Chapter 5

Matrix Representations of $\mathbf{SL}(2, \mathbb{C})$, $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$

This chapter deals with explicit matrix descriptions of the irreducible representations of the groups $\mathbf{SL}(2, \mathbb{C})$, $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$ (unitary representation in the last two cases).

Our presentation (except for Section 5.7) relies heavily on Vilenkin's exposition [39], especially Chapter III.

To the best of our knowledge Vilenkin contains the most detailed presentation of this type of material.

5.1 Irreducible Representations of $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$

In Example 2.8 it was proven that the representations $U_m\colon \mathbf{SU}(2) \to \mathbf{GL}(\mathcal{P}^{\mathbb{C}}_m(2))$ are irreducible.

In Example 2.9 it was proven that the representations $W_\ell\colon \mathbf{SO}(3) \to \mathbf{GL}(\mathcal{P}^{\mathbb{C}}_{2\ell}(2))$ are irreducible.

Recall that since $\mathbf{SU}(2)$ is compact and $\mathcal{P}^{\mathbb{C}}_m(2)$ is finite-dimensional there is an invariant inner product on $\mathcal{P}^{\mathbb{C}}_m(2)$ so we may assume that these representations are unitary.

Proposition 5.1. *Every irreducible unitary representation of* $\mathbf{SU}(2)$ *is equivalent to one of the irreducible unitary representations* $U_m\colon \mathbf{SU}(2) \to \mathbf{U}(\mathcal{P}^{\mathbb{C}}_m(2))$. *Furthermore, every irreducible unitary representation of* $\mathbf{SO}(3)$ *is equivalent to one of the irreducible unitary representations* $W_m\colon \mathbf{SO}(3) \to \mathbf{U}(\mathcal{P}^{\mathbb{C}}_{2m}(2))$.

The key point is to figure out what are the characters χ_{U_m} of the irreducible unitary representations U_m.

We now give a more pleasant description of the irreducible representations of $\mathbf{SO}(3)$ in terms of harmonic polynomials.

5.2 Irreducible Representations of SO(3); Harmonics

Recall that the Laplacian in $\mathbb{R}^n$ is given by

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$$

where $f\colon \mathbb{R}^n \to \mathbb{C}$ is twice differentiable.

The n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ is given by

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

Definition 5.1. Let $\mathcal{P}_k^{\mathbb{C}}(n+1)$ denote the space of *homogeneous polynomials of degree k in $n+1 \geq 2$ variables* with complex coefficients, and let $\mathcal{P}_k^{\mathbb{C}}(S^n)$ denote the *restrictions of homogeneous polynomials* in $\mathcal{P}_k^{\mathbb{C}}(n+1)$ to S^n.

Let $\mathcal{H}_k^{\mathbb{C}}(n+1)$ denote the space of *complex harmonic polynomials*, with

$$\mathcal{H}_k^{\mathbb{C}}(n+1) = \{P \in \mathcal{P}_k^{\mathbb{C}}(n+1) \mid \Delta P = 0\};$$

in the above equation, we view P as a function on $\mathbb{R}^{n+1}$. Harmonic polynomials are sometimes called *solid harmonics*.

Finally, let $\mathcal{H}_k^{\mathbb{C}}(S^n)$ denote the space of *complex spherical harmonics* as the set of restrictions of harmonic polynomials in $\mathcal{H}_k^{\mathbb{C}}(n+1)$ to S^n.

It not hard to prove that the restriction map from $\mathcal{H}_k^{\mathbb{C}}(n+1)$ to $\mathcal{H}_k^{\mathbb{C}}(S^n)$ is a bijection, and thus a linear isomorphism; see Gallier and Quaintance [23] (Section 7.5).

The functions in $\mathcal{H}_k^{\mathbb{C}}(S^n)$, the spherical harmonics, have been studied extensively.

They are the eigenspaces of the Laplacian on the sphere S^n; see Gallier and Quaintance [23] (Chapter 7). We will return to these functions later.

The group $\mathbf{SO}(n+1)$ acts on $\mathcal{P}_k^{\mathbb{C}}(n+1)$ by the (left regular) action

$$
\begin{aligned}
&(\mathbf{R}_Q(P))(x) = P(Q^{-1}x), \\
&Q \in \mathbf{SO}(n+1),\ P \in \mathcal{P}_k^{\mathbb{C}}(n+1),\ x \in \mathbb{R}^{n+1}.
\end{aligned}
$$

Note that the above formula shows that $\mathbf{R}$ is also an action of $\mathbf{SO}(n+1)$ on smooth functions on $\mathbb{R}^{n+1}$.

The action $\mathbf{R}$ on $\mathcal{P}_k^{\mathbb{C}}(n+1)$ is reducible for $k \geq 2$.

For example, we easily check that the subspace of $\mathcal{P}_2^{\mathbb{C}}(n+1)$ generated by the polynomial $x_1^2 + \cdots + x_{n+1}^2$ is invariant.

However this action turns out to be irreducible on $\mathcal{H}_k^{\mathbb{C}}(n+1)$.

This will be shown in Section 6.10.

But first we need to prove that the action of the Laplacian on smooth functions on $\mathbb{R}^{n+1}$ commutes with the action $\mathbf{R}$.

Recall that $\lambda_Q f$ is the function given by
$(\lambda_Q f)(x) = f(Q^{-1}x)$.

Proposition 5.2. *The action of the Laplacian on smooth functions on* $\mathbb{R}^{n+1}$ *commutes with the action* $\mathbf{R}$*; that is, for every smooth function* f *on* $\mathbb{R}^{n+1}$*, for every* $Q \in \mathbf{SO}(n+1)$*, for all* $u \in \mathbb{R}^{n+1}$*, we have*

$$\Delta(\lambda_Q f)(u) = (\Delta f)(Q^{-1}u).$$

As a corollary of Proposition 5.2, the vector space $\mathcal{H}_k^{\mathbb{C}}(n+1)$ is invariant under $\mathbf{R}$, and so $\mathbf{R}\colon \mathbf{SO}(n+1) \to \mathbf{GL}(\mathcal{H}_k^{\mathbb{C}}(n+1))$ is a representation.

Since $\mathbf{SO}(n+1)$ is compact and $\mathcal{H}_k^{\mathbb{C}}(n+1)$ is finite-dimensional, we may assume that $\mathbf{R}$ is unitary.

It is shown in Gallier and Quaintance [23] (Section 7.5) that $\mathcal{H}_k^{\mathbb{C}}(n+1)$ has dimension

$$a_{k,n+1} = \binom{n+k}{k} - \binom{n+k-2}{k-2}$$

if $n \geq 1$, $k \geq 2$, with $a_{0,n+1} = 1$ and $a_{1,n+1} = n$, For $n = 2$, we get $a_{k,3} = 2k+1$.

Here is a list of bases of the homogeneous harmonic polynomials of degree k in three variables up to $k = 4$.

$$\begin{array}{ll}
k=0 & 1 \\
k=1 & x,\ y,\ z \\
k=2 & x^2-y^2,\ x^2-z^2,\ xy,\ xz,\ yz \\
k=3 & x^3-3xy^2,\ 3x^2y-y^3,\ x^3-3xz^2,\ 3x^2z-z^3, \\
 & y^3-3yz^2,\ 3y^2z-z^3,\ xyz \\
k=4 & x^4-6x^2y^2+y^4,\ x^4-6x^2z^2+z^4,\ y^4-6y^2z^2+z^4, \\
 & x^3y-xy^3,\ x^3z-xz^3,\ y^3z-yz^3, \\
 & 3x^2yz-yz^3,\ 3xy^2z-xz^3,\ 3xyz^2-x^3y.
\end{array}$$

To prove that the representations
$\mathbf{R}\colon \mathbf{SO}(n+1) \to \mathbf{U}(\mathcal{H}_k^{\mathbb{C}}(n+1))$ are irreducible we restrict ourselves to the case where $n = 2$.

In order to deal with the case where $n > 2$, we need results from the next chapter.

Since these regular representations map to different spaces, for clarity we index them by k, that is, we write
$\mathbf{R}_k\colon \mathbf{SO}(n+1) \to \mathbf{U}(\mathcal{H}_k^{\mathbb{C}}(n+1))$.

Proposition 5.3. *The representations*
$\mathbf{R}_k\colon \mathbf{SO}(3) \to \mathbf{U}(\mathcal{H}_k^{\mathbb{C}}(3))$ *are irreducible. In fact, the representations* $\mathbf{R}_k\colon \mathbf{SO}(3) \to \mathbf{U}(\mathcal{H}_k^{\mathbb{C}}(3))$ *and*
$W_k\colon \mathbf{SO}(3) \to \mathbf{U}(\mathcal{P}_{2k}^{\mathbb{C}}(2))$ *are equivalent.*

Proposition 5.3 also shows that the representations
$\mathbf{R}_k\colon \mathbf{SO}(3) \to \mathbf{U}(\mathcal{H}_k^{\mathbb{C}}(3))$ form a complete set of irreducible representations of $\mathbf{SO}(3)$.

5.3 Factorization of the Unit Quaternions Using Euler Angles

In order to obtain formulae for the matrix elements of the representations of $\mathbf{SU}(2)$ in terms of special functions, the Jacobi polynomials, it is necessary to understand how to express the unit quaternions in terms of *Euler angles*.

The key fact is that there are three types of unit quaternions, $r_x(\varphi), r_y(\psi), r_z(\theta)$ that define rotations around the x-axis, y-axis, and z-axis, respectively, namely

$$r_x(\varphi/2) = \begin{pmatrix} e^{\frac{i\varphi}{2}} & 0 \\ 0 & e^{-\frac{i\varphi}{2}} \end{pmatrix}, \quad r_y(\psi/2) = \begin{pmatrix} \cos\frac{\psi}{2} & -\sin\frac{\psi}{2} \\ \sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{pmatrix},$$

$$r_z(\theta/2) = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

We immediately check that the rotations corresponding to $r_x(\varphi/2)$, $r_y(\psi/2)$, $r_z(\theta/2)$ under the homomorphism $\rho\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ (see Theorem 1.1) are given by the matrices

$$R_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}, R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix},$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $R_x(\varphi)$ is a *rotation by φ* around the *x-axis* (with the plane orthogonal to the x-axis oriented by (e_2, e_3, e_1)), $R_y(\psi)$ is a *rotation by ψ* around the *$-y$-axis* (with the plane orthogonal to the $-y$-axis oriented by $(e_1, e_3, -e_2)$, or equivalently a rotation by $-\psi$ around the y-axis with the plane orthogonal to the y-axis oriented by (e_3, e_1, e_2)), and $R_z(\theta)$ is a *rotation by θ* around the *z-axis* (with the plane orthogonal to the z-axis oriented by (e_1, e_2, e_3)).

Remark: Beware that a number of authors switch the roles of x and z, in particular Vilenkin [39] and most books on quantum mechanics.

As a consequence, the orientation of the plane normal to the y-axis is flipped. In this case, $R_x(\varphi)$ and $R_z(\varphi)$ are swapped, but $R_y(\psi)$ becomes $R_y(-\psi)$, which is a rotation by ψ around the y-axis (with the plane orthogonal to the y-axis oriented by (e_3, e_1, e_2)).

Vilenkin denotes our matrices r_x, r_y, r_z as $\omega_3, \omega_2, \omega_1$.

The issue of deciding exactly how a quaternion acts on $\mathbb{R}^3$ as a rotation is quite confusing, and we feel that some clarifications are in order.

First we need to decide whether a vector $(x, y, z) \in \mathbb{R}^3$ is represented as a skew-hermitian matrix (a matrix in $\mathfrak{su}(2)$) or as a hermitian matrix.

The first option seems to be followed by most mathematicians and by the computer graphics community.

On the other hand, physicists seem to prefer hermitian matrices to skew-hermtian matrices.

Of course, if S is a skew-hermitian matrix, then iS is a hermitian matrix, and this is the method used to make the conversion, although sometimes $(-i)S$ is used instead.

