for which $(\rho : \sigma_0) = 1$ *are exactly the spaces $\mathcal{H}_k^{\mathbb{C}}(S^n)$ of spherical harmonics on S^n*; see Definition 5.1.

Thus we have a Hilbert sum

$$\mathrm{L}^2(S^n) = \bigoplus_{k \geq 0} \mathcal{H}_k^{\mathbb{C}}(S^n).$$

We also obtain a decomposition of the regular representation $\mathbf{R}\colon \mathbf{SO}(n+1) \to \mathbf{U}(\mathrm{L}^2(S^n))$ into irreducible representations $\mathbf{R}_k\colon \mathbf{SO}(n+1) \to \mathbf{U}(\mathcal{H}_k^{\mathbb{C}}(S^n))$ of $\mathbf{SO}(n+1)$ in the spaces $\mathcal{H}_k^{\mathbb{C}}(S^n)$ of spherical harmonics on S^n.

The above facts are proven in Dieudonné [12] (Chapter XXIII, Section 38).

A different proof is given in Gallier and Quaintance [23] (Chapter 7).

One of the technical results used in these proofs is that

$$\begin{aligned}\mathcal{P}_k^{\mathbb{C}}(n) = \mathcal{H}_k^{\mathbb{C}}(n) \oplus \|x\|^2 \mathcal{H}_{k-2}^{\mathbb{C}}(n) \oplus \cdots \oplus \\ \|x\|^{2j} \mathcal{H}_{k-2j}^{\mathbb{C}}(n) \oplus \cdots \oplus \|x\|^{2[k/2]} \mathcal{H}_{[k/2]}^{\mathbb{C}}(n),\end{aligned}$$

with the understanding that only the first term occurs on the right-hand side when $k < 2$ (the spaces $\mathcal{P}_k^{\mathbb{C}}(n)$ and $\mathcal{H}_k^{\mathbb{C}}(n)$ are described in Definition 5.1).

It is shown in Vilenkin [39] (Chapter IX, Sections 2.10, 2.11) that the irreducible representations $\mathbf{R}_k\colon \mathbf{SO}(n+1) \to \mathbf{U}(\mathcal{H}_k^{\mathbb{C}}(S^n))$ are irreducible representations of class 1 relative to $\mathbf{SO}(n)$ (see Definition **??**) and that they form a complete set of representations of class 1 of $\mathbf{SO}(n+1)$ relative to the subgroup $\mathbf{SO}(n)$;

For $n = 2$, these are actually all the irreducible representations of $\mathbf{SO}(3)$ (see Proposition 5.3).

The space $\mathcal{H}_k^{\mathbb{C}}(S^n)$ is also the *eigenspace associated to the eigenvalue $-k(n+k-1)$ of the Laplacian Δ_{S^n} on S^n*.

The unique zonal spherical function $\omega_\rho = \frac{1}{n_\rho} m_{11}^{(\rho,H)}$ in $\mathcal{H}_k^{\mathbb{C}}(S^n)$ is given in terms of Gegenbaur polynomials; see Gallier and Quaintance [23] (Chapter 7, Sections 3, 5, 6, 7).

6.11 Induced Representations, III; Blattner's Method

6.12 The Borel Construction from a Representation

In this section we explain how the spaces of functions L^α (from Definition 6.7), and the spaces $\mathcal{H}_0$ and $\mathcal{H}^0$ from Section 6.5 and Section 6.11 can be viewed as sections of spaces that are similar to vector bundles but have less structure.

More precisely, such structures have no trivialization maps.

We begin with the simplest situation where we have a group G without any topology on it, a subgroup H of G, a vector space $\mathcal{H}_\sigma$, and a linear representation
$\sigma\colon H \to \mathbf{GL}(\mathcal{H}_\sigma)$.

As usual, write $X = G/H$ and $\pi\colon G \to G/H$ for the quotient map.

Let L^σ be the subspace of $(\mathcal{H}_\sigma)^G$ consisting of all functions $f\colon G \to \mathcal{H}_\sigma$ such that

$$f(gh) = \sigma(h^{-1})(f(g)), \quad \text{for all } g \in G \text{ and all } h \in H.$$

The key point is to construct a space $E = G \times_H \mathcal{H}_\sigma$ together with a surjective map $p\colon E \to X$, such that for every $x \in X = G/H$, the fibre $E_x = p^{-1}(x)$ is isomorphic to the vector space $\mathcal{H}_\sigma$, and the space of sections from X to E is in bijection with L^σ.

This is a special case of the so-called *Borel construction* used to construct a vector bundle from a principal bundle; see Gallier and Quaintance [23] (Chapter 9, Section 9.9).

Definition 6.12. Consider a group G, a subgroup H of G, a vector space $\mathcal{H}_\sigma$, and a linear representation $\sigma\colon H \to \mathbf{GL}(\mathcal{H}_\sigma)$. As usual, write $X = G/H$, $\pi\colon G \to G/H$ for the quotient map, and denote the coset $H = eH$ by x_0. The group H acts on $G \times \mathcal{H}_\sigma$ *on the right* by the action

$$(g, u) \cdot h = (gh, \sigma(h^{-1})(u)), \quad g \in G,\ u \in \mathcal{H}_\sigma,\ h \in H. \tag{act1}$$

The space $E = G \times_H \mathcal{H}_\sigma$ is the orbit space of $G \times \mathcal{H}_\sigma$ under the above action, namely the set of equivalence classes

$$[(g, u)] = \{(gh, \sigma(h^{-1})(u)) \mid h \in H\}\ (g \in G, u \in \mathcal{H}_\sigma)$$

of $G \times \mathcal{H}_\sigma$ under the equivalence relation $\sim$ defined such that for all $g_1, g_2 \in G$ and $u_1, u_2 \in \mathcal{H}_\sigma$,

$$\begin{aligned}(g_1, u_1) \sim (g_2, u_2) \quad &\text{iff} \\ &(\exists h \in H)(g_2 = g_1 h,\ u_2 = \sigma(h^{-1})(u_1)).\end{aligned} \tag{$\sim_1$}$$

The projection $p\colon E \to X$ is defined as $\pi \circ pr_1$, namely for every equivalence class $z = [(g, u)] = \{(gh, \sigma(h^{-1})(u)) \mid h \in H\}$,

$$p(z) = gH = \pi(pr_1(z)).$$

It is immediately verified that the above definition does not depend on the choice of g in the coset gH.

For every $x = gH \in G/H = X$, the fibre $E_x = p^{-1}(x)$ can be given the structure of a vector space isomorphic to $\mathcal{H}_\sigma$.

If we pick a section $r\colon X \to G$, namely a set of representatives $(r_x)_{x\in X}$ (with $r_x \in G$) for the cosets $x \in X = G/H$, with $r_{x_0} = e$,[2] then we can show that the map $\theta_{r_x}\colon \mathcal{H}_\sigma \to E_x$ given by

$$\theta_{r_x}(u) = [(r_x, u)] \tag{θ_{r_x}}$$

is injective.

