Chapter 4

Analysis on Compact Groups and Representations

4.1 Preliminaries About $\mathrm{L}^2(G)$

To avoid technical complications (namely using uncountable Hibert sums) we assume that *our compact groups are metrizable*.

This means that the topology of the compact group G can be defined by a *distance* (a metric).

This is not a severe restriction since *all Lie groups are metrizable*, and we have no need to deal with groups that are not Lie groups.

A nice consequence of assuming that the compact group G is metrizable is that G is *separable*.

This means that G has a *countable dense subset*.

A fundamental result of *Haar* asserts that every compact group (even non-metrizable) has a *left and right-invariant measure λ_G up to a constant*.

This allows us to *integrate functions defined in G*.

For any function $f\colon G \to \mathbb{C}$,

left-invariance means that for any fixed $t \in G$,

$$\int_G f(ts)\, d\lambda_G(s) = \int_G f(s)\, d\lambda_G(s)$$

and *right-invariance* means that for any fixed $t \in G$,

$$\int_G f(st)\, d\lambda_G(s) = \int_G f(s)\, d\lambda_G(s).$$

We also have

$$\int_G f(s^{-1})\, d\lambda_G(s) = \int_G f(s)\, d\lambda_G(s).$$

We may assume that λ_G is normalized so that $\lambda_G(G) = 1$.

We often drop the subscript G in λ_G,

The vector space $\mathrm{L}^1(G)$ is the space of functions $f\colon G \to \mathbb{C}$ such that the integral

$$\int_G f(s)\, d\lambda(s)$$

is defined.

We define the *norm* $\| \; \|_{\mathrm{L}^1(G)}$ by

$$\|f\|_{\mathrm{L}^1(G)} = \int_G |f(s)|\, d\lambda(s).$$

The space $\mathrm{L}^1(G)$ is a *Banach space* (a complete normed vector space).

The vector space $\mathrm{L}^2(G)$ is the space of measurable functions $f\colon G \to \mathbb{C}$ such that the function $s \mapsto |f(s)|^2$ belongs to $\mathrm{L}^1(G)$; that is, the integral

$$\int_G |f(s)|^2 \, d\lambda(s)$$

is defined.

We define an *inner product* $\langle -, - \rangle$ on $\mathrm{L}^2(G)$ by

$$\langle f, g\rangle = \int_G f(s)\overline{g(s)}\, d\lambda(s), \quad f, g \in \mathrm{L}^2(G),$$

and the *norm* $\| \, \|_{\mathrm{L}^2(G)}$ by

$$\|f\|^2_{\mathrm{L}^2(G)} = \langle f, f\rangle = \int_G |f(s)|^2 \, d\lambda(s).$$

Then $\mathrm{L}^2(G)$ with its norm $\|f\|_{\mathrm{L}^2(G)}$ is a *Banach space*, and with its inner product it is a *Hilbert space*.

Because G is *compact*, we have

$$\|f\|_{\mathrm{L}^1(G)} \leq \|f\|_{\mathrm{L}^2(G)}\,,$$

and so

$$\mathrm{L}^2(G) \subseteq \mathrm{L}^1(G).$$

In general, this inclusion is *false* if the space is not compact.

Definition 4.1. The *convolution* $f * g$ of two functions $f, g \in \mathrm{L}^1(G)$ is given by

$$(f{*}g)(s) = \int_G f(t)g(t^{-1}s)\, d\lambda(t) = \int_G f(st^{-1})g(t)\, d\lambda(t).$$

Note that t and t^{-1} occur adjacent to each other.

It can be shown that $f * g \in \mathrm{L}^1(G)$.

Because $\mathrm{L}^2(G) \subseteq \mathrm{L}^1(G)$, it can be shown that if $f, g \in \mathrm{L}^2(G)$, then $f * g \in \mathrm{L}^2(G)$.

This is generally *false* if the space is not compact!

If $f \in \mathrm{L}^2(G)$, then the function f^* given by

$$f^*(s) = \overline{f(s^{-1})}$$

is also in $\mathrm{L}^2(G)$.

We obtain an involution $f \mapsto f^*$.

Given $f \in \mathrm{L}^2(G)$, we define $\check{f}$ as

$$\check{f}(s) = f(s^{-1}), \quad s \in G.$$

Note that

$$f^* = \check{\overline{f}}.$$

With convolution as multiplication this makes $\mathrm{L}^2(G)$ an *involutive Banach algebra*.

Remark: In fact, $L^2(G)$ is a *complete separable Hilbert algebra*.

This means that the following equations hold:

$$\begin{aligned} \langle g^*, f^* \rangle &= \langle f, g \rangle \\ \langle f * g, h \rangle &= \langle g, f^* * h \rangle, \end{aligned}$$

as well as two other technical conditions.