In the first method, we embed $\mathbb{R}^3$ into $\mathfrak{su}(2) \subseteq \mathbb{H}$ using the map

$$\mathrm{su}(x, y, z) = \begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix}, \quad (x, y, z) \in \mathbb{R}^3.$$

Then $q \in \mathbf{SU}(2)$ defines the map ρ_q (on $\mathbb{R}^3$) given by

$$\rho_q(x, y, z) = \mathrm{su}^{-1}(q\,\mathrm{su}(x, y, z)q^*).$$

This is the method used in *this book* and in Gallier and Quaintance [23] (Chapter 15). It is possible to derive an explicit orthogonal matrix corresponding to ρ_q; see Proposition 15.5.

The representation of $\mathbb{R}^3$ as the space of hermitian matrices has several variations, and this is the source of the confusion.

One option is to represent $(x, y, z) \in \mathbb{R}^3$ by the hermitian matrix

$$(-i)\mathrm{su}(x, y, z) = \begin{pmatrix} x & z - iy \\ z + iy & -x \end{pmatrix},$$

A nice feature of this representation is that

$$\begin{pmatrix} x & z - iy \\ z + iy & -x \end{pmatrix} = x\sigma_3 + y\sigma_2 + z\sigma_1,$$

where $\sigma_1, \sigma_2, \sigma_3$ are the *Pauli spin matrices*, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This representation is equivalent to the representation using su and yields the *exact same* rotation ρ_q. See Gallier and Quaintance [23] (Chapter 15).

The second option apparently adopted in most of the quantum mechanics literature is to use a version of $i$su, *except that x and z are swapped and y becomes $-y$.*

Vilenkin [39] (Chapter II, Section 1) uses the map

$$(x_1, y_1, z_1) \mapsto \begin{pmatrix} z_1 & x_1 + iy_1 \\ x_1 - iy_1 & -z_1 \end{pmatrix},$$

so in terms of our embedding,

$$z_1 = x, \quad x_1 = z, \quad y_1 = -y.$$

We can check that the unit quaternions $r_x(\varphi/2), r_y(\psi/2), r_z(\theta/2)$ induce the rotations $R_z(\varphi)$, $R_y(-\psi)$, and $R_x(\theta)$, namely

$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_y(-\psi) = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix},$$

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

These are the rotation matrices used in most books on quantum mechanics, including Sakurai and Napolitano [35].

Using our notation, Vilenkin factors a unit quaternion as

$$q = r_x(\varphi/2) r_z(\theta/2) r_x(\psi/2);$$

see Page 99 of Vilenkin [39]. This quaternion induces the rotation $R_z(\varphi)R_x(\theta)R_z(\psi)$.

Wigner [42] (Page 158) uses the map

$$(x_1, y_1, z_1) \mapsto \begin{pmatrix} -z_1 & x_1 + iy_1 \\ x_1 - iy_1 & z_1 \end{pmatrix},$$

so in terms of our embedding,

$$z_1 = -x, \quad x_1 = z, \quad y_1 = -y.$$

Analogously to the factorization of rotation matrices in terms of the Euler angles, we will prove that every unit quaternion q can be written in the form

$$q = u(\varphi, \theta, \psi) = r_x(\varphi/2) r_z(\theta/2) r_x(\psi/2).$$

Multiplying out the above matrices we get

$$\begin{aligned} u(\varphi, \theta, \psi) &= \begin{pmatrix} e^{\frac{i\varphi}{2}} & 0 \\ 0 & e^{-\frac{i\varphi}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{\frac{i\psi}{2}} & 0 \\ 0 & e^{-\frac{i\psi}{2}} \end{pmatrix} \\ &= \begin{pmatrix} \cos\frac{\theta}{2}\, e^{\frac{i(\varphi+\psi)}{2}} & i\sin\frac{\theta}{2}\, e^{\frac{i(\varphi-\psi)}{2}} \\ i\sin\frac{\theta}{2}\, e^{-\frac{i(\varphi-\psi)}{2}} & \cos\frac{\theta}{2}\, e^{-\frac{i(\varphi+\psi)}{2}} \end{pmatrix}. \end{aligned}$$

The reader can reconfirm by inspection that $u(\varphi, \theta, \psi)^{-1} = u(\varphi, \theta, \psi)^*$.

Proposition 5.4. *Every unit quaternion*

$$q = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1$$

can be expressed as

$$\begin{aligned} q = u(\varphi, \theta, \psi) &= r_x(\varphi/2) r_z(\theta/2) r_x(\psi/2) \\ &= \begin{pmatrix} e^{\frac{i\varphi}{2}} & 0 \\ 0 & e^{-\frac{i\varphi}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{\frac{i\psi}{2}} & 0 \\ 0 & e^{-\frac{i\psi}{2}} \end{pmatrix}. \end{aligned}$$

If $\beta = 0$, we can pick $\theta = 0$ and φ and ψ such that

$$\alpha = e^{i\frac{(\varphi+\psi)}{2}},$$

and in particular, $\psi = 0$.

If $\alpha = 0$, we can pick $\theta = \pi$ and φ and ψ such that

$$\beta = e^{i\frac{(\varphi-\psi+\pi)}{2}},$$

and in particular, $\psi = \pi$.

If $\alpha\beta \neq 0$ and if we require that

$$0 \leq \varphi < 2\pi, \quad 0 < \theta < \pi, \quad -2\pi \leq \psi < 2\pi,$$

then φ and ψ are unique. In this case,

$$\cos\theta = 2|\alpha|^2 - 1, \quad e^{i\varphi} = -\frac{\alpha\beta i}{|\alpha||\beta|}, \quad e^{\frac{i\psi}{2}} = \frac{\alpha}{|\alpha|}\, e^{-\frac{i\varphi}{2}}.$$

An interesting corollary of Proposition 5.4 is the fact that every rotation matrix $Q \in \mathbf{SO}(3)$ can be written in the terms of the Euler angles as a product

$$Q = R_x(\varphi)R_z(\theta)R_x(\psi),$$

namely

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}.$$

But in this case, we may assume that $0 \leq \psi < 2\pi$.

This is because both q and $-q$ define the same rotation ρ_q, but since $e^{i\pi} = e^{-i\pi} = -1$, we have $-r_x(\psi/2) = r_x(\frac{\psi+2\pi}{2})$, so if $-2\pi \leq \psi < 0$, then $0 \leq \psi + 2\pi < 2\pi$ and $Q = R_x(\varphi)R_z(\theta)R_x(\psi + 2\pi)$.

5.4 Multiplication of Quaternions in Terms of Euler Angles

5.5 Dehomogenized Representations of $\mathbf{SL}(2,\mathbb{C})$ and $\mathbf{SU}(2)$

In Example 2.8 we defined the irreducible representations $U_m\colon \mathbf{SU}(2) \to \mathbf{GL}(\mathcal{P}^{\mathbb{C}}_m(2))$ of $\mathbf{SU}(2)$ whose representing spaces are the vector spaces $\mathcal{P}^{\mathbb{C}}_m(2)$ of homogeneous polynomials in two variables.

We also said that it is customary, especially in the physics literature, to index homogeneous polynomials in terms of $\ell = m/2$, which is an integer when m is even but a half integer when m is odd.

In this context, in terms of $\ell = m/2$, a homogeneous polynomial is written as

$$P(z_1, z_2) = \sum_{k=-\ell}^{\ell} c_k z_1^{\ell-k} z_2^{\ell+k},$$

where it is assumed that $\ell + k = j$ where j takes the *integral* values $j = 0, 1, \ldots, 2\ell = m$, so that $\ell - k = 2\ell - (\ell + k) = 2\ell - j$ takes the values $2\ell, 2\ell - 1, \ldots, 0$.

Note that $k = j-\ell = j-m/2$ with $j = 0, 1, \ldots, 2\ell = m$, so k is an integer only if m is even.

If m is odd, say $m = 2h+1$, then $\ell = h + \frac{1}{2}$ and k takes the $2\ell + 1 = m + 1$ values

$$-h - \frac{1}{2},\ -(h-1) - \frac{1}{2}, \ldots, -\frac{1}{2},\ \frac{1}{2},\ 1 + \frac{1}{2}, \ldots, h + \frac{1}{2},$$

and so $k \neq 0$.

If m is even, say $m = 2h$, then $\ell = h$ and k takes the $2\ell + 1 = m + 1$ values

$$-h,\ -(h-1), \ldots, -1,\ 0,\ 1, \ldots, h-1, h.$$

For example, if $\ell = \frac{3}{2}$, then k takes the four values

$$-\frac{3}{2},\ -\frac{1}{2},\ \frac{1}{2},\ \frac{3}{2},$$

and if $\ell = 2$, then k takes the five values

$$-2,\ -1,\ 0,\ 1,\ 2.$$

The representing space is then $\mathcal{P}^{\mathbb{C}}_{2\ell}(2)$ and it has dimension $2\ell + 1$.

Using the standard technique of "dehomogenizing" and "homogenizing" we can use the space of complex polynomials of degree $2\ell + 1$ in *one* variable z instead of the space $\mathcal{P}^{\mathbb{C}}_{2\ell}(2)$ of homogeneous polynomials in *two variables* z_1, z_2.

Given a homogeneous polynomial $P(z_1, z_2)$ of degree $m = 2\ell$, by dehomogenizing we obtain the polynomial $Q(z)$ of degree $m = 2\ell$ given by

$$Q(z) = P(z, 1). \tag{dehomog}$$

So given

$$P(z_1, z_2) = \sum_{k=-\ell}^{\ell} c_k z_1^{\ell-k} z_2^{\ell+k},$$

we obtain

$$Q(z) = \sum_{k=-\ell}^{\ell} c_k z^{\ell-k}. \tag{Q}$$

Observe that due to our indexing scheme, the coefficients of Q have "funny" indices.

For example, for $\ell = 2$, so that $m = 2\ell = 4$,

$$Q(z) = c_{-2}z^4 + c_{-1}z^3 + c_0 z^2 + c_1 z + c_2,$$

and when $\ell = 5/2$, so that $m = 2\ell = 5$, we have

$$Q(z) = c_{-5/2}z^5 + c_{-3/2}z^4 + c_{-1/2}z^3 + c_{1/2}z^2 + c_{3/2}z + c_{5/2}.$$

Conversely, given a polynomial $Q(z)$ of degree $m = 2\ell$, by homogenizing we obtain the homogeneous polynomial $P(z_1, z_2)$ of degree $m = 2\ell$ given by

$$P(z_1, z_2) = z_2^{2\ell} Q\left(\frac{z_1}{z_2}\right). \qquad \text{(homog)}$$

Definition 5.2. Following Vilenkin, we denote the space of polynomials of degree $m = 2\ell$ with complex coefficients in one variable by $\mathcal{P}_\ell^{\mathbb{C}}$.

Note that the "funny" index ℓ is a half integer when m is odd.

We can convert our representations $U_m\colon \mathbf{SU}(2) \to \mathbf{GL}(\mathcal{P}_m^{\mathbb{C}}(2))$ to representations in the spaces $\mathcal{P}_\ell^{\mathbb{C}}$.

Actually, until we use the fact that $\mathbf{SU}(2)$ is compact, we consider representations of $\mathbf{SL}(2,\mathbb{C})$.

Definition 5.3. Given any matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C},\ ad - bc = 1,$$

in $\mathbf{SL}(2,\mathbb{C})$, for any polynomial $Q \in \mathcal{P}_\ell^{\mathbb{C}}$, define $T_\ell(A)(Q(z))$ by

$$T_\ell(A)(Q(z)) = (bz+d)^{2\ell} Q\left(\frac{az+c}{bz+d}\right). \qquad (T_\ell)$$

It is immediately verified that the above formula yields a representation $T_\ell\colon \mathbf{SL}(2,\mathbb{C}) \to \mathbf{GL}(\mathcal{P}_\ell^{\mathbb{C}})$ which yields a representation $T_\ell\colon \mathbf{SU}(2) \to \mathbf{GL}(\mathcal{P}_\ell^{\mathbb{C}})$ when restricted to the subgroup $\mathbf{SU}(2)$ of $\mathbf{SL}(2,\mathbb{C})$.