The map θ_{r_x} is also surjective since for any equivalence class $[(r_x, u)] \in E_x$, by construction, $\theta_{r_x}(u) = [(r_x, u)]$.

Note that the above shows that the equivalence classes in the fibre E_x are the subsets

$$[(r_x, u)] = \{(r_x h, \sigma(h^{-1})(u)) \mid h \in H\}$$

and that any two such classes are disjoint for distinct vectors u_1, u_2 in $\mathcal{H}_\sigma$.

[2] We always assume that for every set of coset representatives $(r_x)_{x\in X}$ we chose $r_{x_0} = e$.

We can transfer the vector space structure on $\mathcal{H}_\sigma$ to E_x using the bijection θ_{r_x}, namely

$$\begin{aligned}[(r_x, u_1)] + [(r_x, u_2)] &= [(r_x, u_1 + u_2)] \\ \lambda[(r_x, u)] &= [(r_x, \lambda u)],\end{aligned}$$

for all $u, u_1, u_2 \in \mathcal{H}_\sigma$, and $\lambda \in \mathbb{C}$.

There is a linear isomorphism between $\mathcal{H}_\sigma$ and E_x (although noncanonical).

Looking ahead, if $\mathcal{H}_\sigma$ is a (separable) Hilbert space, G is locally compact, H is a closed subgroup of G, and and if $\sigma\colon H \to \mathbf{U}(\mathcal{H}_\sigma)$ is a unitary representation, then the map θ_{r_x} can be used to transfer the Hilbert space structure of $\mathcal{H}_\sigma$ to the fibre E_x by setting

$$\langle [(r_x, u_1)], [(r_x, u_2)] \rangle = \langle u_2, u_2 \rangle, \quad u_1, u_2 \in \mathcal{H}_\sigma.$$

The fact that the space L^σ in realized by the space of sections of E is shown in the next proposition.

Definition 6.13. A *section of E* is any function $s\colon X \to E$ such that $p \circ s = \mathrm{id}_X$ where p is the projection $p\colon E \to X$, or equivalently a function $s\colon X \to E$ such that $s(x) \in E_x$ for every $x \in X = G/H$. The set of sections $s\colon X \to E$ is denoted $\Gamma(E)$.

Remark: At this stage $X = G/H$ is just a set without any topology so a section is just a function. Later when G and H are locally compact groups it will make sense to consider continuous sections.

Given a set of coset representatives $(r_x)_{x \in X}$, recall from Definition 6.4 (Equation (u)) that we define $u(g, x)$ as

$$u(g, x) = r_{g \cdot x}^{-1} g r_x,$$

and that by Equation (s) we have

$$g = r_x u(g, x_0), \quad s \in G, \quad x = gH.$$

Proposition 6.14. *Let $E = G \times_H \mathcal{H}_\sigma$, $X = G/H$, $p \colon E \to X$, and $\sigma \colon H \to \mathbf{GL}(\mathcal{H}_\sigma)$, as in Definition 6.12. Also let L^σ be the set consisting of all functions $f \colon G \to \mathcal{H}_\sigma$ such that*

$$f(gh) = \sigma(h^{-1})(f(g)), \quad \text{for all } g \in G \text{ and all } h \in H.$$

The maps $\mathcal{S} \colon L^\sigma \to \Gamma(E)$ and $\mathcal{L} \colon \Gamma(E) \to L^\sigma$ are defined as follows.

Pick any set of representatives $(r_x)_{x\in X}$ *(with* $r_x \in G$*) for the cosets* $x \in X = G/H$*. For every function* $f\colon G \to \mathcal{H}_\sigma$*, for any coset* $x = r_x H$*, define the section* $\mathcal{S}(f)$ *by*

$$\mathcal{S}(f)(x) = [(r_x, f(r_x))], \qquad (\mathcal{S})$$

and for every section $s\colon X \to E$*, for every coset* $x = r_x H$*, if* $s(x) = [(r_x, u)]$ *for some* $u \in \mathcal{H}_\sigma$*, define the function* $\mathcal{L}(s)$ *on* G *by*

$$\mathcal{L}(s)(r_x h) = \sigma(h^{-1})(u), \quad h \in H \qquad (\mathcal{L})$$

or equivalently

$$\mathcal{L}(s)(g) = \sigma(u(g, x_0)^{-1})(u), \quad g \in G. \qquad (\mathcal{L}')$$

Then $\mathcal{S}(f)$ does not depend on the set of coset representatives $(r_x)_{x\in X}$, $\mathcal{S}(f) \in \Gamma(E)$, $\mathcal{L}(s) \in L^\sigma$, and $\mathcal{S}$ and $\mathcal{L}$ are mutual inverses. Therefore $\mathcal{S}$ is a bijection between L^σ and $\Gamma(E)$.

Remark: If we use the isomorphisms $\theta_{r_x} \colon \mathcal{H}_\sigma \to E_x$ given by

$$\theta_{r_x}(u) = [(r_x, u)], \quad u \in \mathcal{H}_\sigma,$$

then the maps $\mathcal{S} \colon L^\sigma \to \Gamma(E)$ and $\mathcal{L} \colon \Gamma(E) \to L^\sigma$ are defined as follows.

For every function $f\colon G \to \mathcal{H}_\sigma$ in L^σ and for any coset $x = r_x H$,

$$\mathcal{S}(f)(x) = \theta_{r_x}(f(r_x)), \tag{$\mathcal{S}_2$}$$

and for every section $s\colon X \to E$ and any $g \in G$,

$$\mathcal{L}(s)(g) = \sigma(u(g, x_0)^{-1})(\theta_{r_x}^{-1}(s(gH))). \tag{$\mathcal{L}_2$}$$

The isomorphisms θ_{r_x} are omitted by some authors but this is not quite right.

The last important ingredient is that G acts (on the left) on $E = G \times_H \mathcal{H}_\sigma$ in an *equilinear* fashion.

Definition 6.14. Under the same conditions as in Definition 6.12, we define a left action of G on $E = G \times_H \mathcal{H}_\sigma$ by

$$g_1 \cdot [(g, u)] = [(g_1 g, u)], \quad g_1, g \in G, \ u \in \mathcal{H}_\sigma.$$

That this action is equilinear means the following.

Proposition 6.15. *Under the same conditions as in Definition 6.14, the following facts hold:*

(1) The action of G on $E = G \times_H \mathcal{H}_\sigma$ is equivariant, which means that

$$p(g \cdot [(g_1, u)]) = g \cdot p([(g_1, u)]), \quad g, g_1 \in G, \, u \in \mathcal{H}_\sigma.$$

The action on the right-hand side is the action on cosets in G/H given by $g \cdot (g_2 H) = (g g_2)H$.