This is important because there is a structure theorem for complete separable Hilbert algebras which is the deep reason behind the Peter–Weyl theorem.

The *Peter–Weyl theorem* has two parts.

Peter–Weyl I describes $\mathrm{L}^2(G)$ as a Hilbert sum of finite-dimensional minimal two-sided ideals which have (orthogonal) Hilbert bases of functions.

A subspace $\mathfrak{a}$ of $\mathrm{L}^2(G)$ is a *two-sided ideal* if

$$f * g * h \in \mathfrak{a}, \quad g \in \mathfrak{a},\ f, h \in \mathrm{L}^2(G).$$

The two-sided ideal $\mathfrak{a}$ is *minimal* if it does not contain any proper two-sided ideal.

A *left ideal* is a subspace $\mathfrak{l}$ of $\mathrm{L}^2(G)$ such that

$$f * g \in \mathfrak{l}, \quad g \in \mathfrak{l},\ f \in \mathrm{L}^2(G).$$

The functions in $\mathfrak{a}$ make up a matrix which turns out to define an *irreducible representation of G*.

Peter–Weyl II decomposes a unitary representation of G as a Hilbert sum of spaces with nice properties.

Definition 4.2. Let H be a Hilbert space and let $(H_\alpha)_{\alpha\in\Lambda}$ (with Λ a countable index set) be a family of closed subspaces of H satisfying the following conditions:

(1) For all $\alpha \neq \beta$, the subspaces H_α and H_β are orthogonal.

(2) The algebraic direct sum $\bigoplus_{\alpha\in\Lambda} H_\alpha$ is dense in H.

This means that every $h \in H$ is the limit of a sequence of finite sums $S_n \in \bigoplus_{\alpha\in\Lambda} H_\alpha$ that converge to h as n tends to ∞.

Then we say that H is the *Hilbert sum* $(H_\alpha)_{\alpha\in\Lambda}$.

If each H_α has dimension 1, and if we pick some nonzero element $h_\alpha \in H_\alpha$ then we say that $(h_\alpha)_{\alpha\in\Lambda}$ is a *Hilbert basis* of H.

Remark: We will only consider countable index sets Λ. This implies that a Hilbert sum is *separable*.

There is a more convenient characterization of Hilbert bases involving certain series but since the index set Λ is not necessarily ordered (although in bijection with $\mathbb{N}$) it is necessary to generalize the notion of convergence of a series.

Definition 4.3. Given a normed vector space E (say, a Hilbert space), for any nonempty countable index set Λ, we say that a family $(u_\ell)_{\ell\in\Lambda}$ of vectors in E is *summable with sum $v \in E$* iff for every $\epsilon > 0$, there is some *finite* subset I of Λ, such that,

$$\left\| v - \sum_{j\in J} u_j \right\| < \epsilon \tag{conv}$$

for every *finite* subset J with $I \subseteq J \subseteq \Lambda$.

We say that the family $(u_\ell)_{\ell\in\Lambda}$ is *summable* iff there is some $v \in E$ such that $(u_\ell)_{\ell\in\Lambda}$ is summable with sum v. In this case we write

$$v = \sum_{\ell\in\Lambda} u_\ell.$$

and we call v the *sum (or limit)* of the series.

The notion of convergence of a series $\sum_{\ell \in \Lambda} u_\ell$ defined in Definition 4.3 is quite strong in the sense that *it is independent of any ordering chosen on* Λ.

Indeed let $\omega \colon \mathbb{N} \to \Lambda$ be any bijection. We obtain the "ordinary" series

$$\sum_{k=0}^{\infty} v_k, \quad v_k = u_{\omega(k)}.$$

Then for any $\epsilon > 0$, if I is any finite subset $I \subseteq \Lambda$ satisfying Condition (conv), there is some $N > 0$ such that

$$I \subseteq \{\omega(0), \ldots, \omega(N)\},$$

so for all $n \geq N$, since

$$I \subseteq J_n = \{\omega(0), \ldots, \omega(n)\},$$

we have

$$\left\| v - \sum_{k=0}^{n} u_{\omega(k)} \right\| < \epsilon,$$

which means that the series

$$\sum_{k=0}^{\infty} v_k = \sum_{k=0}^{\infty} u_{\omega(k)}$$

converges to v.

We say that the series is *commutatively convergent*.

For example, given a function $f \in \mathrm{L}^2(\mathbb{T})$ and if its Fourier coefficients are c_k, then the Fourier series

$$\sum_{k=-\infty}^{\infty} c_k e^{ik\theta}$$

is summable with limit f.

A summable series may *fail* to be absolutely convergent!