Note that the above formula for $T_\ell(A)(Q(z))$ is *not* what we would obtain directly from the representation U_ℓ.

We are using Vilenkin's formula to facilitate comparison with his exposition; see

Vilenkin [39] (Chapter III, Section 2.1) and Kosmann-Schwarzbach [30].

With our version we define the representations T_ℓ as

$$T_\ell(A)(Q(z)) = (-cz + a)^{2\ell} Q\left(\frac{dz - b}{-cz + a}\right).$$

In its homogeneous form, Vilenkin's version of the representation U_ℓ is

$$U_\ell^v(A)(Q(z_1,z_2)) = Q(az_1 + cz_2, bz_1 + dz_2).$$

Observe that

$$\begin{pmatrix} az_1 + cz_2 \\ bz_1 + dz_2 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = A^\top \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

but in our case

$$\begin{pmatrix} dz_1 - bz_2 \\ -cz_1 + az_2 \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = A^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

We immediately check that if

$$Y = \begin{pmatrix} b & d \\ -a & -c \end{pmatrix},$$

then

$$YA^\top = A^{-1}Y,$$

and $\det(Y) = ad - bc = 1$.

Then Y defines a linear isomorphism of $\mathcal{P}^{\mathbb{C}}_{2\ell}(2)$ given by $Q(z_1, z_2) \mapsto Q(bz_1 + dz_2, -az_1 - cz_2)$, and this map is an equivalence between the representations U_ℓ and U_ℓ^v (we leave the details as an exercise).

We also leave it as an exercise (using the dehomogenization and the homogenization maps, which are linear isomorphisms) to check that the representation $U_{2\ell}\colon \mathbf{SL}(2,\mathbb{C}) \to \mathbf{GL}(\mathcal{P}^{\mathbb{C}}_{2\ell}(2))$ is equivalent to the representation $T_\ell\colon \mathbf{SL}(2,\mathbb{C}) \to \mathbf{GL}(\mathcal{P}^{\mathbb{C}}_{\ell})$ and similarly the representation $U_{2\ell}\colon \mathbf{SU}(2) \to \mathbf{GL}(\mathcal{P}^{\mathbb{C}}_{2\ell}(2))$ is equivalent to the representation $T_\ell\colon \mathbf{SU}(2) \to \mathbf{GL}(\mathcal{P}^{\mathbb{C}}_{\ell})$.

In particular, the representations $T_\ell\colon \mathbf{SU}(2) \to \mathbf{GL}(\mathcal{P}^{\mathbb{C}}_{\ell})$ form a complete set of irreducible representations of $\mathbf{SU}(2)$.

5.6 The Lie Algebra Representation Associated with T_ℓ

5.7 Irreducible Lie Algebra Representations of $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{su}(2)$

5.8 $\mathbf{SU}(2)$-Invariant Hermitian Inner Product on $\mathcal{P}_\ell^{\mathbb{C}}$

We now restrict our attention to the representations T_ℓ of $\mathbf{SU}(2)$.

Our goal is to find explicitly an $\mathbf{SU}(2)$-invariant hermitian inner product on $\mathcal{P}_\ell^{\mathbb{C}}$.

Because $\mathbf{SU}(2)$ is compact, such an inner product must exist.

If such an invariant hermitian inner product $\langle -, - \rangle$ exists, in particular it must be invariant for the matrices $T_\ell(r_x(\varphi/2))$, $T_\ell(r_y(\theta/2))$ and $T_\ell(r_z(\psi/2))$, so we assert such invariance and deduce consequences by taking derivatives.

In fact the proof shows that is suffices to assert invariance for the matrices $T_\ell(r_x(\varphi/2))$ and $T_\ell(r_y(\theta/2))$.

First we need to figure out what is $T_\ell(r_x(\varphi/2))(z^{\ell-k})$.

Proposition 5.5. *Each polynomial $z^{\ell-k}$ is an eigenvector of $T_\ell(r_x(\varphi/2))$ for the eigenvalue $e^{-ik\varphi}$, that is,*

$$T_\ell(r_x(\varphi/2))(z^{\ell-k}) = e^{-ik\varphi} z^{\ell-k}. \qquad (*_1)$$

Thus in the basis $(z^{\ell-k})_{-\ell\leq k\leq \ell}$, the matrix of $T_\ell(r_x(\varphi/2))$ is the diagonal matrix

$$\begin{pmatrix} e^{i\ell\varphi} & & & & \\ & e^{i(\ell-1)\varphi} & & & \\ & & \ddots & & \\ & & & e^{-i(\ell-1)\varphi} & \\ & & & & e^{-i\ell\varphi} \end{pmatrix}.$$

Next we need to state the invariance of the inner product for $T_\ell(r_x(\varphi/2))$ and $T_\ell(r_y(\varphi/2))$.

After some labor we find that the $2\ell+1$ polynomials

$$\frac{z^{\ell-k}}{\sqrt{(\ell-k)!(\ell+k)!}},$$

form an orthonormal basis of $\mathcal{P}_\ell^{\mathbb{C}}$ for an invariant hermitian inner product on $\mathbf{SU}(2)$ which is uniquely determined by setting $\langle 1,1\rangle = (2\ell)!$.

This is an important result that we record below.

Proposition 5.6. *In Vilenkin's notation, the polynomials*

$$\psi_k(z) = \frac{z^{\ell-k}}{\sqrt{(\ell-k)!(\ell+k)!}}, \quad -\ell \leq k \leq \ell \qquad (*_{10})$$

form an orthonormal basis of $\mathcal{P}_\ell^{\mathbb{C}}$ *for a unique invariant hermitian inner product on* $\mathbf{SU}(2)$*.*

The ψ_k *are the unit-length eigenvectors of the linear map* $T_\ell(r_x(\varphi/2))$*.*

Actually, it is remarkable that if we define a hermitian inner product on $\mathcal{P}_\ell^{\mathbb{C}}$ by requiring that the polynomials ψ_k form an orthonormal basis, then this inner product is $\mathbf{SU}(2)$ invariant.

The proof of this fact relies on two standard facts of Lie group theory about the relationship between a representation and its derivative.

Proposition 5.7. *The hermitian inner product on* $\mathcal{P}_\ell^{\mathbb{C}}$ *making the basis* (ψ_k) *orthonormal is* $\mathbf{SU}(2)$*-invariant.*

5.9 Matrices of the Irreducible Representations of $\mathbf{SL}(2,\mathbb{C})$

We now use the basis (ψ_k) to find various expressions for the matrix entries of the matrix $t^{(\ell)}(A)$ representing $T_\ell(A)$ in this basis.

We give $\mathcal{P}_\ell^{\mathbb{C}}$ the hermitian inner product making (ψ_k) an orthonormal basis. In this section we consider an arbitrary matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C},\ \alpha\delta - \beta\gamma = 1$$

in $\mathbf{SL}(2,\mathbb{C})$.

The special case of $\mathbf{SU}(2)$ is considered in later sections. In this latter case these matrices are unitary.

We use $\alpha, \beta, \gamma, \delta$ instead of a, b, c, d to make it easier to follow Vilenkin's exposition. Since the ψ_k form an orthonormal basis, we have

$$\begin{aligned} t_{jk}^{(\ell)}(A) &= \langle T_\ell(A)(\psi_k),\ \psi_j\rangle \\ &= \frac{\langle T_\ell(A)(z^{\ell-k}),\ z^{\ell-j}\rangle}{\sqrt{(\ell-j)!(\ell+j)!(\ell-k)!(\ell+k)!}}. \qquad (*_{21}) \end{aligned}$$

By (T_ℓ) we have

$$\begin{aligned} T_\ell(A)(z^{\ell-k}) &= (\beta z+\delta)^{2\ell}\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)^{\ell-k} \\ &= (\alpha z+\gamma)^{\ell-k}(\beta z+\delta)^{\ell+k}, \end{aligned}$$

so we obtain

$$t_{jk}^{(\ell)}(A) = \frac{\langle (\alpha z+\gamma)^{\ell-k}(\beta z+\delta)^{\ell+k},\ z^{\ell-j}\rangle}{\sqrt{(\ell-j)!(\ell+j)!(\ell-k)!(\ell+k)!}}. \qquad (*_{22})$$

The expression on the right-hand side can be "doctored on" in various ways.

The first brute-force method is to use the binomial formula together with the orthogonality of $z^{\ell-j}$ and $z^{\ell-k}$ for $j \neq k$ and the formulae

$$\langle z^{\ell-k},\, z^{\ell-k}\rangle = (\ell-k)!(\ell+k)!, \quad -\ell \leq k \leq \ell.$$

We get

$$t_{jk}^{(\ell)}(A) = \sqrt{\frac{(\ell-j)!(\ell+j)!}{(\ell-k)!(\ell+k)!}} \sum_{h=M}^{N} \binom{\ell-k}{\ell-j-h}\binom{\ell+k}{h} \alpha^{\ell-j-h}\beta^{h}\gamma^{j+h-k}\delta^{\ell+k-h} \qquad (*_{23})$$

with $M = \max(0, k-j)$, $N = \min(\ell-j, \ell+k)$.

This can be somewhat simplified as

$$t^{(\ell)}_{jk}(A) = \sqrt{(\ell-j)!(\ell+j)!(\ell-k)!(\ell+k)!} \\ \times \sum_{h=M}^{N} (h!(\ell-j-h)!(\ell+k-h)!(j-k+h)!)^{-1} \\ \alpha^{\ell-j-h}\beta^{h}\gamma^{j+h-k}\delta^{\ell+k-h}, \qquad (*_{24})$$

also with $M = \max(0, k-j)$, $N = \min(\ell - j, \ell + k)$.

It is understood that if any of $\alpha, \beta, \gamma, \delta$ is zero, then the corresponding exponent must be zero.

Of course, since $\alpha\delta - \beta\gamma = 1$, at most two of these coefficients must be nonzero.

Using the factorization of A as the product of an upper triangular matrix and a lower triangular matrix, Vilenkin obtains simpler formulae; see Vilenkin [39] (Chapter III, Section 3.2).

In particular, if $\delta \neq 0$, then we have the following fomula that will be needed later:

$$t^{(\ell)}_{jk}(A) = \sqrt{\frac{(\ell-j)!(\ell-k)!}{(\ell+j)!(\ell+k)!}} \times \sum_{h=\max(j,k)}^{\ell} \frac{(\ell+h)!}{(\ell-h)!(h-j)!(h-k)!}\, \beta^{h-j}\gamma^{h-k}\delta^{j+k}. \qquad (*_{25})$$

If $\beta = \gamma = 0$, then $\alpha\delta = 1$,

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix},$$

and $t^{(\ell)}_{jk}(A)$ is the diagonal matrix with

$$t^{(\ell)}_{kk}(A) = \alpha^{-2k} = \delta^{2k}.$$

Another strategy is to use Taylor's formula. Recall that for polynomial $P(z)$ of degree m we have

$$P(z) = \sum_{j=0}^{m} \frac{P^{(j)}(0)}{j!} z^j,$$

where $P^{(k)}(0)$ is the value of the kth derivative of P at $z = 0$.

Proposition 5.8. *With respect to the orthonormal basis* (ψ_k) *of* $\mathcal{P}_\ell^{\mathbb{C}}$, *the entries in the matrix* $t^{(\ell)}(A)$ *are given by the formulae below.*

$$t_{jk}^{(\ell)}(A) = \sqrt{\frac{(\ell+j)!}{(\ell-k)!(\ell+k)!(\ell-j)!}} \frac{d^{\ell-j}}{z^{\ell-j}}[(\alpha z+\gamma)^{\ell-k}(\beta z+\delta)^{\ell+k}]_{z=0}. \qquad (*_{29})$$

If $\alpha\beta \neq 0$, *then*

$$t_{jk}^{(\ell)}(A) = \sqrt{\frac{(\ell+j)!}{(\ell-k)!(\ell+k)!(\ell-j)!}} \frac{\beta^{k-j}}{\alpha^{k+j}} \frac{d^{\ell-j}}{dz^{\ell-j}}[z^{\ell-k}(z+1)^{\ell+k}]_{z=\beta\gamma}. \qquad (*_{30})$$

5.10 Euler Angles Matrix Representations of T_ℓ

The "best" formula is obtained by using the Euler angles.