(2) The restriction of the action of G to the fibre E_x is a linear isomorphism between E_x and $E_{g\cdot x}$ ($x \in G/H$). In particular, every fibre E_x is isomorphic to E_{x_0}.

Observe that *the action of H on the fibre E_{x_0} is a representation $\sigma^0\colon H \to \mathbf{GL}(E_{x_0})$ equivalent to the representation $\sigma\colon H \to \mathbf{GL}(\mathcal{H}_\sigma)$*, since E_{x_0} consists of the equivalence classes of the form $[(e, u)]$ (recall that $r_{x_0} = e$), with $u \in \mathcal{H}_\sigma$, so for every $h \in H$,

$$h \cdot [(e, u)] = [(h, u)] = [(hh^{-1}, \sigma(h)(u))] = [(e, \sigma(h)(u))].$$

The linear isomorphisms between the fibres E_{x_0} and E_x induce representations $\sigma^x\colon H \to \mathbf{GL}(E_x)$ equivalent to the representation $\sigma\colon H \to \mathbf{GL}(\mathcal{H}_\sigma)$.

6.13 Induced Representations and G-Bundles

Next what we would like to do is to show *how induced representations can be recovered from certain kinds of vector bundles (actually a more basic notion of vector bundle) equipped with an equilinear action*.

Such construction is given as an exercise in Dieudonné [11] (Chapter XXII, Section 3, Problem 16).

It is also discussed in Kirillov [28] (Section 13) and sketched without details in Folland [19] (Chapter 6).

Following Kirillov, we adopt the terminology of G-bundle.

First we introduce a weaker notion that we call pre-G-bundle (for the lack of a better name).

Definition 6.15. Let G be a group, H be a subgroup of G, E be some set, and let $p\colon E \to X$ be a surjective map, where as usual we write $X = G/H$. We say that E (really $p\colon E \to X$) is a *pre-G-bundle* if there is an *equivariant* left action $\cdot$ of G on E, which means that

$$p(g \cdot z) = g \cdot p(z), \quad g \in G,\ z \in E.$$

Proposition 6.16. *If $p\colon E \to X$ is a pre-G-bundle, then for every $x \in X = G/H$, for every $g \in G$, the map $z \mapsto g \cdot z$ ($z \in E_x$) is a bijection from E_x to $E_{g \cdot x}$.*

We finally come to the desired concept by requiring that the fibres are vector spaces and that the bijections between fibres are linear isomorphisms.

The key concept is the notion of *equilinear action* which occurs in Dieudonné [15], Chapter XIX, Section 1.

Definition 6.16. Let G be a group, H be a subgroup of G, E be some set, and let $p\colon E \to X$ be a surjective map, where as usual we write $X = G/H$. We say that E (really $p\colon E \to X$) is a *G-bundle* if each fibre E_x ($x \in X = G/H$) is a vector space and if there is an *equilinear* left action $\cdot$ of G on E, which means that:

(1) The action is equivariant, that is,

$$p(g \cdot z) = g \cdot p(z), \quad g \in G,\ z \in E.$$

(2) For every $x \in X = G/H$, for every $g \in G$, the map $z \mapsto g \cdot z$ ($z \in E_x$) is a linear isomorphism between E_x and $E_{g \cdot x}$.

Let x_0 denote the coset $H = eH$.

Proposition 6.16 implies that every fibre is isomorphic to E_{x_0}.

Then the restriction of the action of G to H on the fibre E_{x_0}, for simplicity also denoted as E_0, maps E_0 to E_0 (since $h \cdot x_0 = x_0$ for all $h \in H$).

Since the maps $z \mapsto h \cdot z$ ($z \in E_0$) are linear isomorphisms, *we have a representation $\sigma \colon H \to \mathbf{GL}(E_0)$* given by

$$\sigma(h)(z) = h \cdot z. \qquad (\sigma)$$

Observe that $E = G \times_H \mathcal{H}_\sigma$ with the projection $p \colon E \to X$ is a G-bundle.

If E is an abstract G-bundle as in Definition 6.16, then *the fibre E_0 plays the role of the vector space $\mathcal{H}_\sigma$* which occurs in the representation $\sigma\colon H \to \mathbf{GL}(\mathcal{H}_\sigma)$ and is involved in the construction of the G-bundle $G \times_H \mathcal{H}_\sigma$.

So it is natural to also refer to E_0 as $\mathcal{H}_\sigma$, which we will do except when confusion arises

Let $(r_x)_{x\in X}$ be any set of coset representatives of $X = G/H$.

The map σ_x from E_x to itself given by

$$\sigma_x(h)(z) = (r_x h r_x^{-1})\cdot z = r_x\cdot\sigma(h)(r_x^{-1}\cdot z), h \in H, z \in E_x$$

is a linear isomorphism of E_x, in other words, a representation $\sigma_x\colon H \to \mathbf{GL}(E_x)$.

The representation $\sigma_x \colon H \to \mathbf{GL}(E_x)$ is equivalent to the representation $\sigma \colon H \to \mathbf{GL}(E_0)$ *via* the linear isomorphism from E_0 to E_x given by $z \mapsto r_x \cdot z$.

It is easy to see that if another set of coset representatives $(r_x h_x)_{x \in X}$ is used, then

$$\sigma_x(h)(z) = (r_x h_x h h_x^{-1} r_x^{-1}) \cdot z;$$

in other words, we obtain a representation equivalent to $\sigma \colon H \to \mathbf{GL}(E_0)$, where the linear isomorphism from E_0 to E_x is given by $z \mapsto r_x h_x \cdot z$.

Consequently, the sections in $\Gamma(E)$, called *feature fields* in group equivariant deep learning in computer vision, are *functions whose domain transforms under the action of G and whose codomain transforms by representations of H equivalent to $\sigma \colon H \to \mathbf{GL}(E_0)$;*
more precisely each fibre E_x transforms under the representation σ_x.

The space L^σ and the representation of G in L^σ induced by $\sigma\colon H \to \mathbf{GL}(E_0)$ can be recovered from the G-vector bundle as we now explain.

Definition 6.17. Let $p\colon E \to X$ be a G-bundle, with $X = G/H$. As before, let $x_0 = H$, let E_0 be the fibre $E_0 = p^{-1}(x_0)$, and let $\sigma\colon H \to \mathbf{GL}(E_0)$ be the representation given by $\sigma(h)(z) = h \cdot z$ for all $z \in E_0$ and all $h \in H$. Also let L^σ be the set consisting of all functions $f\colon G \to E_0$ such that

$$f(gh) = \sigma(h^{-1})(f(g)) = h^{-1} \cdot f(g), \quad \text{for all } g \in G \text{ and all } h \in H. \qquad (*_{\dagger_1})$$

There is an action of G on the set $\Gamma(E)$ of section $s\colon X \to E$ given by

$$(g \cdot s)(x) = g \cdot (s(g^{-1} \cdot x)), \qquad g \in G,\ x \in X. \quad (\dagger_\Gamma)$$

In the above equation, G acts on $X = G/H$ in $g^{-1} \cdot x$, and G acts on E in $g \cdot (s(g^{-1} \cdot x))$.