If we use the bijection $\omega\colon \mathbb{N} \to \mathbb{Z}$ given by

$$\omega(k) = \begin{cases} h & \text{if } k = 2h \\ -(h+1) & \text{if } k = 2h+1, \end{cases}$$

so $\mathbb{Z}$ has the ordering

$$0, -1, 1, -2, 2, -3, 3, \ldots, -(h+1), h+1, \ldots,$$

and we have the partial sums

$$S_{2n} = \sum_{k=-n}^{n} c_k e^{ik\theta}.$$

Given a Hibert basis of a separable Hilbert space H the notion of Fourier coefficients is defined as follows.

Definition 4.4. Let H be a separable Hilbert space and let $(u_\ell)_{\ell\in\Lambda}$ be a Hilbert basis of H (where Λ is countable). For any $v \in H$, the *Fourier coefficients of v* are given by

$$c_\ell = \frac{\langle v, u_\ell\rangle}{\|u_\ell\|^2}, \quad \ell \in \Lambda.$$

The next proposition states properties characterizing Hilbert bases (total orthogonal families). See Gallier and Quaintance, Vol I, Appendix D.

Proposition 4.1. *Let H be a separable Hilbert space, and let $(u_\ell)_{\ell\in\Lambda}$ be an orthogonal family in H. The following properties are equivalent:*

(1) The family $(u_\ell)_{\ell\in\Lambda}$ is a Hilbert basis (that is, its linear span is dense in H).

(2) For every vector $v \in H$, if $(c_\ell)_{\ell\in\Lambda}$ are the Fourier coefficients of v, then the family $(c_\ell u_\ell)_{\ell\in\Lambda}$ is summable and $v = \sum_{\ell\in\Lambda} c_\ell u_\ell$.

(3) For every vector $v \in H$, we have the Parseval identity:

$$\|v\|^2 = \sum_{\ell\in\Lambda} |c_\ell|^2.$$

(4) For every vector $u \in H$, if $\langle u, u_\ell\rangle = 0$ for all $\ell \in \Lambda$, then $u = 0$.

A Hilbert basis is *not* a basis in the algebraic sense because in general, $\sum_{\ell\in\Lambda} c_\ell u_\ell$ is an infinite sum, and not a finite linear combination.

There is also a version of Proposition 4.1 for Hilbert sums.

Recall that if H is a Hilbert space and if W is a *closed* subspace of E (*closed and convex* will also do), then for any $v \in H$, the *orthogonal projection* $p_W(v)$ *of* v *onto* W is well-defined.

It is the unique vector $w \in W$ such that $v - w$ is orthogonal to W.

Proposition 4.2. *Let H be a separable Hilbert space, and let $(H_\ell)_{\ell \in \Lambda}$ be a family of closed pairwise orthogonal subspaces of H. The following properties are equivalent:*

(1) The family $(H_\ell)_{\ell \in \Lambda}$ is a Hilbert sum (that is, $\bigoplus_{\ell \in \Lambda} H_\ell$ is dense in H).

(2) For every vector $v \in H$, if $(v_\ell)_{\ell \in \Lambda}$ is the family of orthogonal projections of v onto the H_ℓ, then the family $(v_\ell)_{\ell \in \Lambda}$ is summable and $v = \sum_{\ell \in \Lambda} v_\ell$.

(3) For every vector $v \in H$, we have the Parseval identity:

$$\|v\|^2 = \sum_{\ell \in \Lambda} \|v_\ell\|^2 .$$

Example 4.1. The family of functions $(e^{im\theta})_{m\in\mathbb{Z}}$ is a Hilbert basis of $\mathrm{L}^2(\mathbb{T})$.

Later on we will discuss an interesting Hilbert basis of $\mathrm{L}^2(\mathbb{R})$ involving the *Hermite polynomials*.

A Hilbert basis of $\mathrm{L}^2(\mathbf{SO}(3))$ involves the irreducible unitary representations of $\mathbf{SO}(3)$.

More generally, if G is a separable compact group, a Hilbert basis of $\mathrm{L}^2(G)$ involves the irreducible unitary representations of G.

This is what Peter–Weyl I says!

4.2 The Peter–Weyl Theorem, I

The theorem below is the first of several theorems describing the structure of the involutive Banach algebra $L^2(G)$, where G is a metrizable compact group.

By Proposition **??**, the Banach algebra $L^2(G)$ is a complete separable Hilbert algebra, so Theorem **??** is applicable and yields most of a deep theorem first proved by Peter and Weyl (1927); see Theorem 4.3.

The version of both parts of the Peter–Weyl theorem presented here is due to Jean Dieudonné.