We now restrict ourselves to $\mathbf{SU}(2)$, although it possible to handle the more general case; see Vilenkin [39] (Chapter III, Sections 3.3–3.9).

By Proposition 5.4 every matrix $q \in \mathbf{SU}(2)$, where

$$q = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1,$$

can be expressed as

$$\begin{aligned} q &= u(\varphi, \theta, \psi) = r_x(\varphi/2) r_z(\theta/2) r_x(\psi/2) \\ &= \begin{pmatrix} e^{\frac{i\varphi}{2}} & 0 \\ 0 & e^{-\frac{i\varphi}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{\frac{i\psi}{2}} & 0 \\ 0 & e^{-\frac{i\psi}{2}} \end{pmatrix} \end{aligned}$$

with

$$0 \leq \varphi < 2\pi, \quad 0 \leq \theta \leq \pi, \quad -2\pi \leq \psi < 2\pi.$$

Furthermore, if $\alpha\beta \neq 0$ and if we require that $0 < \theta < \pi$, then φ, θ, ψ are unique.

Since T_ℓ is a representation we have

$$T_\ell(q) = T_\ell(r_x(\varphi/2))T_\ell(r_z(\theta/2))T_\ell(r_x(\psi/2)).$$

We also proved that the polynomials in the basis $(\psi_k(z))$ are eigenvectors of $T_\ell(r_x(\varphi/2))$ and $T_\ell(r_x(\psi/2))$, namely (by $(*_1)$)

$$\begin{aligned} T_\ell(r_x(\varphi/2))\psi_k(z) &= e^{-ik\varphi}\psi_k(z) \\ T_\ell(r_x(\psi/2))\psi_k(z) &= e^{-ik\psi}\psi_k(z). \end{aligned}$$

Proposition 5.9. *For any matrix $q \in \mathbf{SU}(2)$ expressed in terms of the Euler angles as $q = u(\varphi, \theta, \psi) = r_x(\varphi/2) r_z(\theta/2) r_x(\psi/2)$, with respect to the orthonormal basis (ψ_k) of $\mathcal{P}_\ell^{\mathbb{C}}$, we have*

$$t_{jk}^{(\ell)}(q) = e^{-i(j\varphi + k\psi)}\, t_{jk}^{(\ell)}(r_z(\theta/2)). \qquad (*_{31})$$

Thus we are left with finding an explicit expression for the matrix $t^{(\ell)}(r_z(\theta/2))$,

Definition 5.4. Define the matrix $t^{(\ell)}(\theta)$ as $t^{(\ell)}(\theta) = t^{(\ell)}(r_z(\theta/2))$, with

$$r_z(\theta/2) = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

If $\theta = \pi$, then $r_z(\pi/2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, and by $(*_{27})$ we know that $t^{(\ell)}(\pi)$ is the anti-diagonal matrix with $t^{(\ell)}_{jk}(\pi) = 0$ if $j \neq k$ and $t^{(\ell)}_{j\,-j}(\pi) = i^{2\ell}$.

If $\theta = 0$, then $r_z(0)$ is the identity matrix I_2, and $t^{(\ell)}(0)$ is the identity matrix $I_{2\ell+1}$.

If $0 \leq \theta < \pi$, then we can find the matrix $t^{(\ell)}(\theta)$ using Equation $(*_{25})$ in we which we set $\alpha = \delta = \cos\frac{\theta}{2} \neq 0$ (since $0 \leq \theta < \pi$), and $\beta = \gamma = i\sin\frac{\theta}{2}$.

We obtain the following formula.

Proposition 5.10. *The elements of the matrix* $t^{(\ell)}(\theta) = t^{(\ell)}(r_z(\theta/2))$ $(0 \leq \theta < \pi)$ *are given by the formula*

$$t^{(\ell)}_{jk}(\theta) = i^{-(j+k)}\sqrt{\frac{(\ell-j)!(\ell-k)!}{(\ell+j)!(\ell+k)!}}\left(\cos\frac{\theta}{2}\right)^{j+k} \times \sum_{h=\max(j,k)}^{\ell} \frac{(\ell+h)!\, i^{2h}}{(\ell-h)!(h-j)!(h-k)!}\left(\sin\frac{\theta}{2}\right)^{2h-(j+k)}. \qquad (*_{32})$$

If ℓ *is a half integer, then* h *is also a half integer.*

For $\theta = 0$*, we must have* $h = j = k$*, and* $t^{(\ell)}(0)$ *is the identity matrix* $I_{2\ell+1}$*, as we already know.*

If we assume that $0 < \theta < \pi$, then we obtain the following formula given in Vilenkin:

$$
\begin{aligned}
t^{(\ell)}_{jk}(\theta) = i^{-(j+k)} & \sqrt{\frac{(\ell-j)!(\ell-k)!}{(\ell+j)!(\ell+k)!}} \left(\cot\frac{\theta}{2}\right)^{j+k} \\
& \times \sum_{h=\max(j,k)}^{\ell} \frac{(\ell+h)!\, i^{2h}}{(\ell-h)!(h-j)!(h-k)!} \left(\sin\frac{\theta}{2}\right)^{2h}.
\end{aligned}
\tag{$*_{33}$}
$$

If we recall from (†) that if $j = -k$ then

$$\frac{(\ell+h)!}{(\ell-h)!(h+k)!(h-k)!} = \binom{\ell+h}{2h}\binom{2h}{h-k},$$

we obtain

$$\begin{aligned} t^{(\ell)}_{k-k}(\theta) &= t^{(\ell)}_{-kk}(\theta) \\ &= \sum_{h=\max(-k,k)}^{\ell} \binom{\ell+h}{2h}\binom{2h}{h-k} i^{2h} \left(\sin\frac{\theta}{2}\right)^{2h}. \end{aligned} \qquad (*_{34})$$

Even though this equation was derived assuming that $\theta < \pi$, it is still correct for $\theta = \pi$, namely the following equation holds

$$\sum_{h=\max(-k,k)}^{\ell} \binom{\ell+h}{2h}\binom{2h}{h-k} i^{2h} = i^{2\ell},$$

or equivalently, since we may assume that $k \geq 0$,

$$\sum_{h=k}^{\ell} (-1)^{\ell-h} \binom{\ell+h}{2h}\binom{2h}{h-k} = 1. \qquad (\dagger\dagger)$$

Jocelyn showed that this equation can be proven using an identity due to Euler.

Because there is a surjective homomorphism $\rho\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ whose kernel is $\{I, -I\}$ (see Theorem 1.1), Proposition 2.8, Proposition 5.1, and the fact that the representation $U_{2\ell}\colon \mathbf{SU}(2) \to \mathbf{GL}(\mathcal{P}^{\mathbb{C}}_{2\ell}(2))$ is equivalent to the representation $T_\ell\colon \mathbf{SU}(2) \to \mathbf{GL}(\mathcal{P}^{\mathbb{C}}_\ell)$ (see the end of Section 5.5), imply that the irreducible unitary representations of $\mathbf{SO}(3)$ are of the form $W_\ell\colon \mathbf{SO}(3) \to \mathbf{U}(\mathcal{P}^{\mathbb{C}}_\ell)$, with

$$W_\ell(\rho_q) = T_\ell(q) \quad q \in \mathbf{SU}(2), \quad \ell \in \mathbb{N},$$

and where $T_{\ell'}\colon \mathbf{SU}(2) \to \mathbf{U}(\mathcal{P}^{\mathbb{C}}_{\ell'})$ are the irreducible unitary representations of $\mathbf{SU}(2)$ (with ℓ' a half integer or an integer).

So the irreducible representations of $\mathbf{SO}(3)$ constitute *only half* of the representations of $\mathbf{SU}(2)$, those that correspond to nonnegative *integer values* of ℓ.

Therefore, all the formulae obtained for the matrices $t_{jk}^{(\ell)}(q)$ apply and *the matrix $w_{jk}^{(\ell)}(\rho_q)$ associated with the unitary map $W_\ell(\rho_q)$ is $t_{jk}^{(\ell)}(q)$, with $\ell \in \mathbb{N}$.*

Remarkably, if $q \in \mathbf{SU}(2)$ is expressed in terms of the Euler angles as $q = u(\varphi, \theta, \psi) = r_x(\varphi/2)r_z(\theta/2)r_x(\psi/2)$, then the corresponding rotation matrix $R = \rho_q$ is given by $R = R_x(\varphi)R_z(\theta)R_x(\psi)$, where we may assume that $0 \leq \varphi < 2\pi$, $0 \leq \theta \leq \pi$, $0 \leq \psi < 2\pi$ (see Section 5.3).

Consequently, if we express a rotation matrix $R \in \mathbf{SO}(3)$ in terms of Euler angles as $R = R_x(\varphi)R_z(\theta)R_x(\psi)$, we find that the matrix $w^{(\ell)}(R)$ associated with the unitary map $W_\ell(R)$ is $t^{(\ell)}(u(\varphi, \theta, \psi))$, with $\ell \in \mathbb{N}$.

Using Proposition 5.9 and since by Definition 5.4, $t^{(\ell)}(\theta) = t^{(\ell)}(r_z(\theta/2))$, we obtain the following result.

Proposition 5.11. *For any matrix $R \in \mathbf{SO}(3)$ expressed in terms of the Euler angles as $R = R_x(\varphi)R_z(\theta)R_x(\psi)$, with respect to the orthonormal basis (ψ_k) of $\mathcal{P}_\ell^{\mathbb{C}}$, the matrix $w^{(\ell)}(R)$ of the unitary map $W_\ell(R)$ associated with the irreducible representation $W_\ell\colon \mathbf{SO}(3) \to \mathbf{U}(\mathcal{P}_\ell^{\mathbb{C}})$ is given by*

$$w_{jk}^{(\ell)}(R) = e^{-i(j\varphi + k\psi)}\, t_{jk}^{(\ell)}(\theta), \quad \ell \in \mathbb{N}. \qquad (*_{31'})$$

Formula $(*_{31'})$ still gives the matrix elements $T_\ell(q)$ (with $q \in \mathbf{SU}(2)$) of the irreducible representation T_ℓ of $\mathbf{SU}(2)$ when ℓ is a positive half integer, but this is *not* a representation of $\mathbf{SO}(3)$.

This point is a notorious source of confusion.

The functions $e^{-i(j\varphi+k\psi)}\, t^{(\ell)}_{jk}(\theta)$ arise in quantum mechanics, but physicists prefer the functions $t^{(\ell)}_{jk}(\theta)$ to be real.

In his famous book first published in German in 1931 and then in English in 1959 (translated by J.J. Griffin), E. Wigner [42] introduced the matrices $d^{\ell}(\theta)$ given by

$$d^{\ell}_{jk}(\theta) = (-1)^{j-k} i^{j-k} t^{(\ell)}_{jk}(\theta).$$

The reason for the factor $(-1)^{j-k} i^{j-k}$ is that by using Formula $(*_{24})$ with $\alpha = \delta = \cos\frac{\theta}{2}$ and $\beta = \gamma = i\sin\frac{\theta}{2}$, we obtain

$$t^{(\ell)}_{jk}(\theta) = i^{j-k}\sqrt{(\ell-j)!(\ell+j)!(\ell-k)!(\ell+k)!}$$
$$\times \sum_{h=M}^{N} (-1)^h (h!(\ell-j-h)!(\ell+k-h)!(j-k+h)!)^{-1}$$
$$\left(\cos\frac{\theta}{2}\right)^{2\ell+k-j-2h} \left(\sin\frac{\theta}{2}\right)^{2h+j-k}$$

with $M = \max(0, k-j)$, $N = \min(\ell-j, \ell+k)$ and $0 \leq \theta \leq \pi$.