Define the maps $\mathcal{S}$ and $\mathcal{L}$ as follows.

For every function $f\colon G \to E_0 \in L^\sigma$, for every coset $x \in X = G/H$ and any coset representative $r_x \in G$ of x, let

$$\mathcal{S}(f)(x) = r_x \cdot f(r_x), \tag{$\mathcal{S}_3$}$$

where the action is the action of G on E.

For every section $s\colon X \to E$, for every $g \in G$, let

$$\mathcal{L}(s)(g) = g^{-1} \cdot s(gH) = g^{-1} \cdot s(g \cdot x_0), \tag{$\mathcal{L}_3$}$$

where the action is the action of G on E.

Proposition 6.17. *The following facts hold.*

(1) *The map* $\mathcal{S}(f)$ *is independent of the choice of the representative* g *chosen in the coset* $x = gH = g\cdot x_0$ *and* $\mathcal{S}(f) \in \Gamma(E)$*; that is,*

$$\mathcal{S}(f)(gH) = \mathcal{S}(f)(g \cdot x_0) = g \cdot f(g), \quad g \in G. \quad (\mathcal{S}_3'')$$

(2) *We have* $\mathcal{L}(s) \in L^\sigma$.

(3) *The maps* $\mathcal{S}\colon L^\sigma \to \Gamma(E)$ *and* $\mathcal{L}\colon \Gamma(E) \to L^\sigma$ *are mutual inverses. Thus* $\mathcal{S}$ *is an isomorphism between* L^σ *and* $\Gamma(E)$.

We can also recover the representation $\mathrm{Ind}_H^G\,\sigma\colon G \to \mathbf{GL}(L^\sigma)$ induced by the representation $\sigma\colon H \to \mathbf{GL}(E_0)$.

Proposition 6.18. *Define the map* $\rho\colon G \to \mathbf{GL}(L^\sigma)$ *by*

$$\rho(g)(f) = \mathcal{S}^{-1}(g\cdot\mathcal{S}(f)) = \mathcal{L}(g\cdot\mathcal{S}(f)), \quad g \in G, f \in L^\sigma. \tag{$\dagger_2$}$$

In the above equation, $\mathcal{S}(f) \in \Gamma(E)$ *and the action of* G *is the action of* G *on* $\Gamma(E)$ *from Definition 6.17. For all* $g, g_1 \in G$ *and all* $f \in L^\sigma$*, we have*

$$[\rho(g)(f)](g_1) = f(g^{-1}g_1), \tag{$\dagger_3$}$$

that is, $\rho\colon G \to \mathbf{GL}(L^\sigma)$ *is the representation* $\mathrm{Ind}_H^G\,\sigma$ *induced from the representation* $\sigma\colon H \to \mathbf{GL}(E_0)$*.*

6.14 Hermitian G-Bundles

The above definitions and constructions can be adapted to deal with unitary representations.

In this case, G is a locally compact group, H is a closed subgroup of G, and $\sigma\colon H \to \mathbf{U}(\mathcal{H}_\sigma)$ is a unitary representation, where $\mathcal{H}_\sigma$ is a separable Hilbert space.

As we explained earlier, up to linear isomorphisms, we can endow the fibres E_x of the G-bundle $E = G \times_H \mathcal{H}_\sigma$ with a Hilbert space structure so that each fibre E_x is isometric to $\mathcal{H}_\sigma$ *via* a unitary isomorphism.

The action of G on E has the property that each map $z \mapsto g \cdot z$ from the fibre E_x to the fibre $E_{g\cdot x}$ is *unitary*.

Definition 6.18. Let G be a locally compact group, H a closed subgroup of G, E be some topological Hausdorff space, and let $p\colon E \to X$ be a surjective *continuous* map, where as usual we write $X = G/H$. We say that E (really $p\colon E \to X$) is a *hermitian G-bundle* if each fibre E_x ($x \in X = G/H$) is a separable Hilbert space and if there is an *equilinear continuous* left action $\cdot\colon G \times E \to E$ of G on E, which means that:

(1) The action is equivariant, that is,

$$p(g \cdot z) = g \cdot p(z), \quad g \in G,\ z \in E.$$

(2) For every $x \in X = G/H$, for every $g \in G$, the map $z \mapsto g \cdot z$ ($z \in E_x$) is a *unitary isomorphism* between E_x and $E_{g \cdot x}$.

Let x_0 denote the coset $H = eH$.

Every fibre is isomorphic to E_{x_0} and the restriction of the action of G to H on the fibre E_{x_0}, for simplicity also denoted as E_0, maps E_0 to E_0 (since $h \cdot x_0 = x_0$ for all $h \in H$).

Since the action of G on E is continuous, for every $z \in E_0$ the map $h \mapsto h \cdot z$ is a continuous map from H to E_0, and since the maps $z \mapsto h \cdot z$ ($z \in E_0$) are unitary, we have a *unitary representation* $\sigma \colon H \to \mathbf{U}(E_0)$ given by

$$\sigma(h)(z) = h \cdot z. \qquad (\sigma)$$

If the fibres are finite-dimensional vector spaces equipped with hermitian inner products, we say that E has *finite rank*, and the common dimension of these vector spaces is called the *rank* of E.

Assume that E is a hermitian G-bundle of rank n, and pick some orthonormal basis $(e_1, \ldots, e_n)$ of E_0.

Since the map $z \mapsto g \cdot z$ ($z \in E_0, g \in G$) is a unitary map from E_0 to $E_{g \cdot x_0}$, the n-tuple $(g \cdot e_1, \ldots, g \cdot e_n)$ is an orthonormal basis of $E_{g \cdot x_0}$.

Inspired by Section 6.1 we make the following definition.

Definition 6.19. Let E be a hermitian G-bundle of rank n and pick some orthonormal basis $(e_1, \ldots, e_n)$ of E_0.

The Hilbert space $\mathrm{L}^2(G; E_0)$ consists of all functions $f \colon G \to E_0$ such that $f = f_1 e_1 + \cdots + f_n e_n$, where the f_i are functions in $\mathrm{L}^2(G)$; equivalently, $\mathrm{L}^2(G; E_0)$ is the finite Hilbert sum

$$\mathrm{L}^2(G; E_0) = \bigoplus_{i=1}^{n} \mathrm{L}^2(G) e_i.$$

The inner product of two functions $f = \sum_{i=1} f_i e_i$ and $g = \sum_{i=1} g_i e_i$ is

$$\langle f, g\rangle = \sum_{i=1}^{n} \int_G f_i(s)\overline{g_i(s)}\, d\lambda_G(s),$$

where λ_G is a left Haar measure on G.