They are discussed in Vol V of his treatise on analysis and also rely on results from Vol II of the same book Dieudonné [13, 10] ($\approx$ 1975).

These are the stronger versions that we are aware of.

We also made use of material from Hewitt and Ross (Vol II) and Folland.

Theorem 4.3. *(Peter–Weyl theorem, I) Let G be a metrizable compact group. The complete Hilbert algebra $\mathrm{L}^2(G)$ is the Hilbert sum*

$$\mathrm{L}^2(G) = \bigoplus_{\rho \in R} \mathfrak{a}_\rho$$

of a finite or countably infinite family $(\mathfrak{a}_\rho)_{\rho \in R}$ of Hilbert algebras $\mathfrak{a}_\rho$ of finite dimension n_ρ^2, where each $\mathfrak{a}_\rho$ is a minimal two-sided ideal of $\mathrm{L}^2(G)$ isomorphic to the matrix algebra $\mathrm{M}_{n_\rho}(\mathbb{C})$, and $\mathfrak{a}_h \mathfrak{a}_k = (0)$ for all $h \neq k$ $(h, k \in R)$.

What this means is that each ideal $\mathfrak{a}_\rho$ has an orthogonal basis of functions $\left(m_{ij}^{(\rho)}\right)_{1 \leq i,j \leq n_\rho}$ satisfying certain properties stated in Theorem 4.4 so that the map from $\mathfrak{a}_\rho$ to $\mathrm{M}_{n_\rho}(\mathbb{C})$ given by $\sum_{i,j} \lambda_{ij} m_{ij}^{(\rho)} \mapsto (\lambda_{ij})_{1 \leq i,j \leq n_\rho}$ is an algebra isomorphism.

The elements of $\mathfrak{a}_\rho$ *are classes of continuous functions on* G*; the unit element of* $\mathfrak{a}_\rho$ *is the class of a continuous function* u_ρ *such that* $\overline{\check{u}}_\rho = u_\rho$*, and the orthogonal projection of* $\mathrm{L}^2(G)$ *onto* $\mathfrak{a}_\rho$ *is the map* $f \mapsto f * u_\rho = u_\rho * f$*, for every* $f \in \mathcal{L}^2(G)$*.*

Furthermore, for every $f \in \mathcal{L}^2(G)$*, we have*

$$f = \sum_{\rho \in R} f * u_\rho,$$

where the series on the right-hand side is commutatively convergent.

Our next goal is to get a better understanding of the structure of the algebras $\mathfrak{a}_\rho$ by decomposing them as finite Hilbert sums of minimal left ideals $\mathfrak{l}_j$, and by choosing some Hilbert bases in these ideals.

The following result reveals that some representations are hidden in the Hilbert sum of the $\mathfrak{a}_\rho$.

Theorem 4.4. *Each $\mathfrak{a}_\rho$ contains a basis of functions $(m_{ij}^{(\rho)})_{1\leq i,j\leq n_\rho}$ such that the following properties hold. We will also write m_{ij} instead of $m_{ij}^{(\rho)}$.*

(1) For every j with $1 \leq j \leq n_\rho$, the $(m_{ij})_{1\leq i\leq n_\rho}$ form an orthogonal basis of a minimal left ideal $\mathfrak{l}_j$, and the $(m_{ij})_{1\leq i,j\leq n_\rho}$ form an orthogonal basis of $\mathfrak{a}_\rho = \bigoplus_{j=1}^{n_\rho} \mathfrak{l}_j$.

(2) We have $m_{ji} = \check{\overline{m}}_{ij}$ and $m_{ij} * m_{hk} = \delta_{jh}\, m_{ik}$.

(3) We have $\langle m_{ij}, m_{ij}\rangle = n_\rho$ and $m_{ij}(e) = n_\rho\, \delta_{ij}$, for all i, j with $1 \leq i, j \leq n_\rho$, and $u_\rho = \sum_{j=1}^{n_\rho} m_{jj}$.

The family of functions

$$\left(\frac{1}{\sqrt{n_\rho}} m_{ij}^{(\rho)}\right)_{1\leq i,j\leq n_\rho,\ \rho\in R}$$

is a Hilbert basis of $\mathrm{L}^2(G)$.

(4) *For every* $s \in G$, *if we define the* $n_\rho \times n_\rho$ *matrix* $M_\rho(s)$ *by*

$$M_\rho(s) = \left(\frac{1}{n_\rho} m_{ij}^{(\rho)}(s)\right),$$

then these matrices are invertible and satisfy the equations

$$M_\rho(st) = M_\rho(s)M_\rho(t) \text{ and } M_\rho(s^{-1}) = (M_\rho(s))^*.$$

Thus, the map $s \mapsto M_\rho(s)$ *is a continuous unitary representation in matrix form* $M_\rho \colon G \to \mathbf{U}(n_\rho)$ *of* G *in* $\mathbb{C}^{n_\rho}$, *for the standard hermitian inner product* $\sum_{j=1}^{n_\rho} \alpha_j \overline{\beta}_j$.