When we multiply the above expression by $(-1)^{j-k} i^{j-k}$, we obtain the term

$$(-1)^{j-k} i^{j-k} i^{j-k} = (-1)^{j-k} i^{2(j-k)} = (-1)^{j-k} (-1)^{j-k} = +1.$$

The above amounts to performing the following operations on the matrix $t^{(\ell)}(\theta)$: multiply the jth row by $(-1)^j i^j$ and multiply the kth column by $(-1)^{-k} i^{-k}$.

The resulting matrix $d^{(\ell)}(\theta)$ remains unitary. In fact, it becomes a *real orthogonal matrix*.

Definition 5.5. The *Wigner's d-matrices $d^{(\ell)}(\theta)$* are given by

$$d^{(\ell)}_{jk}(\theta) = \sqrt{(\ell - j)!(\ell + j)!(\ell - k)!(\ell + k)!}$$
$$\times \sum_{h=M}^{N} (-1)^h (h!(\ell - j - h)!(\ell + k - h)!(j - k + h)!)^{-1}$$
$$\left(\cos\frac{\theta}{2}\right)^{2\ell+k-j-2h} \left(\sin\frac{\theta}{2}\right)^{2h+j-k} \quad (*_{35})$$

with $M = \max(0, k - j)$, $N = \min(\ell - j, \ell + k)$;

see Wigner [42], Formula 15.27.

The d-matrices $d^{(\ell)}(\theta)$ are real orthogonal matrices.

However, beware that besides the fact that the indices ℓ, j, k, h are denoted j, μ', μ, κ and the angles φ, θ, ψ are denoted α, β, γ, the angles α, β, γ *have a different meaning*.

Indeed, Wigner factors a unit quaternion as $q = r_x(-\alpha/2)r_y(\beta/2)r_x(-\gamma/2)$ (where r_x and r_y are defined in Section 5.3), and the x-axis and the z-axis are swapped, which means that in our notation, the rotation matrix R associated with q is

$$R = R_z(-\alpha)R_y(\beta)R_z(-\gamma).$$

Wigner uses $r_y(\beta/2)$ instead of $r_z(\beta/2)$ because it is a real matrix.

As a consequence, *Wigner's $\mathcal{D}$-matrices* (see Wigner [42], Formula 15.8 and Formula 15.27) are the matrices $\mathcal{D}^{(\ell)}$ given by

$$\mathcal{D}^{(\ell)}_{jk}(\alpha, \beta, \gamma) = e^{i(j\alpha + k\gamma)} d^{(\ell)}_{jk}(\beta).$$

As earlier, the matrices $\mathcal{D}^{(\ell)}$ correspond to the irreducible unitary representations U_ℓ of $\mathbf{SU}(2)$ when ℓ assumes all nonnegative integer and half integer values, and when ℓ is restricted to be a nonnegative integer, they correspond to the irreducible unitary representations W_ℓ of $\mathbf{SO}(3)$.

According to Wigner, the method for determining the irreducible representations of $\mathbf{SO}(3)$ as the irreducible representations of $\mathbf{SU}(2)$ corresponding to nonnegative *integer values* of ℓ is due to H. Weyl, who also discovered the irreducible representations of $\mathbf{SU}(2)$.

The irreducible representations of $\mathbf{SU}(2)$ corresponding to half integer values of ℓ are often called *double-valued representations* of $\mathbf{SO}(3)$, an unfortunate terminology since they are *not* representations of $\mathbf{SO}(3)$, but instead representations of $\mathbf{SU}(2)$.

Wigner's sign conventions is not always the sign convention used in the physics literature.

5.11 Representations of $\mathbf{SL}(2,\mathbb{C})$ and $\mathbf{SU}(2)$ Using Finite Fourier Series

There is one more method for computing the matrix elements $t_{jk}^{(\ell)}(A)$ (where $A \in \mathbf{SL}(2,\mathbb{C})$) based on integration.

The idea is to use another representing space for the representation T_ℓ, namely the vector space (of dimension $2\ell+1$) of finite Fourier series

$$\Phi(e^{i\varphi}) = \sum_{k=-\ell}^{\ell} c_k e^{-ik\varphi},$$

with $c_k \in \mathbb{C}$.

Observe that if $Q(z)$ is the polynomial of degree 2ℓ given by

$$Q(z) = \sum_{k=-\ell}^{\ell} c_k z^{\ell-k}$$

so that the powers appears in the order $z^{2\ell}, z^{2\ell-1}, \ldots, z, 1$, the Fourier series $\Phi(e^{i\varphi})$ with the same coefficients is given by

$$\Phi(e^{i\varphi}) = e^{-i\ell\varphi} Q(e^{i\varphi}).$$

Denote the space of Fourier series of dimension $2\ell + 1$ as $\mathfrak{F}_\ell$.

We would like to define a representation of $\mathbf{SL}(2,\mathbb{C})$ in $\mathfrak{F}_\ell$.

Definition 5.6. The map $\mathcal{T}_\ell\colon \mathbf{SL}(2,\mathbb{C}) \to \mathbf{GL}(\mathfrak{F}_\ell)$ is defined by

$$\mathcal{T}_\ell(A)(\Phi(e^{i\varphi})) = e^{-i\ell\varphi}(ae^{i\varphi}+c)^\ell(be^{i\varphi}+d)^\ell\Phi\left(\frac{ae^{i\varphi}+c}{be^{i\varphi}+d}\right) \tag{$\mathcal{T}_\ell$}$$

for every matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2,\mathbb{C})$$

and every Fourier series $\Phi(e^{i\varphi}) \in \mathfrak{F}_\ell$.

It is easily verified that $\mathcal{T}_\ell\colon \mathbf{SL}(2,\mathbb{C}) \to \mathbf{GL}(\mathfrak{F}_\ell)$ is a representation.

It can be shown that the representation $\mathcal{T}_\ell\colon \mathbf{SL}(2,\mathbb{C}) \to \mathbf{GL}(\mathfrak{F}_\ell)$ is equivalent to the representation $T_\ell\colon \mathbf{SL}(2,\mathbb{C}) \to \mathbf{GL}(\mathcal{P}_\ell^{\mathbb{C}})$

Proposition 5.12. *The matrix elements $t_{jk}^{(\ell)}(A)$ are given by the following formula:*

$$t_{jk}^{(\ell)}(A) = \frac{1}{2\pi}\sqrt{\frac{(\ell-j)!(\ell+j)!}{(\ell-k)!(\ell+k)!}} \int_0^{2\pi} (ae^{i\varphi}+c)^{\ell-k}(be^{i\varphi}+d)^{\ell+k}\, e^{i(j-\ell)\varphi}\, d\varphi. \qquad (*_{37})$$

We obtain another useful formula for computing $t_{jk}^{(\ell)}(\theta)$ by applying the above formula to the matrix

$$r_z(\theta/2) = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \in \mathbf{SU}(2).$$

We get

$$t^{(\ell)}_{jk}(\theta) = \frac{1}{2\pi}\sqrt{\frac{(\ell-j)!(\ell+j)!}{(\ell-k)!(\ell+k)!}}\int_0^{2\pi}\left(\cos\frac{\theta}{2}e^{i\varphi} + i\sin\frac{\theta}{2}\right)^{\ell-k} \left(i\sin\frac{\theta}{2}e^{i\varphi} + \cos\frac{\theta}{2}\right)^{\ell+k} e^{i(j-\ell)\varphi}\, d\varphi,$$

and since $e^{-i\ell\varphi} = e^{-\frac{i(\ell+k)\varphi}{2}}e^{-\frac{i(\ell-k)\varphi}{2}}$, the above formula is also written as stated below.

Proposition 5.13. *The matrix elements* $t^{(\ell)}_{jk}(\theta)$ *($0 \leq \theta \leq \pi$) are given by the following formula:*

$$t^{(\ell)}_{jk}(\theta) = \frac{1}{2\pi}\sqrt{\frac{(\ell-j)!(\ell+j)!}{(\ell-k)!(\ell+k)!}} \times \int_0^{2\pi} \left(\cos\frac{\theta}{2}e^{\frac{i\varphi}{2}} + i\sin\frac{\theta}{2}e^{-\frac{i\varphi}{2}}\right)^{\ell-k} \left(i\sin\frac{\theta}{2}e^{\frac{i\varphi}{2}} + \cos\frac{\theta}{2}e^{-\frac{i\varphi}{2}}\right)^{\ell+k} e^{ij\varphi}\, d\varphi. \quad (*_{38})$$

For small values of ℓ, this equation is quite practical.

For example, here is a list of the matrices $t^{\ell}(\theta)$ for $\ell = 0, 1/2, 1, 3/2$ as in Vilenkin [39] (Chapter III, Section 3.7).

$$t^{(0)}(\theta) = (1), \quad t^{(1/2)}(\theta) = r_z(\theta/2) = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix},$$

$$t^{(1)}(\theta) = \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{i}{\sqrt{2}}\sin\frac{\theta}{2} & -\sin^2\frac{\theta}{2} \\ \frac{i}{\sqrt{2}}\sin\frac{\theta}{2} & \cos\theta & \frac{i}{\sqrt{2}}\sin\frac{\theta}{2} \\ -\sin^2\frac{\theta}{2} & \frac{i}{\sqrt{2}}\sin\frac{\theta}{2} & \cos^2\frac{\theta}{2} \end{pmatrix},$$

and

$$t^{(3/2)}(\theta) =$$
$$\begin{pmatrix} \cos^3\frac{\theta}{2} & i\sqrt{3}\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} & -\sqrt{3}\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} & -i\sin^3\frac{\theta}{2} \\ i\sqrt{3}\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} & \cos^3\frac{\theta}{2} - 2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2} & 2i\cos^2\frac{\theta}{2}\sin\frac{\theta}{2} - i\sin^3\frac{\theta}{2} & -\sqrt{3}\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} \\ -\sqrt{3}\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} & 2i\cos^2\frac{\theta}{2}\sin\frac{\theta}{2} - i\sin^3\frac{\theta}{2} & \cos^3\frac{\theta}{2} - 2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2} & i\sqrt{3}\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} \\ -i\sin^3\frac{\theta}{2} & -\sqrt{3}\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} & i\sqrt{3}\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} & \cos^3\frac{\theta}{2} \end{pmatrix}.$$

5.12 Matrix Elements of $T_\ell(q)$ and Jacobi Polynomials

In this section we assume again that $q \in \mathbf{SU}(2)$ is given in terms of the Euler angles as
$q = u(\varphi, \theta, \psi) = r_x(\varphi/2)r_z(\theta/2)r_x(\psi/2)$.

Since $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$ and $\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1$, for $0 \leq \theta \leq \pi$, we have $0 \leq \cos\frac{\theta}{2} \leq 1$ and $0 \leq \sin\frac{\theta}{2} \leq 1$, so

$$\begin{aligned} \cos\frac{\theta}{2} &= \sqrt{\frac{1+\cos\theta}{2}} \\ \sin\frac{\theta}{2} &= \sqrt{\frac{1-\cos\theta}{2}} \\ \cot\frac{\theta}{2} &= \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}, \end{aligned} \qquad (*_{39})$$

with $\theta > 0$ for the third formula.

Thus we see that $t_{jk}^{(\ell)}(\theta)$ is a function of $\cos\theta$ for $0 \leq \theta < \pi$.

Therefore there is a function $P_{jk}^{\ell}(z)$ such that

$$t_{jk}^{(\ell)}(\theta) = P_{jk}^{\ell}(\cos\theta), \quad 0 \leq \theta < \pi,$$

and $(*_{31})$ is also written as

$$t_{jk}^{(\ell)}(q) = e^{-i(j\varphi + k\psi)}\, P_{jk}^{\ell}(\cos\theta).$$

By Equation $(*_{32})$ and the above trigonometric identities we obtain the following result.