Let L^σ be the subspace of $\mathrm{L}^2(G; E_0)$ given by

$$\begin{aligned} L^\sigma = \{f \in \mathrm{L}^2(G; E_0) \mid f(gh) = \sigma(h^{-1})(f(g)),\\ \text{for all } g \in G \text{ and all } h \in H\}. \end{aligned} \tag{$\dagger_4$}$$

It is easy to check that L^σ is closed in $\mathrm{L}^2(G; E_0)$, so it is a Hilbert space.

If $A(h)$ is the unitary matrix representing $\sigma(h)$ with respect to the basis $(e_1, \ldots, e_n)$, with

$$\sigma(h)(e_j) = \sum_{i=1}^{n} a_{ij} e_i,$$

we leave it as an exercise to prove that the condition $f(gh) = \sigma(h^{-1})(f(g))$ translates into

$$\begin{pmatrix} f_1(gh) \\ \vdots \\ f_n(gh) \end{pmatrix} = A(h)^* \begin{pmatrix} f_1(g) \\ \vdots \\ f_n(g) \end{pmatrix}.$$

Since $(g \cdot e_1, \ldots, g \cdot e_n)$ is an orthonormal basis of $E_{g \cdot x_0}$ for every g, every section $s \colon X \to E$ is uniquely determined by n functions $s_i \colon X \to \mathbb{C}$ defined by

$$s(g \cdot x_0) = s_1(g \cdot x_0)(g \cdot e_1) + \cdots + s_n(g \cdot x_0)(g \cdot e_n), \quad g \in G.$$

By analogy with the definition of $\mathrm{L}^2(G; E_0)$ we have the following definition.

Definition 6.20. Let E be a hermitian G-bundle of rank n and pick some orthonormal basis $(e_1, \ldots, e_n)$ of E_0. The subspace $\mathrm{L}^2(X; E)$ of $\Gamma(E)$ is defined as the Hilbert space of sections $s\colon X \to E$ that can be expressed as

$$s(g\cdot x_0) = s_1(g\cdot x_0)(g\cdot e_1) + \cdots + s_n(g\cdot x_0)(g\cdot e_n), \quad g \in G$$

for some functions $s_i \in \mathrm{L}^2_\mu(X)$, where μ is the G-invariant measure (unique up to a scalar) on $X = G/H$ induced by λ_G.

Then for two sections $s, t \in \mathrm{L}^2(X; E)$ determined by the n-tuples $(s_1, \ldots, s_n)$ and $(t_1, \ldots, t_n)$ of functions in $\mathrm{L}^2_\mu(X)$, the inner product is given by

$$\langle s, t\rangle = \sum_{i=1}^{n} \int_X s_i(x)\overline{t_i(x)}\, d\mu(x).$$

Note that the induced representation $\rho\colon G \to \mathbf{GL}(L^\sigma)$ of Proposition 6.18 is now a unitary representation $\rho\colon G \to \mathbf{U}(L^\sigma)$.

A *difficulty that arises because sections now belong to* $\mathrm{L}^2(X; E)$ *and functions in* L^σ *now belong to* $\mathrm{L}^2(G; E_0)$ is that, in general, if $r\colon X \to G$ is a section specifying a set of coset representatives of $X = G/H$, the maps $\mathcal{L}$ and $\mathcal{S}$ as defined by

$$\mathcal{L}(s)(g) = (\sigma(u(g, x_0)))^{-1}(r_x^{-1}\cdot s(g\cdot x_0)), \quad x = gH = g\cdot x_0 \qquad (\mathcal{L}_3')$$

and

$$\mathcal{S}(f)(g \cdot x_0) = r_x \cdot \sigma(u(g, x_0))(f(g)) \qquad (\mathcal{S}_3')$$

may yield a function $\mathcal{L}(s)$ *not* in L^σ for some section $s \in \mathrm{L}^2(X; E)$, or a section $\mathcal{S}(f)$ *not* in $\mathrm{L}^2(X; E)$ for some function $f \in L^\sigma$.

If they do for all $s \in \mathrm{L}^2(X; E)$ and all $f \in L^\sigma$, they are mutual inverse maps from L^σ to $\mathrm{L}^2(X; E)$ so we can figure out what is the induced representation $\Pi\colon G \to \mathbf{U}(\mathrm{L}^2(X; E))$ from the definition of the representation $\rho\colon G \to \mathbf{U}(L^\sigma)$ using the fact that the following diagram commutes

$$\begin{array}{ccc}
L^\sigma & \xrightarrow{\rho(g)} & L^\sigma \\
\mathcal{L} \uparrow & & \downarrow S \\
\mathrm{L}^2(X; E) & \xrightarrow[\Pi_g]{} & \mathrm{L}^2(X; E)
\end{array}$$

for every $g \in G$.

For any $g \in G$, any $x \in X = G/H$, and any $s \in \mathrm{L}^2(X; E)$, since $\rho(g)(f) = \mathcal{L}(g \cdot \mathcal{S}(f))$ for any $f \in L^\sigma$, we have

$$\begin{aligned}(\Pi_g(s))(x) &= [\mathcal{S}(\rho(g)(\mathcal{L}(s))](x) = [\mathcal{S}(\mathcal{L}(g \cdot \mathcal{S}(\mathcal{L}(s))))](x) \\ &= (g \cdot s)(x) = g \cdot s(g^{-1} \cdot x),\end{aligned}$$

which we record as

$$(\Pi_g(s))(x) = g \cdot s(g^{-1} \cdot x). \tag{$\dagger_5$}$$

Unravelling the definitions we get

$$g = r_x u(g, g^{-1} \cdot x)(r_{g^{-1} \cdot x})^{-1}, \tag{$\dagger_6$}$$

and substituting the right-hand side expression for the leftmost occurrence of g in $g \cdot s(g^{-1} \cdot x)$ we deduce that

$$(\Pi_g(s))(x) = r_x \cdot \sigma(u(g, g^{-1} \cdot x))((r_{g^{-1} \cdot x})^{-1} \cdot s(g^{-1} \cdot x)). \tag{$\dagger_7$}$$

If we compare with Formula (Π_s^α) in Definition 6.2, namely,

$$(\Pi_g^\alpha(s))(x) = \alpha(g, g^{-1} \cdot x)(s(g^{-1} \cdot x)) \qquad (\Pi_s^\alpha)$$

since $\alpha(g, y) = \sigma(u(g, y))$ is a cocycle, we have

$$(\Pi_g^\sigma(s))(x) = \sigma(u(g, g^{-1} \cdot x))(s(g^{-1} \cdot x)), \qquad (\Pi_s^\sigma)$$

and as we explained earlier, in the case of the G-bundle $G \times_H \mathcal{H}_\sigma$, the map $z \mapsto r_x \cdot z$ sends the fibre E_0 to the fibre E_x and the map $x \mapsto g^{-1} \cdot x$ sends the fibre $E_{g\cdot x}$ to the fibre E_0, and since σ is a representation in E_0, we see that $(\dagger_7)$ and (Π_s^σ) are indeed equivalent.