We will show later that the representations M_ρ are *irreducible* and that they constitute *all* irreducible unitary representations fo G (up to equivalence).

The (complex) constant functions form a two-sided ideal in $\mathrm{L}^2(G)$, and thus must be an ideal of the form $\mathfrak{a}_{\rho_0}$.

Definition 4.5. The ideal $\mathfrak{a}_{\rho_0}$ is called the *trivial ideal*.

The corresponding representation M_{ρ_0} is one-dimensional, and $M_{\rho_0}(s) = 1$ for all $s \in G$. In other words, M_{ρ_0} is the *trivial representation* of G.

For every $\rho \neq \rho_0$, we have

$$\int_G m_{ij}^{(\rho)}(s)\, d\lambda(s) = 0. \qquad (*_{\rho\neq\rho_0})$$

In Section 6.9 we will need the following result.

Proposition 4.5. *For any unitary* $n_\rho \times n_\rho$ *matrix* P, *for every* $s \in G$, *let* $Q_\rho(s) = P^* M_\rho(s) P$. *The matrices* $Q_\rho(s) = (q_{ij}(s))$ *define* n_ρ^2 *functions* $q_{ij} \in \mathfrak{a}_\rho$ *which are linear combinations of the functions* m_{ij}, *where* $m_{ij}(s) \in M_\rho(s)$, *and satisfy the following properties:*

(1) *The* $(q_{ij})_{1 \leq i,j \leq n_\rho}$ *form an orthogonal basis of* $\mathfrak{a}_\rho$.

(2) *We have* $q_{ji} = \check{\overline{q}}_{ij}$ *and* $q_{ij} * q_{hk} = \delta_{jh}\, q_{ik}$.

(3) *We have* $\langle q_{ij}, q_{ij} \rangle = n_\rho$ *and* $q_{ij}(e) = n_\rho \delta_{ij}$, *for all* i, j *with* $1 \leq i, j \leq n_\rho$,

(4) *The map* $s \mapsto Q_\rho(s)$ *is unitary representation in matrix form* $Q_\rho \colon G \to \mathbf{U}(n_\rho)$ *of* G *in* $\mathbb{C}^{n_\rho}$, *equivalent to the unitary representation* $M_\rho \colon G \to \mathbf{U}(n_\rho)$.

(5) If $\mathfrak{l}_j$ is the minimal left ideal of $\mathfrak{a}_\rho$ spanned by the jth column M_ρ^j of M_ρ,

$$\mathfrak{l}_j = \bigoplus_{i=1}^{n_\rho} \mathbb{C} m_{ij}^{(\rho)},$$

then the jth column of $Q_\rho = P^* M_\rho P$ spans a minimal ideal $\mathfrak{l}_j^Q$ of $\mathfrak{a}_\rho$ (of dimension n_ρ) given by

$$\mathfrak{l}_j^Q = \left\{ \sum_{h=1}^{n_\rho} p_{hj} \left(\sum_{k=1}^{n_\rho} \mu_k m_{kh} \right) \;\middle|\; \mu = (\mu_1, \ldots, \mu_{n_\rho}) \in \mathbb{C}^{n_\rho} \right\},$$

where every $\sum_{k=1}^{n_\rho} \mu_k m_{kh} \in \mathfrak{l}_h$ is a linear combination of the entries of the hth column of M_ρ involving the same scalars $(\mu_1, \ldots, \mu_{n_\rho})$ for all $h = 1, \ldots, n_\rho$.

4.3 Characters of Compact Groups

As in the previous section, we work with metrizable compact groups.

Since we have the Peter–Weyl theorem and Theorem 4.4 at our disposal, it will be fairly easy to prove the properties of characters of these groups.

Definition 4.6. Let G be a (metrizable) compact group. With the notations of Section 4.3, for every $\rho \in R$, define the *character χ_ρ of G associated with the ideal $\mathfrak{a}_\rho$* as the function given by

$$\chi_\rho(s) = \frac{1}{n_\rho} u_\rho(s) = \frac{1}{n_\rho} \sum_{j=1}^{n_\rho} m_{jj}^{(\rho)}(s) = \mathrm{tr}(M_\rho(s)),$$

for all $s \in G$.

The character χ_{ρ_0} associated with $\mathfrak{a}_{\rho_0}$ is the constant function $\chi_{\rho_0}(s) = 1$ for all $s \in G$, called the *trivial character* of G.