Proposition 5.14. *The polynomial* $P^{\ell}_{jk}(z)$ *($-1 < z \leq 1$) given by*

$$P^{\ell}_{jk}(z) = i^{-(j+k)}\sqrt{\frac{(\ell-j)!(\ell-k)!}{(\ell+j)!(\ell+k)!}}\,\left(\frac{1+z}{2}\right)^{\frac{j+k}{2}} \times \sum_{h=\max(j,k)}^{\ell} \frac{(\ell+h)!\,i^{2h}}{(\ell-h)!(h-j)!(h-k)!}\,\left(\frac{1-z}{2}\right)^{\frac{2h-(j+k)}{2}} \tag{$*_{40}$}$$

has the property that

$$t^{(\ell)}_{jk}(\theta) = P^{\ell}_{jk}(\cos\theta), \quad 0 \leq \theta < \pi, \tag{$*_{41}$}$$

and

$$t^{(\ell)}_{jk}(q) = e^{-i(j\varphi+k\psi)}\,P^{\ell}_{jk}(\cos\theta). \tag{$*_{42}$}$$

If ℓ is a half integer, then h is also a half integer.

It is understood that if $z = 1$, then $P^{\ell}_{jk}(1) = 1$ iff $j = k$, and $P^{\ell}_{jk}(1) = 0$ otherwise.

Proposition 5.15. *If $0 < \theta < \pi$, so that $-1 < z < 1$, then we have*

$$P^{\ell}_{jk}(z) = \frac{(-1)^{\ell-k} i^{k-j}}{2^{\ell}} \sqrt{\frac{(\ell+j)!}{(\ell-k)!(\ell+k)!(\ell-j)!}} \times (1+z)^{-\frac{(j+k)}{2}} (1-z)^{\frac{k-j}{2}} \frac{d^{\ell-j}}{dy^{\ell-j}} [(1-y)^{\ell-k}(1+y)^{\ell+k}]_{y=z}. \quad (*_{43})$$

The polynomials $P^{\ell}_{jk}(z)$ enjoy some symmetry relations.

Formula $(*_{43})$ also reveals a relationship with the Jacobi polynomials.

Definition 5.7. The *Jacobi polynomials* $P_h^{\lambda,\mu}(z)$, with $\lambda, \mu \in \mathbb{R}$, $h \in \mathbb{N}$, are defined by the formula

$$P_h^{\lambda,\nu}(z) = \frac{(-1)^h}{2^h h!}(1-z)^{-\lambda}(1+z)^{-\mu}\,\frac{d^h}{dz^h}[(1-z)^{\lambda+h}(1+z)^{\mu+h}]. \tag{Ja}$$

Proposition 5.16. *The polynomials $P_{jk}^{\ell}(z)$ and the Jacobi polynomials are related by the equation*

$$P_{\ell-j}^{j-k,k+j}(z) = 2^j i^{k-j}\sqrt{\frac{(\ell-k)!(\ell+k)!}{(\ell-j)!(\ell+j)!}} \\ (1-z)^{\frac{k-j}{2}}(1+z)^{-\frac{(k+j)}{2}}P_{jk}^{\ell}(z). \qquad (*_{45})$$

As we noted earlier, if ℓ is a half integer then j and k cannot be zero.

If ℓ is an integer, then $j = 0$ or $k = 0$ is allowed, and so $\lambda = 0$ and $\mu = 0$ are also allowed.

In this case the Jacobi polynomial $P_\ell^{0,0}(z)$, simply denoted as $P_\ell(z)$, is given by

$$P_\ell(z) = \frac{(-1)^\ell}{2^\ell \ell!} \frac{d^\ell}{dz^\ell}(1 - z^2)^\ell,$$

or equivalently

$$P_\ell(z) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dz^\ell}(z^2 - 1)^\ell.$$

This is a *Legendre* polynomial.

Similarly, if ℓ is an integer, then for $k = 0$ the polynomials $P_{m0}^\ell(z)$ are related to polynomials $P_\ell^m(z)$ known as the associated Legendre polynomials.

Definition 5.8. The *Legendre polynomial* $P_\ell(z)$ are defined by

$$P_\ell(z) = \frac{(-1)^\ell}{2^\ell \ell!} \frac{d^\ell}{dz^\ell}(1 - z^2)^\ell,$$

and the *associated Legendre polynomials*

are defined by

$$\begin{aligned} P_\ell^m(z) &= \frac{(-1)^{m+\ell}}{2^\ell \ell!} (1 - z^2)^{\frac{m}{2}} \frac{d^{m+\ell}}{dz^{m+\ell}}(1 - z^2)^\ell \\ &= (-1)^m (1 - z^2)^{\frac{m}{2}} \frac{d^m}{dz^m} P_\ell(z), \end{aligned}$$

with $\ell, m \in \mathbb{N}$.

Some authors omit the sign $(-1)^m$ in the definition of the associated Legendre polynomials.

We see immediately that

$$P^{\ell}_{00}(z) = P_{\ell}(z). \tag{$*_{46}$}$$

It is not hard to show that

$$P^{j}_{\ell}(z) = i^{j}\sqrt{\frac{(\ell+j)!}{(\ell-j)!}}\, P^{\ell}_{j0}(z). \tag{$*_{47}$}$$

See Vilenkin [39] (Chapter III, Section 3.9).

Since by $(*_{42})$ we have

$$t^{(\ell)}_{j0}(q) = e^{-ij\varphi}\, P^{\ell}_{j0}(\cos\theta),$$

we obtain

$$t^{(\ell)}_{j0}(q) = i^{-j}\sqrt{\frac{(\ell-j)!}{(\ell+j)!}}\, e^{-ij\varphi}\, P^{j}_{\ell}(\cos\theta), \quad -\ell \leq j \leq \ell. \tag{$*_{48}$}$$

Recall that ℓ is an integer.

Following Vilenkin [39] (Chapter III, Section 2.7) we show how the the function $t_{j0}^{(\ell)}(q)$ (with $q = r_x(\varphi/2)r_z(\theta/2)$), which does not depend on ψ, can be viewed as a function on the sphere S^2.

5.13 Harmonic Functions on the Sphere S^2

First recall that the group $\mathbf{SO}(3)$ acts transitively in the sphere S^2 and that the stabilizer of the point $e_1 = (1, 0, 0)$ is the subgroup H_x of rotations

$$R_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

around the x-axis, so the sphere S^2 is homeomorphic to the quotient space $\mathbf{SO}(3)/H_x$.

It follows that the functions $f \in \mathrm{L}^2(\mathbf{SO}(3))$ such that $f(RQ) = f(R)$ for all $R \in \mathbf{SO}(3)$ and all $Q \in H_x$ correspond bijectively to the functions in $\mathrm{L}^2(S^2)$.

From Section 5.3, since every rotation R can be factored as

$$R = R_x(\varphi)R_z(\theta)R_x(\psi),$$

with $R_x(\varphi), R_x(\psi) \in H_x$, we see that a representative of the left coset RH_x is given by

$$R_x(\varphi)R_z(\theta).$$

Therefore the points of S^2 are the orbit of $e_1 = (1, 0, 0)$ under all rotations $R_x(\varphi)R_z(\theta)$.

But the group H_x corresponds to the subgroup Ω_x defined below.

Definition 5.9. The subgroup Ω_x of $\mathbf{SU}(2)$ is given by

$$\Omega_x = \left\{ H(t) = r_x(t/2) = \begin{pmatrix} e^{\frac{it}{2}} & 0 \\ 0 & e^{-\frac{it}{2}} \end{pmatrix} \;\middle|\; 0 \leq t \leq 2\pi \right\}. \tag{Ω_x}$$

In fact we claim that $\mathbf{SU}(2)/\Omega_x$ is a homogeneous space homeomorphic to S^2 so that the functions $f \in \mathrm{L}^2(\mathbf{SU}(2))$ such that $f(qH) = f(q)$ for all $q \in \mathbf{SU}(2)$ and all $H \in \Omega_x$ also correspond bijectively to the functions in $\mathrm{L}^2(S^2)$.

The group $\mathbf{SU}(2)$ acts on the sphere S^2 by rotations, which means that for any skew-hermitian matrix

$$X = \begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix}, \quad (x, y, z) \in S^2$$

and any $q \in \mathbf{SU}(2)$, we have the action

$$q \cdot X = qXq^*.$$

Since this action is a rotation of S^2, it is transitive.

We easily showe that the stabilizer of $e_1 = (1, 0, 0)$ is indeed the subgroup Ω_x.

From Section 5.3, since every unit quaternion q can be factored as

$$q = r_x(\varphi/2)r_z(\theta/2)r_x(\psi/2),$$

with $r_x(\varphi/2), r_x(\psi/2) \in \Omega_x$, we see that a representative of the left coset $q\Omega_x$ is given by

$$r_x(\varphi/2)r_z(\theta/2).$$

Therefore the points of S^2 are the orbit of $e_1 = (1, 0, 0)$ under all rotations $r_x(\varphi/2)r_z(\theta/2)$, and from Section 5.3, since the corresponding rotation matrices are

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

by reading of the first column of the matrix Q, we see that the corresponding orbit points on the sphere S^2 have coordinates

$$(\cos\theta,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi).$$

According to the physical convention, the spherical coordinates of a point p with respect to the (azimuthal) angle φ measured from the x-axis in the xy-plane and (polar) angle θ measured from the z-axis in the plane containing the z-axis an passing through the point p are given by

$$(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta).$$

Thus we see that the coordinates

$$(\cos\theta,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi)$$

are “funny” spherical coordinates for which the x-axis and the z-axis are swapped and φ is changed to $\pi/2-\varphi$.

Following Vilenkin (Chapter III, Section 3.10) we make the following definition.

Definition 5.10. For any j such that $-\ell \leq j \leq \ell$, the function $t^{(\ell)}_{j0}(q)$ which does not depend on ψ (with $q = r_x(\varphi/2)r_z(\theta/2)r_x(\psi/2)$), can be viewed as a function on the sphere S^2, and is denoted $Y_{\ell j}(\varphi, \theta)$, with $0 \leq \varphi < 2\pi$ and $0 \leq \theta < \pi$. The function $Y_{\ell j}(\varphi, \theta)$ is called a *spherical function*.

Observe that the $2\ell + 1$ functions $Y_{\ell j}(\varphi, \theta) = t^{(\ell)}_{j0}(q)$ $(-\ell \leq j \leq \ell)$ constitute the *middle column* of the matrix $t^{(\ell)}(q)$.

In view of Proposition 5.11 and $(*_{41})$, for any matrix $R \in \mathbf{SO}(3)$ expressed in terms of the Euler angles as $R = R_x(\varphi)R_z(\theta)R_x(\psi)$, with respect to the orthonormal basis (ψ_k) of $\mathcal{P}_\ell^{\mathbb{C}}$, the matrix $w^{(\ell)}(R)$ of the unitary map $W_\ell(R)$ associated with the irreducible representation $W_\ell \colon \mathbf{SO}(3) \to \mathbf{U}(\mathcal{P}_\ell^{\mathbb{C}})$ is given by

$$w_{jk}^{(\ell)}(R) = e^{-i(j\varphi + k\psi)}\, t_{jk}^{(\ell)}(\theta) = e^{-i(j\varphi + k\psi)}\, P_{jk}^{\ell}(\theta) = t_{jk}^{(\ell)}(q),$$
$$\ell \in \mathbb{N},$$

where $q = r_x(\varphi/2)r_z(\theta/2)r_x(\psi/2)$. In particular, for $k = 0$ we see that

$$w_{j0}^{(\ell)}(R) = t_{j0}^{(\ell)}(q) = Y_{\ell j}(\varphi, \theta).$$

Thus we have shown the following result.