We still have the issue that, in general, the representation Π may not be continuous.

This depends on the existence of suitable sections $r\colon X \to G$.

A case where continuous sections exist is when $G = N \rtimes H$ is a *semi-direct product with N abelian*; see Section **??**.

6.15 Hermitian G-Vector Bundles

A way to deal with the problem that continuous sections $r\colon X \to G$ may not exist is to assume that $p\colon E \to G/H$ is locally trivial, namely to *assume the existence of local trivializations*.

In other words we assume that E is a vector bundle.

We will recall the definition of vector bundles and principal bundles below.

Vector bundles and principal bundles are discussed in Gallier and Quaintance [23], Bott and Tu [3], Morita [34], Bröcker and tom Dieck [5], Duistermaat and Kolk [17] and Dieudonné [14].

To avoid technical complications *we assume that G is a Lie group and that H is a closed Lie subgroup of G*.

Now because G is a Lie group and H is a closed Lie subgroup of G, the quotient space $X = G/H$ is a smooth manifold and $\pi\colon G \to G/H$ is a principal H-bundle, whose definition is recalled below; see Gallier and Quaintance [23] (Section 9.9, Proposition 9.2) and Duistermaat and Kolk [17] (Appendix A).

Definition 6.21. A *principal H-bundle* is a quadruple $\xi = (\mathcal{E}, \pi, \mathcal{E}/H, H)$, where $\mathcal{E}$ be a smooth manifold, H is Lie group, and $\cdot\colon \mathcal{E} \times H \to \mathcal{E}$ is a smooth right action of H on $\mathcal{E}$ satisfying the following properties:

(1) The right action of H on $\mathcal{E}$ is free;

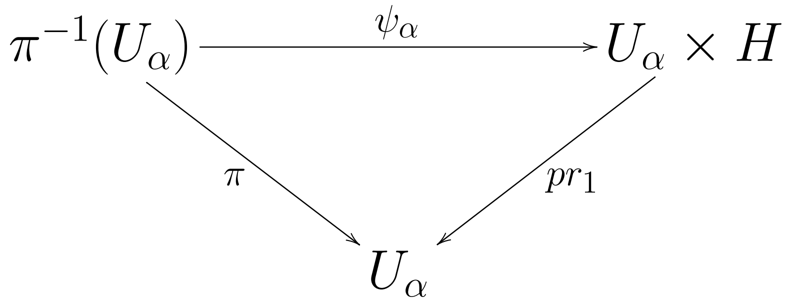
(2) The orbit space $X = \mathcal{E}/H$ is a smooth manifold under the quotient topology, and the projection $\pi\colon \mathcal{E} \to \mathcal{E}/H$ is smooth;

(3) There is some open cover $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of $X = \mathcal{E}/H$ and a family $\psi = (\psi_\alpha)_{\alpha \in I}$ of diffeomorphisms called *(local) trivializations*

$$\psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times H,$$

such that

(a) (local triviality) the diagram



commutes.

(b) Every map $\psi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times H$ is an equivariant diffeomorphism, which means that

$$\psi_\alpha(z \cdot h) = \psi_\alpha(z) \cdot h,$$

for all $z \in \pi^{-1}(U_\alpha)$ and all $h \in H$, where the right action of H on $U_\alpha \times H$ is $(x, h_1) \cdot h = (x, h_1 h)$.

Observe that if $\psi_\alpha(z) = (x, h_1)$, then since $\psi_\alpha(z) \cdot h = (x, h_1 h)$, we have $pr_1(\psi_\alpha(z) \cdot h) = pr_1(\psi_\alpha(z)) = x = \pi(z)$.

Recall that the action $\cdot \colon \mathcal{E} \times H \to \mathcal{E}$ is *free* if it acts without fixed points, that is, for every $h \in H$, if $h \neq 1$, then $x \cdot h \neq x$ for all $x \in \mathcal{E}$.

By Conditions (a) and (b) and the definition of the right action of H on $U_\alpha \times H$, for all $z \in \pi^{-1}(U_\alpha)$ and all $h \in H$, we have

$$\pi(z{\cdot}h) = pr_1(\psi_\alpha(z{\cdot}h)) = pr_1(\psi_\alpha(z){\cdot}h) = pr_1(\psi_\alpha(z)) = \pi(z),$$

so for any $x \in X = \mathcal{E}/H$ and any $z \in \mathcal{E}_x = \pi^{-1}(x)$, we have $z \cdot h \in \mathcal{E}_x$.

In fact, for any $z \in \mathcal{E}_x$, it can be shown that

$$\mathcal{E}_x = \{z \cdot h \mid h \in H\},$$

namely *the orbits of the right action of H on $\mathcal{E}$ are the fibres $\mathcal{E}_x$, with $x \in X$*.

Since the action of H on $\mathcal{E}$ is free, the action of H on $\mathcal{E}_x$ is also free.

For all α, β such that $U_\alpha \cap U_\beta \neq \emptyset$, for every $x \in U_\alpha \cap U_\beta$, we have a diffeomorphism

$$\psi_{\alpha,x} \circ \psi_{\beta,x}^{-1} \colon H \longrightarrow H,$$

which yields the map $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{Diff}(H)$ called a *transition map* given by

$$g_{\alpha\beta}(x) = \psi_{\alpha,x} \circ \psi_{\beta,x}^{-1}, \quad x \in U_\alpha \cap U_\beta.$$

Intuitively, the transition functions express how the fibre $\mathcal{E}_x$ *twists* as x moves in $U_\alpha \cap U_\beta$.

From the definition above, the isomorphism $\psi_\alpha \circ \psi_\beta^{-1} \colon (U_\alpha \cap U_\beta) \times H \to (U_\alpha \cap U_\beta) \times H$ is given by

$$(\psi_\alpha \circ \psi_\beta^{-1})(x, h) = (x, g_{\alpha\beta}(x)(h)), \quad x \in U_\alpha \cap U_\beta, h \in H.$$

A priori, the map $g_{\alpha\beta}(x)$ is a diffeomorphism of the Lie group H, but because the transition maps ψ_α are equivariant, it is shown in Gallier and Quaintance [23] (Chapter 9, Proposition 9.21) that $g_{\alpha\beta}(x)$ is the left translation by $g_{\alpha\beta}(x)(1) \in H$, that is,

$$g_{\alpha\beta}(x)(h) = g_{\alpha\beta}(x)(1)h, \quad x \in U_\alpha \cap U_\beta,\ h \in H.$$

Since the group of left translations of H (the maps $L_h \colon H \to H$ given by $L_h(h_1) = hh_1$ $(h, h_1 \in H)$) is isomorphic to H, we usually view the map $g_{\alpha\beta}(x)$ as a the element $g_{\alpha\beta}(x)(1)$ of H.