Proposition 4.6. *The following properties hold.*

(1) Every character χ_ρ is a continuous central function, which means that

$$\chi_\rho(sts^{-1}) = \chi_\rho(t) \qquad \text{for all } s, t \in G.$$

(2) We have

$$\chi_\rho(s^{-1}) = \overline{\chi_\rho(s)} \qquad \text{for all } s \in G.$$

(3) We have

$$\chi_\rho * \chi_{\rho'} = 0 \quad \text{whenever } \rho \neq \rho', \quad \text{and} \quad \chi_\rho * \chi_\rho = \frac{1}{n_\rho} \chi_\rho.$$

(4) The family of characters $(\chi_\rho)_{\rho \in R}$ forms a Hilbert basis of the center of $\mathrm{L}^2(G)$, which means that:

(a) We have

$$\langle \chi_\rho, \chi_{\rho'} \rangle = \int \chi_\rho(s)\overline{\chi_{\rho'}(s)}\, d\lambda(s) = 0 \ \ \textit{whenever } \rho \neq \rho'$$
$$\langle \chi_\rho, \chi_\rho \rangle = \int |\chi_\rho(s)|^2\, d\lambda(s) = 1.$$

(b) For every function $f \in \mathrm{L}^2(G)$ *we have*

$$f = \sum_{\rho \in R} n_\rho\, (f * \chi_\rho),$$

and for every central function $f \in \mathrm{L}^2(G)$ *we have*

$$f = \sum_{\rho \in R} \langle f, \chi_\rho \rangle \chi_\rho.$$

(c) We have

$$\int \chi_\rho(s)\, d\lambda(s) = 0 \qquad \textit{for all } \rho \neq \rho_0.$$

(d) For all $s \in G$*, we have*

$$\chi_\rho(s) = \mathrm{tr}(M_\rho(s)),$$

and

$$\chi_\rho(e) = n_\rho.$$

Observe that unlike the characters of a locally compact *abelian* group G, which take their values in $\mathbf{U}(1) \cong \mathbf{T}$, the characters χ_ρ of a compact *not necessarily abelian* group G take their values in $\mathbb{C}$.

For instance $\chi_\rho(e) = n_\rho$, and in general, $n_\rho > 1$.

Also, the characters χ_ρ are *not homomorphims*.

In general, $\chi_\rho(st) \neq \chi_\rho(s)\chi_\rho(t)$.

Since $\overline{f * g} = \overline{f} * \overline{g}$, the function which maps the class of a function $f \in \mathcal{L}^2(G)$ to the class of its complex conjugate $\overline{f}$ is a semilinear bijection of $\mathrm{L}^2(G)$, and an automorphism of its ring structure (under convolution).

Definition 4.7. The above automorphism of $\mathrm{L}^2(G)$ maps every ideal $\mathfrak{a}_\rho$ into the minimal two-sided ideal

$$\overline{\mathfrak{a}_\rho} = \{\overline{f} \mid f \in \mathfrak{a}_\rho\}$$

that we denote by $\mathfrak{a}_{\overline{\rho}}$.

The map $\mathfrak{a}_\rho \mapsto \mathfrak{a}_{\overline{\rho}}$ permutes the indices of R but leaves the Hilbert sum unchanged, namely $\mathrm{L}^2(G)$ is the Hilbert sum of both families $(\mathfrak{a}_\rho)_{\rho\in R}$ and $(\mathfrak{a}_{\overline{\rho}})_{\rho\in R} = (\overline{\mathfrak{a}_\rho})_{\rho\in R}$.

If (as usual), given a complex matrix $X = (x_{ij})$, we denote by $\overline{X}$ the matrix $(\overline{x}_{ij})$, then we have

$$\overline{M_\rho(s)} = M_{\overline{\rho}}(s) \qquad \text{for all } s \in G,$$

and as a consequence, since $u_\rho(s) = n_\rho \mathrm{tr}(M_\rho(s)) = n_\rho \chi_\rho(s)$, we have

$$\overline{u_\rho} = u_{\overline{\rho}} \quad \text{and} \quad \overline{\chi_\rho} = \chi_{\overline{\rho}}.$$

Thus the equation $\mathfrak{a}_\rho = \mathfrak{a}_{\overline{\rho}}$ is equivalent to saying that the character χ_ρ only takes *real values*.

4.4 The Peter–Weyl Theorem, II

In this section we prove the second part of the Peter–Weyl theorem which has to do with unitary representations.

In particular, we prove the important result that the unitary representations $s \mapsto M_\rho(s)$ of G discussed in Theorem 4.4 are irreducible (in fact, all of them, up to equivalence).