Proposition 5.17. *The following facts hold.*

(1) *For any matrix $R \in \mathbf{SO}(3)$ expressed as $R = R_x(\varphi)R_z(\theta)R_x(\psi)$ in terms of the Euler angles, with respect to the orthonormal basis (ψ_k) of $\mathcal{P}_\ell^{\mathbb{C}}$, the matrix $w^{(\ell)}(R)$ of the unitary map $W_\ell(R)$ associated with the irreducible representation $W_\ell \colon \mathbf{SO}(3) \to \mathbf{U}(\mathcal{P}_\ell^{\mathbb{C}})$ is equal to the matrix $t^{(\ell)}(q)$ of the unitary map $T_\ell(q)$ associated with the irreducible representation $T_\ell \colon \mathbf{SU}(2) \to \mathbf{U}(\mathcal{P}_\ell^{\mathbb{C}})$, where $q = r_x(\varphi/2)r_z(\theta/2)r_x(\psi/2)$ ($\ell \in \mathbb{N}$).*

(2) *Viewed as functions on S^2, the $2\ell + 1$ functions $t_{j0}^{(\ell)}(q)$ (with $q = r_x(\varphi/2)r_z(\theta/2)r_x(\psi/2)$) constitute the middle column of the matrix $t^{(\ell)}(q)$ and the $2\ell+1$ functions $w_{j0}^{(\ell)}(R)$ (with $R = R_x(\varphi)R_z(\theta)R_x(\psi)$) constitute the middle column of the matrix $w^{(\ell)}(R)$.*

(3) Viewed as a function on S^2 in spherical coordinates

$$(x, y, z) = (\cos\theta,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi),$$

we have

$$Y_{\ell j}(x, y, z) = Y_{\ell j}(\varphi, \theta) = t^{(\ell)}_{j0}(q) = w^{(\ell)}_{j0}(R),$$

with $q = r_x(\varphi/2)r_z(\theta/2)$ and $R = R_x(\varphi)R_z(\theta)$.

As we observed earlier, the matrices $t^{(\ell)}(\theta)$, and so the polynomials $P^{\ell}_{jk}(z)$, are not all real.

And indeed Equation ($*_{48}$) shows that the functions $Y_{\ell j}(\varphi,\theta) = t^{(\ell)}_{j0}(q)$ are not all real.

A way to fix this is to multiply $Y_{\ell j}(\varphi,\theta)$ by i^j.

It turns out that $i^j\sqrt{2\ell+1}\,Y_{\ell j}(\varphi,\theta)$ is a function known as the classical spherical harmonic, (unfortunately) denoted $Y^j_\ell(\theta,\varphi)$.

Definition 5.11. The function $Y^j_\ell(\theta,\varphi)$ called *Laplace spherical harmonic* by Dieudonné is given by

$$Y^j_\ell(\theta,\varphi) = \sqrt{\frac{(2\ell+1)(\ell-j)!}{(\ell+j)!}}e^{-ij\varphi}P^j_\ell(\cos\theta).$$

If we recall that the motivation for introducing the Wigner d-matrices was to *deal with real orthogonal matrices instead of complex unitary matrices*, we can use the Wigner d-matrices instead of the matrices $t^{(\ell)}(\theta)$, but there is an annoying sign issue.

Wigner defines his d-matrices as

$$d^{\ell}_{jk}(\theta) = (-1)^{j-k} i^{j-k} t^{(\ell)}_{jk}(\theta),$$

so for $k = 0$, the factor i^j makes the term real, but now we have the extra factor $(-1)^j$, so the middle column of the d-matrix consists of the entries $(-1)^j P^j_\ell(\cos\theta)$ instead of $P^j_\ell(\cos\theta)$.

The remedy is to redefine the Wigner d-matrices by omitting the factor $(-1)^{j-k}$ in the above formula, or equivalently to define the Wigner $\mathcal{D}$-matrix
$\mathcal{D}^{(\ell)}(R) = \mathcal{D}^{(\ell)}(\varphi,\theta,\psi)$ as follows.

Definition 5.12. The *Wigner $\mathcal{D}$-matrix $\mathcal{D}^{(\ell)}(R)$* is defined as

$$\begin{aligned}\mathcal{D}^{(\ell)}_{jk}(R) = \mathcal{D}^{(\ell)}_{jk}(\varphi,\theta,\psi) &= e^{-i(j\varphi+k\psi)}(-1)^{j-k}d^{(\ell)}_{jk}(\theta)\\ &= e^{-i(j\varphi+k\psi)}i^{j-k}t^{(\ell)}_{jk}(\theta),\end{aligned}$$

where $R = R_x(\varphi)R_z(\theta)R_x(\psi)$.

Of course the Wigner $\mathcal{D}$-matrix $\mathcal{D}^{(\ell)}$ defines an irreducible representation $\mathcal{D}^{(\ell)}\colon \mathbf{SO}(3) \to \mathbf{U}(\mathcal{P}^{\mathbb{C}}_{\ell})$ equivalent to the irreducible representation $W_\ell\colon \mathbf{SO}(3) \to \mathbf{U}(\mathcal{P}^{\mathbb{C}}_{\ell})$.

Also now the *middle column of $\mathcal{D}^{(\ell)}(\varphi,\theta,\psi)$* consists of the rescaled functions $1/\sqrt{2\ell+1}\, Y^j_\ell(\theta,\varphi)$, as desired.

Note that Sakurai and Napolitano [35] also add the factor $(-1)^{j-k}$ in their definition of the $\mathcal{D}$-matrix.

We will prove in Section 5.15 that the family of functions $(Y_\ell^j(\theta,\varphi))_{\ell\in\mathbb{N},\, -\ell\leq j\leq \ell}$ forms a Hilbert basis for the functions in $\mathrm{L}^2(S^2)$.

There is another property of the functions $Y_\ell^j(\theta,\varphi)$ worth stating because it plays a role in equivariant deep learning in cnns.

Here we assume that $Y_\ell^j(\theta,\varphi)$ is viewed as a function on $\mathbf{SO}(3)/H_x$.

Since the group $\mathbf{SO}(3)$ acts on S^2, it is natural to wonder how the function $\lambda_R Y_\ell^j$ is related to Y_ℓ^j, for $R\in\mathbf{SO}(3)$.

Here is more natural to write $Y_\ell^j(x,y,z)$, where $(x,y,z) \in S^2$ are expressed in spherical coordinates in terms of the Euler angles φ and θ as in Proposition 5.17.

Proposition 5.18. *Denote the column vector consisting of the $2\ell+1$ functions Y_ℓ^j by Y_ℓ ($\ell \in \mathbb{N}$). For every rotation $R \in \mathbf{SO}(3)$ expressed as $R = R_x(\varphi)R_z(\theta)R_x(\psi)$, we have*

$$\begin{aligned} Y_\ell(R\cdot(x,y,z)) &= \mathcal{D}^{(\ell)}(R)Y_\ell(x,y,z) \\ &= \mathcal{D}^{(\ell)}(\varphi,\theta,\psi)Y_\ell(x,y,z), \quad (x,y,z) \in S^2. \end{aligned}$$

As a corollary, we also have

$$\begin{aligned} \overline{Y_\ell}(R^{-1}\cdot(x,y,z)) &= (\mathcal{D}^{(\ell)}(R))^\top\, \overline{Y_\ell}(x,y,z) \\ &= (\mathcal{D}^{(\ell)}(\varphi,\theta,\psi))^\top\, \overline{Y_\ell}(x,y,z),\ (x,y,z) \in S^2. \end{aligned}$$

In special case where $j = 0$ the function $t_{00}^{(\ell)}(q) = P_\ell(\cos\theta)$ depends only on θ and is called a *zonal spherical function*.

More properties of the Legendre and Jacobi polynomials and functional relations and generating functions for the functions $P_{jk}^{\ell}(z)$, can be found in Vilenkin [39], Chapter III, Sections 3-5.

5.14 Integration on $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$

In this section we derive explicit formulae for the normalized Haar measures on $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$ when these groups are parametrized by the Euler angles. Technically, these parametrizations are injective only on open subsets of $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$, but the complements of these open sets have measure zero so from the point of view integration we obtain formulae for integrating all functions in $\mathrm{L}^2(\mathbf{SU}(2))$ and all functions in $\mathrm{L}^2(\mathbf{SO}(3))$ (respectively equipped with these left and right invariant Haar measures).

As a first step we will need to derive a formula for an $\mathbf{SU}(2)$-invariant volume form on $\mathbf{SU}(2)$ as a pull-back of the $\mathbf{SO}(4)$-invariant volume form ω_{S^3} on S^3. The reader may want to review volume forms and integration on manifolds before reading this section. These topics are covered in Gallier and Quaintance [23] (Chapter 4 and 6).

5.15 Fourier Series of Functions in $\mathrm{L}^2(\mathbf{SU}(2))$, $\mathrm{L}^2(\mathbf{SO}(3))$ and $\mathrm{L}^2(S^2)$

In the preceding sections we computed explicitly several matrix representations $t^{(\ell)}(q)$ for the irreducible representations $T_\ell \colon \mathbf{SU}(2) \to \mathbf{U}(\mathcal{P}_\ell^{\mathbb{C}})$ with respect to an invariant hermitian inner product on $\mathcal{P}_\ell^{\mathbb{C}}$.

In terms of the general results presented in Sections 4.2–4.4, especially Theorem 4.4, $\rho = \ell$, $n_\rho = 2\ell + 1$, $M_\ell(q) = t^{(\ell)}(q)$, and since

$$M_\ell(q) = \left(\frac{1}{n_\ell} m_{ij}^{(\ell)}(q)\right),$$

the functions $m_{ij}^{(\ell)}(q)$ are given by $m_{ij}^{(\ell)}(q) = (2\ell{+}1)t_{ij}^{(\ell)}(q)$, where ℓ ranges through the set $R = \{0, 1/2, 1, 3/2, 2, 5/2, 3, \ldots\}$ of all nonnegative integer and half integer values.

By Peter–Weyl I (Theorem 4.3), the $n_\ell^2 = (2\ell+1)^2$ functions $\frac{1}{\sqrt{n_\ell}} m_{ij}^{(\ell)} = \sqrt{2\ell+1}\, t_{ij}^{(\ell)}$ in the matrix $\sqrt{2\ell+1}\, t^{(\ell)}$ form an orthonormal basis of the minimal two-sided ideal $\mathfrak{a}_\ell$ arising in the Hilbert sum

$$\mathrm{L}^2(\mathbf{SU}(2)) = \bigoplus_{\ell} \mathfrak{a}_\ell,$$

and thus the family of functions

$$\left(\sqrt{2\ell+1}\, t_{ij}^{(\ell)}\right)_{-\ell \leq i,j \leq \ell,\, \ell \in R}$$

with $R = \{0, 1/2, 1, 3/2, 2, \ldots\}$, is a Hilbert basis of $\mathrm{L}^2(\mathbf{SU}(2))$.