Another technical issue is that Definition 6.21 is too restrictive because it does not allow for the addition of compatible local trivializations.

We can fix this problem as follows.

Definition 6.22. Let $\xi = (\mathcal{E}, \pi, X, H)$ be principal bundle, with $X = \mathcal{E}/H$. Given a trivializing cover $\{(U_\alpha, \psi_\alpha)\}$ for ξ, for any open U of X and any diffeomorphism

$$\varphi\colon \pi^{-1}(U) \to U \times H,$$

we say that (U, φ) is *compatible with the trivializing cover* $\{(U_\alpha, \psi_\alpha)\}$ iff whenever $U \cap U_\alpha \neq \emptyset$, there is some smooth map $g_\alpha\colon U \cap U_\alpha \to H$, so that

$$\varphi \circ \psi_\alpha^{-1}(x, h) = (x, g_\alpha(x)(h)),$$

for all $x \in U \cap U_\alpha$ and all $h \in H$.

Two trivializing covers are *equivalent* iff every local trivialization of one cover is compatible with the other cover.

This is equivalent to saying that the union of two trivializing covers is a trivializing cover.

Definition 6.22 yields the official definition of a principal bundle $\xi = (\mathcal{E}, \pi, X, H)$ in which $\{(U_\alpha, \psi_\alpha)\}$ is an equivalence class of trivializing covers.

As for manifolds, given a trivializing cover $\{(U_\alpha, \psi_\alpha)\}$, the set of all bundle charts compatible with $\{(U_\alpha, \psi_\alpha)\}$ is a maximal trivializing cover equivalent to $\{(U_\alpha, \psi_\alpha)\}$.

In the special case where $\mathcal{E}$ is equal to a Lie group G and H is a closed Lie subgroup of G, as we said above, $(G, \pi\colon G \to X, X, H)$ is principal H-bundle (with $X = G/H$).

Definition 6.23. A *hermitian vector bundle with fibre $\mathcal{H}$* is a quadruple $\xi = (\mathcal{E}, p, X, \mathcal{H})$, where $\mathcal{E}$ and X are smooth manifold, $p\colon \mathcal{E} \to X$ is a surjective smooth map, and $\mathcal{H}$ is a finite-dimensional complex vector space with a hermitian inner product, such that the following conditions hold:

(1) Each fibre $\mathcal{E}_x$ ($x \in X$) is a finite-dimensional space equipped with a hermitian inner product $\langle -, - \rangle_x$.

(2) There is some open cover $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of X and a family $\varphi = (\varphi_\alpha)_{\alpha \in I}$ of diffeomorphisms called *(local) trivializations*

$$\varphi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times \mathcal{H},$$

such that:

(a) (local triviality) the diagram

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times \mathcal{H} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U_\alpha & \end{array}$$

commutes.

(b) For every $x \in U_\alpha$, the map $\varphi_{\alpha,x} \colon \mathcal{E}_x \to \mathcal{H}$ is a unitary isomorphism.

Since the maps $\varphi_{\alpha,x}$ are unitary, the maps

$$\varphi_{\alpha,x} \circ \varphi_{\beta,x}^{-1} \colon \mathcal{H} \longrightarrow \mathcal{H},$$

are also unitary, so the transition maps are of the form $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathbf{U}(\mathcal{H})$.

We also need to be able to add compatible trivializations.

Definition 6.24. Let $\xi = (\mathcal{E}, p, X, \mathcal{H})$ be a hermitian vector bundle. Given a trivializing cover $\{(U_\alpha, \varphi_\alpha)\}$ for ξ, for any open U of X and any diffeomorphism

$$\varphi \colon p^{-1}(U) \to U \times \mathcal{H},$$

we say that (U, φ) is *compatible with the trivializing cover* $\{(U_\alpha, \varphi_\alpha)\}$ iff whenever $U \cap U_\alpha \neq \emptyset$, there is some smooth map $g_\alpha \colon U \cap U_\alpha \to \mathbf{U}(\mathcal{H})$, so that

$$\varphi \circ \varphi_\alpha^{-1}(x, u) = (x, g_\alpha(x)(u)),$$

for all $x \in U \cap U_\alpha$ and all $u \in \mathcal{H}$. Two trivializing covers are *equivalent* iff every local trivialization of one cover is compatible with the other cover.

This is equivalent to saying that the union of two trivializing covers is a trivializing cover.

The official definition of a hermitian vector bundle $\xi = (\mathcal{E}, p, X, \mathcal{H})$ requires $\{(U_\alpha, \varphi_\alpha)\}$ to be an equivalence class of trivializing covers.

As earlier, given a trivializing cover $\{(U_\alpha, \varphi_\alpha)\}$, the set of all bundle charts compatible with $\{(U_\alpha, \varphi_\alpha)\}$ is a maximal trivializing cover equivalent to $\{(U_\alpha, \varphi_\alpha)\}$.

Technically, a hermitian vector bundle has the property that the hermitian inner product $\langle -, - \rangle_x$ on the fibre $\mathcal{E}_x$ *varies smoothly with* $x \in X$.

This is formalized as follows. For any open subset U of X, a *frame over* U is an n-tuple $(s_1, \ldots, s_n)$ of smooth sections $s_i \colon U \to \mathcal{E}$ such that $(s_1(x), \ldots, s_n(x))$ is a basis of the fibre $\mathcal{E}_x$ for all $x \in U$ (where n is the dimension of $\mathcal{H}$ and all the $\mathcal{E}_x$).

The hermitian inner products $\langle -, - \rangle_x$ have the property that for every U_α, for every frame $(s_1, \ldots, s_n)$ over U_α, the maps

$$x \mapsto \langle s_i(x), s_j(x) \rangle_x, \quad 1 \leq i, j \leq n, \; x \in U_\alpha,$$

are smooth. For details, see Gallier and Quaintance [23] (Section 9.8) and Morita [34] (Section 5.1).

Remark: Since X is a manifold, for any local trivialization $\varphi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times \mathcal{H}$, of ξ, since U_α is an open subset of X, there is some open set V in the maximal atlas defining X such that $V \subseteq U_\alpha$ and a chart $\theta\colon V \to \mathbb{R}^m$ (where m is the dimension of the manifold X), so if $\rho\colon \mathcal{H} \to \mathbb{R}^n$ is an isomorphism (obtained by picking a basis on $\mathcal{H}$), the map

$$(\theta \times \rho) \circ \varphi_\alpha\colon p^{-1}(V) \to \mathbb{R}^{m+n}$$

is a chart of $\mathcal{E}$ viewed as a manifold.