Given a unitary representation $V\colon G \to \mathbf{U}(H)$, for any fixed $x \in H$, we need to integrate the function

$$s \mapsto f(s)V(s)(x), \quad s \in G,$$

where $f\colon G \to \mathbb{C} \in \mathrm{L}^1(G)$, but the above function is vector-valued with values in the infinite-dimensional Hilbert space H.

However, the theory of integration only applies to functions with values in $\mathbb{C}$.

A way around this difficulty uses the notion of *weak integral*, which relies on the *Riesz representation theorem* of continuous linear forms on a Hilbert space.

If H is a Hilbert space, its *dual* H' is the vector space of *continuous linear forms* $\varphi\colon H \to \mathbb{C}$.

We also define $\overline{H}'$ as the space of *continuous semi-linear forms* $\varphi\colon H \to \mathbb{C}$, which are the continuous functions such that for all $x, y \in H$ and all $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\varphi(x+y) &= \varphi(x) + \varphi(y)\\ \varphi(\lambda x) &= \overline{\lambda}\varphi(x).\end{aligned}$$

Theorem 4.7. *(Riesz representation theorem) Let H be a Hilbert space.*

(1) The mapping $\flat\colon H \to H'$ defined such that for every $x \in H$, the linear form $\flat(x)$ is given by

$$\flat(x)(y) = \langle y, x\rangle, \quad y \in H,$$

is a semi-linear continuous bijection. Thus for every continuous linear form $\varphi \in H'$, there is a unique $x \in H$ such that

$$\varphi(y) = \flat(x)(y) = \langle y, x\rangle, \quad y \in H.$$

(2) The mapping $\flat^l\colon H \to \overline{H}'$ defined such that for every $x \in H$, the semi-linear form $\flat^l(x)$ is given by

$$\flat^l(x)(y) = \langle x, y\rangle, \quad y \in H,$$

is a continuous bijection. Thus for every continuous semi-linear linear form $\varphi \in \overline{H}'$, there is a unique $x \in H$ such that

$$\varphi(y) = \flat^l(x)(y) = \langle x, y\rangle, \quad y \in H.$$

Appying the Riesz representation theorem for every fixed $x \in H$ to the continuous semi-linear map

$$\varphi(y) = \int f(s)\langle V(s)(x), y\rangle \, d\lambda(s), \quad y \in H,$$

the unique vector $\widetilde{V}(f)(x) \in H$ such that

$$\langle \widetilde{V}(f)(x), y\rangle = \int f(s)\langle V(s)(x), y\rangle \, d\lambda(s) \text{ for all } y \in H$$

is called the *weak integral* of the function $s \mapsto V(s)(x)$ from G to H with respect to the measure $f d\lambda$, and we write

$$\widetilde{V}(f)(x) = \int_G f(s)V(s)(x) \, d\lambda(s).$$

We also denote $\widetilde{V}(f)$ by $V_{\mathrm{ext}}(f)$.

Recall fom Definition 4.6 that $u_\rho(s) = n_\rho \chi_\rho(s)$.

Theorem 4.8. *(Peter–Weyl theorem, II) Let G be a metrizable compact group, and let $V\colon G \to \mathbf{U}(H)$ be a unitary representation of G in a separable Hilbert space H.*

(1) For every $\rho \in R$, the map π_ρ^V given by

$$\begin{aligned}\pi_\rho^V(x) = V_{\mathrm{ext}}(\overline{u_\rho})(x) &= \int_G \overline{u_\rho}(s)V(s)(x)\, d\lambda(s) \\ &= n_\rho \int_G \overline{\chi_\rho}(s)V(s)(x)\, d\lambda(s), \;\; x \in H, \quad \text{(proj)}\end{aligned}$$

is an orthogonal projection of H onto a closed subspace E_ρ (which may be reduced to (0)), and H is the Hilbert sum

$$H = \bigoplus_{\rho\in R,\, E_\rho\neq(0)} E_\rho$$

of the $E_\rho \neq (0)$.

(2) *Every subspace* $E_\rho \neq (0)$ *is invariant under* V*, and the restriction* V_ρ *of* V *to* E_ρ *is a finite or countably infinite Hilbert sum of irreducible representations, all equivalent to* M_ρ*, viewed as a representation* $M_\rho \colon G \to \mathbf{U}(\mathbb{C}^{n_\rho})$.

Thus E_ρ *is a finite or countably infinite Hilbert sum of* d_ρ *finite-dimensional subspaces* $E_\rho^{k_\rho}$ *(where* $d_\rho = \infty$ *is possible),*

$$E_\rho = \bigoplus_{k_\rho=1}^{d_\rho} E_\rho^{k_\rho},$$

and each $E_\rho^{k_\rho}$ *is isomorphic to* $\mathbb{C}^{n_\rho}$.