By the results of Section 4.7 on the Fourier transform and the Fourier cotransform, by Definition 4.19 of the Fourier transform $\mathcal{F}(f)$ and Equation (FI) (see also Theorem 4.26),

$$f(s) = \sum_{\rho \in R} n_\rho \operatorname{tr}\left(\mathcal{F}(f)(\rho) M_\rho(s)\right) \qquad f \in \mathrm{L}^2(G),\, s \in G,$$

since $M_\ell(q) = t^{(\ell)}(q)$, for every $\ell \in R$, the $(2\ell+1) \times (2\ell+1)$ matrix $\alpha^{(\ell)} = \mathcal{F}(f)(\ell)$ of Fourier coefficients of $f \in \mathrm{L}^2(\mathbf{SU}(2))$ is given by

$$\alpha^{(\ell)} = \int_{\mathbf{SU}(2)} f(q)(t^{(\ell)}(q))^* \, d\nu(q),$$

where ν is the normalized Haar measure on $\mathbf{SU}(2)$, and by the Fourier inversion formula (FI) we have

$$f(q) = \sum_{\ell \in R} (2\ell+1) \operatorname{tr}\big(\alpha^{(\ell)} t^{(\ell)}(q)\big), \quad q \in \mathbf{SU}(2).$$

Written in terms of matrix elements, we obtain the equations

$$\alpha_{jk}^{(\ell)} = \int_{\mathbf{SU}(2)} f(q)\overline{t_{kj}^{(\ell)}(q)}\, d\nu(q) \tag{FC1}$$

and

$$f(q) = \sum_{\ell \in R} (2\ell+1) \sum_{j=-\ell}^{\ell} \sum_{k=-\ell}^{\ell} \alpha_{kj}^{(\ell)}\, t_{jk}^{(\ell)}(q), \quad q \in \mathbf{SU}(2). \tag{FS1}$$

Using the Euler angles, Proposition 5.14 (in particular, $(*_{41}), (*_{42})$), namely

$$\begin{aligned} t^{(\ell)}_{jk}(q) &= t^{(\ell)}_{jk}(u(\varphi, \theta, \psi)) = e^{-i(j\varphi + k\psi)}\, t^{(\ell)}_{jk}(\theta) \\ &= e^{-i(j\varphi + k\psi)}\, P^{\ell}_{jk}(\cos\theta), \quad \ell \in R, \end{aligned}$$

Proposition **??** (formula for the Haar measure on $\mathbf{SU}(2)$), and the fact that $\overline{P^{\ell}_{jk}(\cos\theta)} = (-1)^{j-k} P^{\ell}_{jk}(\cos\theta)$ (left as an exercise), by swapping j and k in (FC1), we obtain the following series expansion for the functions in $\mathrm{L}^2(\mathbf{SU}(2))$.

Proposition 5.19. *Every function $f \in \mathrm{L}^2(\mathbf{SU}(2))$ expressed in terms of the Euler angles $(0 \leq \varphi < 2\pi, 0 \leq \theta < \pi, -2\pi \leq \psi < 2\pi)$ can be written as the Fourier series*

$$\begin{aligned} & f(u(\varphi,\theta,\psi)) \\ & = \sum_{\ell \in R} (2\ell+1) \sum_{j=-\ell}^{\ell} \sum_{k=-\ell}^{\ell} \alpha_{kj}^{(\ell)} \, e^{-i(j\varphi+k\psi)} P_{jk}^{\ell}(\cos\theta), \end{aligned} \tag{FS2}$$

where the Fourier coefficients are given by

$$\begin{aligned} \alpha_{kj}^{(\ell)} = \frac{(-1)^{j-k}}{16\pi^2} \int_{-2\pi}^{2\pi} \int_{0}^{2\pi} \int_{0}^{\pi} & f(u(\varphi,\theta,\psi)) \\ & e^{i(j\varphi+k\psi)} P_{jk}^{\ell}(\cos\theta) \sin\theta \, d\theta \, d\varphi \, d\psi. \end{aligned} \tag{FC2}$$

Recall that $u(\varphi,\theta,\psi) = r_x(\varphi/2) r_z(\theta/2) r_x(\psi/2) \in \mathbf{SU}(2)$.

The above discussion applies to $\mathbf{SO}(3)$ and its irreducible representations $W_\ell\colon \mathbf{SO}(3) \to \mathbf{U}(\mathcal{P}_\ell^{\mathbb{C}})$, which are now indexed by the set $\mathbb{N}$ of natural numbers.

By Peter–Weyl I (Theorem 4.3), the $n_\ell^2 = (2\ell+1)^2$ functions $\frac{1}{\sqrt{n_\ell}} m_{ij}^{(\ell)} = \sqrt{2\ell+1}\, w_{ij}^{(\ell)}$ in the matrix $\sqrt{2\ell+1}\, w^{(\ell)}$, where $w^{(\ell)}(R)$ is the matrix associated with $W^\ell(R)$ for $R \in \mathbf{SO}(3)$, form an orthonormal basis of the minimal two-sided ideal $\mathfrak{a}_\ell$ arising in the Hilbert sum

$$\mathrm{L}^2(\mathbf{SO}(3)) = \bigoplus_{\ell} \mathfrak{a}_\ell,$$

and thus the family of functions

$$\left(\sqrt{2\ell+1}\, w_{ij}^{(\ell)}\right)_{-\ell \le i,j \le \ell,\, \ell \in \mathbb{N}}$$

is a Hilbert basis of $\mathrm{L}^2(\mathbf{SO}(3))$.

It follows that for every $\ell \in \mathbb{N}$, the $(2\ell+1)\times(2\ell+1)$ matrix $\alpha^{(\ell)} = \mathcal{F}(f)(\ell)$ of Fourier coefficients of $f \in \mathrm{L}^2(\mathbf{SO}(3))$ is given by

$$\alpha^{(\ell)} = \int_{\mathbf{SO}(3)} f(R)(w^{(\ell)}(R))^* \, d\nu_0(R),$$

where ν_0 is the normalized Haar measure on $\mathbf{SO}(3)$, and by the Fourier inversion formula (FI) we have

$$f(R) = \sum_{\ell\in\mathbb{N}} (2\ell+1)\,\mathrm{tr}\big(\alpha^{(\ell)} w^{(\ell)}(R)\big), \quad R \in \mathbf{SO}(3).$$

Written in terms of matrix elements, we obtain the equations

$$\alpha_{jk}^{(\ell)} = \int_{\mathbf{SO}(3)} f(R)\overline{w_{kj}^{(\ell)}(R)}\, d\nu_0(R) \tag{FC1'}$$

and

$$f(R) = \sum_{\ell\in\mathbb{N}} (2\ell+1) \sum_{j=-\ell}^{\ell} \sum_{k=-\ell}^{\ell} \alpha_{kj}^{(\ell)}\, w_{jk}^{(\ell)}(q), \quad R \in \mathbf{SO}(3). \tag{FS1'}$$

Using the Euler angles, Proposition 5.14 (in particular, $(*_{41}), (*_{42})$), Proposition **??** (formula for the Haar measure on $\mathbf{SO}(3)$), that by Proposition 5.11 we have

$$\begin{aligned} w^{(\ell)}_{jk}(R_0(\varphi,\theta,\psi)) &= e^{-i(j\varphi+k\psi)}\, t^{(\ell)}_{jk}(\theta) \\ &= e^{-i(j\varphi+k\psi)}\, P^{\ell}_{jk}(\cos\theta), \quad \ell \in \mathbb{N}, \end{aligned}$$

and using the fact that
$\overline{P^{\ell}_{jk}(\cos\theta)} = (-1)^{j-k} P^{\ell}_{jk}(\cos\theta)$ (left as an exercise), we obtain the following series expansion for the functions in $\mathrm{L}^2(\mathbf{SO}(3))$.

Let $R_0(\varphi,\theta,\psi) = R_x(\varphi)R_z(\theta)R_x(\psi) \in \mathbf{SO}(3)$.

Proposition 5.20. *Every function $f \in \mathrm{L}^2(\mathbf{SO}(3))$ expressed in terms of the Euler angles $(0 \leq \varphi < 2\pi, 0 \leq \theta < \pi, 0 \leq \psi < 2\pi)$ can be written as the Fourier series*

$$\begin{aligned} & f(R_0(\varphi,\theta,\psi)) \\ & = \sum_{\ell \in \mathbb{N}} (2\ell+1) \sum_{j=-\ell}^{\ell} \sum_{k=-\ell}^{\ell} \alpha_{kj}^{(\ell)}\, e^{-i(j\varphi+k\psi)} P_{jk}^{\ell}(\cos\theta), \end{aligned} \tag{FS2'}$$

where the Fourier coefficients are given by

$$\begin{aligned} \alpha_{kj}^{(\ell)} = \frac{(-1)^{j-k}}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi} & f(R_0(\varphi,\theta,\psi)) \\ & e^{i(j\varphi+k\psi)} P_{jk}^{\ell}(\cos\theta) \sin\theta \, d\theta \, d\varphi \, d\psi. \end{aligned} \tag{FC2'}$$

Remarks:

(1) If the functions f are real-valued, it may be preferable to use the Wigner d-matrices $d^{(\ell)}(\theta)$ of Definition 5.5, which are real orhogonal, instead of the complex matrices $t^{(\ell)}(\theta)$, which amounts to using $(-1)^{j-k}i^{j-k}t^{(\ell)}_{jk}(\theta)$ instead of $t^{(\ell)}_{jk}(\theta)$, that is, the real polynomials $(-1)^{j-k}i^{j-k}P^{\ell}_{jk}$ instead of P^{ℓ}_{jk} in (FS2') and (FC2'). This is common practice in computer vision.

(2) A variant of the definition of the Fourier transform and of the Fourier cotransform occurs in the computer vision community. In these formula, $w^{(\ell)}(R)$ is replaced by $(w^{(\ell)}(R))^*$, namely

$$\alpha^{(\ell)} = \int_{\mathbf{SO}(3)} f(R) w^{(\ell)}(R)\, d\nu_0(R),$$

and

$$f(R) = \sum_{\ell\in\mathbb{N}} (2\ell+1)\,\mathrm{tr}\big(\alpha^{(\ell)}(w^{(\ell)}(R))^*\big), \quad R \in \mathbf{SO}(3).$$

Our version is consistent with the definition of the Fourier transform in the case where G is abelian.

We can also obtain the following Fourier series expansion for every function $f \in \mathrm{L}^2(S^2)$ in terms of the associated Legendre functions,

$$f(\varphi,\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\sum_{j=-\ell}^{\ell}\beta_\ell^j\, e^{-ij\varphi}P_\ell^j(\cos\theta), \qquad \text{(FS8)}$$

with

$$\beta_\ell^j = \frac{1}{4\pi}\frac{(\ell-j)!}{(\ell+j)!}\int_0^{2\pi}\int_0^{\pi} f(\varphi,\theta)e^{ij\varphi}P_\ell^j(\cos\theta)\sin\theta\, d\theta\, d\varphi. \qquad \text{(FC8)}$$

We also have the Parseval identity

$$\int_0^{2\pi}\int_0^{\pi} |f(\varphi,\theta)|^2 d\nu = \sum_{\ell=0}^{\infty}(2\ell+1)\sum_{j=-\ell}^{\ell}\frac{(\ell+j)!}{(\ell-j)!}|\beta_\ell^j|^2, \tag{PS2}$$

where $d\nu = (1/4)\sin\theta\, d\theta\, d\varphi$ is the normalized measure on S^2 in spherical coordinates; among other sources, see Gallier and Quaintance [23] (Section 6.4).

Recall from Definition 5.10 and $(*_{48})$ that

$$Y_{\ell j}(\varphi,\theta) = t^{(\ell)}_{j0}(q) = i^{-j}\sqrt{\frac{(\ell-j)!}{(\ell+j)!}}\, e^{-ij\varphi}\, P^j_\ell(\cos\theta), \quad -\ell \leq j \leq \ell,$$

with $\ell \in \mathbb{N}$, so we have

$$\sqrt{\frac{(2\ell+1)(\ell-j)!}{(\ell+j)!}} e^{-ij\varphi} P^j_\ell(\cos\theta) = i^j\sqrt{2\ell+1}\, Y_{\ell j}(\varphi,\theta),$$

for $\ell \in \mathbb{N}$ and $-\ell \leq j \leq \ell$, and in view of (FS8) and (FS2), *the above functions form a Hilbert basis for the functions in* $\mathrm{L}^2(S^2)$.

As we explained just after Proposition 5.17, the functions $i^j \sqrt{2\ell + 1}\, Y_{\ell j}(\varphi, \theta)$ are (a version of) the *Laplace spherical harmonics* $Y_\ell^j(\theta, \varphi)$, namely

$$Y_\ell^j(\theta, \varphi) = \sqrt{\frac{(2\ell + 1)(\ell - j)!}{(\ell + j)!}} e^{-ij\varphi} P_\ell^j(\cos\theta).$$

Remark: Some authors include $1/\sqrt{4\pi}$ in the leading constant.

The associated Legendre functions can be computed starting with the Legendre polynomials using some recurrence equations; see Gallier and Quaintance [23] (Section 7.3).