We can now define a hermitian G-vector bundle as a hermitian G-bundle which is also a hermitian vector bundle in the special case where $X = G/H$.

Definition 6.25. Let G be a Lie group, H a closed subgroup of G, E a smooth manifold, $\mathcal{H}$ a finite-dimensional complex vector space equipped with a hermitian inner product, and let $p\colon E \to X$ be a surjective *smooth* map, where as usual we write $X = G/H$. We say that E, more precisely $(E, p, X, \mathcal{H}, G)$, is a *hermitian G-vector bundle with fibre $\mathcal{H}$* if

(1) Each fibre E_x ($x \in X = G/H$) is a finite-dimensional vector space equipped with a hermitian inner product $\langle -, -\rangle_x$ and there is an *equilinear smooth* left action $\cdot$ of G on E.

(2) There is some open cover $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of $X = G/H$ and a family $\varphi = (\varphi_\alpha)_{\alpha \in I}$ of diffeomorphisms called *(local) trivializations*

$$\varphi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times \mathcal{H},$$

such that:

(a) (local triviality) the diagram

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times \mathcal{H} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U_\alpha & \end{array}$$

commutes.

(b) For every $x \in U_\alpha$, the map $\varphi_{\alpha,x} \colon \mathcal{E}_x \to \mathcal{H}$ is a unitary isomorphism.

As in the case of a hermitian vector bundle, we require that the hermitian inner product $\langle -, - \rangle_x$ varies smoothly with x.

If $\xi = (\mathcal{E}, \pi, X, H)$ is a principal H-bundle (with $X = \mathcal{E}/H$) and $\sigma\colon H \to \mathbf{U}(\mathcal{H}_\sigma)$ is a unitary representation of H in a finite-dimensional hermitian vector space $\mathcal{H}_\sigma$, then the Borel construction of Section 6.12 can be adapted to produce a hermitian vector bundle $(E, p\colon E \to X, X, \mathcal{H}_\sigma)$, with $E = \mathcal{E} \times_H \mathcal{H}_\sigma$ and $X = \mathcal{E}/H$.

The following theorem is a special case of a construction in which $\mathcal{H}_\sigma$ is replaced by any smooth manifold F equipped with a smooth left action of H on F (technically, an effective action);[3] see Dieudonné [14] (Theorem 16.14.7).

[3]Recall that an action $\cdot\colon H \times F \to F$ is effective if for any $h \in H$, if $h \cdot x = x$ for all $x \in F$, then $h = 1$.

Theorem 6.19. *Let $\xi = (\mathcal{E}, \pi, X, H)$ be a principal H-bundle (with $X = \mathcal{E}/H$) and let $\sigma\colon H \to \mathbf{U}(\mathcal{H}_\sigma)$ be a unitary representation of H in a finite-dimensional hermitian vector space $\mathcal{H}_\sigma$. Consider the right action of H on $\mathcal{E} \times \mathcal{H}_\sigma$ given by*

$$(z, u) \cdot h = (z \cdot h, \sigma(h^{-1})(u)), \quad z \in \mathcal{E},\ u \in \mathcal{H}_\sigma,\ h \in H. \tag{act2}$$

Here the right action of H on $\mathcal{E}$ is the action arising from the fact that $\mathcal{E}$ is a principal H-bundle, so it is free, and thus the action (act2) is also free. Then the orbit space $E = \mathcal{E} \times_H \mathcal{H}_\sigma$ is a smooth manifold. Furthermore, the following properties hold.

(1) The quadruple $(E, p, \mathcal{E}/H, \mathcal{H}_\sigma)$ is a hermitian vector bundle with fibre $\mathcal{H}_\sigma$, where the projection $p\colon \mathcal{E} \times_H \mathcal{H}_\sigma \to \mathcal{E}/H$ is given by $p([(z, u)]) = \pi(z)$, $z \in \mathcal{E}, u \in \mathcal{H}_\sigma$, and with $\pi\colon \mathcal{E} \to \mathcal{E}/H$.

(2) If $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ is an open cover of $X = \mathcal{E}/H$ and $\psi = (\psi_\alpha)_{\alpha\in I}$ is a family of local trivializations $\psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times H$ for $\mathcal{E}$, then for any smooth section $s\colon U_\alpha \to \pi^{-1}(U_\alpha)$, the inverse φ_α^{-1} of a local trivialization $\varphi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times \mathcal{H}_\sigma$ of E is given by

$$\varphi_\alpha^{-1}(x, u) = [(s(x), u)], \quad x \in U_\alpha,\ u \in \mathcal{H}_\sigma.$$

(3) For every $z \in \mathcal{E}$, the map $u \mapsto [(z, u)]$ is a unitary map from $\mathcal{H}_\sigma$ to the fibre $E_{\pi(z)} = p^{-1}(\pi(z))$.

(4) For any $x \in X$, for any fixed $x_0 \in \mathcal{E}_x = \pi^{-1}(x)$, the map given by

$$h\cdot[(x_0, u)] = [(x_0, h\cdot u)] = [(x_0, \sigma_h(u))], h \in H, u \in \mathcal{H}_\sigma,$$

is a unitary representation of H on the fibre $E_x = p^{-1}(x)$.

If we pick the section $s_\alpha \colon U_\alpha \to \pi^{-1}(U_\alpha)$ to be the special section given by

$$s_\alpha(x) = \psi_\alpha^{-1}(x, 1), \quad x \in U_\alpha, \qquad (\dagger_{10})$$

then we can figure out what are the corresponding transition functions. We find hat

$$(\varphi_\alpha \circ \varphi_\beta^{-1})(x, u) = (x, g_{\alpha\beta}(x)(1) \cdot u),$$

which shows that the transition functions of the vector bundle E are also given by the $g_{\alpha\beta}(x)(1) \in H$, except that this time H acts on $\mathcal{H}$ (on the left).

Going back to the original definition of the action of H on $\mathcal{H}$ given by the unitary representation σ, we have

$$(\varphi_\alpha \circ \varphi_\beta^{-1})(x, u) = (x, \sigma(g_{\alpha\beta}(x)(1))(u)).$$

Combining what we did in Section 6.12 with Theorem 6.19 we obtain the following result.

Theorem 6.20. *Let G be a Lie group, H a closed Lie subgroup of G, and $\sigma\colon H \to \mathbf{U}(\mathcal{H}_\sigma)$ a unitray representation in a finite-dimensional hermitian vector space. Then $(E, p, X, \mathcal{H}_\sigma, G)$ is a hermitian G-vector bundle, with $E = G \times_H \mathcal{H}_\sigma$ and $X = G/H$.*