More precisely, each subrepresentation $V_\rho^{k_\rho} \colon G \to \mathbf{U}(E_\rho^{k_\rho})$ *is equivalent to the irreducible representation* $M_\rho \colon G \to \mathbf{U}(\mathbb{C}^{n_\rho})$.

Let us emphasize that Theorem 4.8 proves that *every* representation M_ρ is irreducible, which is not at all obvious from their definition.

Theorem 4.8 also shows that every irreducible unitary representation of G is equivalent to some representation of the form M_ρ, and M_ρ is not equivalent to $M_{\rho'}$ for $\rho \neq \rho'$.

Definition 4.8. Let G be a locally compact (metrizable, separable) group. A sequence of unitary representations $(U_\rho \colon G \to \mathbf{U}(H_\rho))_{\rho \in R}$ of G where R is some index set (possibly infinite) is called a *complete set of irreducible unitary representations of G* if

(1) Each unitary representation $U_\rho \colon G \to \mathbf{U}(H_\rho)$ is irreducible.

(2) Any two representations U_ρ and $U_{\rho'}$ with $\rho \neq \rho'$ are inequivalent.

(3) Every irreducible unitary representation $V : G \to \mathbf{U}(H)$ of G is equivalent to some representation U_ρ (necessarily unique).

Consequently $(M_\rho)_{\rho \in R}$ is a complete set of unitary irreducible representations of G in a separable Hilbert space.

When we deal with more than one group G (say also a closed subgroup of G) we use the notation $R(G)$ instead of R.

If the compact group G is abelian, then every algebra $\mathfrak{a}_\rho$ is abelian, and since it is simple, it must be one-dimensional.

Therefore, every unitary representation of a (metrizable) compact abelian group is a finite or a countably infinite Hilbert sum of *one-dimensional* representations.

It is customary to introduce the following terminology.

Definition 4.9. With the notations of Theorem 4.8, if $V: G \to \mathbf{U}(H)$ is a unitary representation of G in a separable Hilbert space H, and if $H = \bigoplus_{\rho \in R, E_\rho \neq (0)} E_\rho$ is the Hilbert sum induced by the projections $\pi_\rho^V = V_{\mathrm{ext}}(\overline{u_\rho})$, with

$$\pi_\rho^V(x) = n_\rho \int_G \overline{\chi_\rho}(s) V(s)(x)\, d\lambda(s), \quad x \in H$$

whenever $E_\rho \neq (0)$ and $V_\rho: G \to \mathbf{U}(E_\rho)$ is the corresponding representation, we say that the irreducible representation M_ρ is *contained* in the representation V. If

$$E_\rho = \bigoplus_{k_\rho = 1}^{d_\rho} E_\rho^{k_\rho}$$

is finite-dimensional of dimension $d_\rho n_\rho > 0$ (recall that each subspace $E_\rho^{k_\rho}$ is isomorphic to $\mathbb{C}^{n_\rho}$) we say that M_ρ is *contained d_ρ times* in V (or *infinitely many times* if E_ρ is infinite-dimensional).

We also call d_ρ the *multiplicity* of M_ρ in V_ρ. The representations M_ρ such that $d_\rho > 0$ are called the *irreducible components* of the representation V.

If we consider the left regular representation $\mathbf{R}$ of G in $\mathrm{L}^2(G)$, then the projection $\pi_\rho^{\mathbf{R}}$ is given by

$$\begin{aligned}\pi_\rho^{\mathbf{R}}(f) &= \int \overline{u_\rho}(s)\mathbf{R}_s(f)\, d\lambda(s)\\ &= \int \overline{u_\rho}(s)\lambda_s(f)\, d\lambda(s) = \overline{u_\rho} * f = \overline{u_\rho * \overline{f}},\end{aligned}$$

so $E_\rho = \mathfrak{a}_{\overline{\rho}} = \overline{\mathfrak{a}_\rho}$ for all $\rho \in R$, and Theorem 4.8 says that on $\mathfrak{a}_{\overline{\rho}}$, the representation $\mathbf{R}$ splits into n_ρ irreducible representations all equivalent to M_ρ.

We can view these representation as acting on the columns of $M_{\overline{\rho}} = \overline{M_\rho}$, which span n_ρ minimal left ideals $\mathfrak{l}_j^{(\overline{\rho})}$ of $\mathfrak{a}_{\overline{\rho}}$; that is,

$$\mathfrak{a}_{\overline{\rho}} = \bigoplus_{j=1}^{n_\rho} \mathfrak{l}_j^{(\overline{\rho})} \quad \text{and} \quad \mathfrak{l}_j^{(\overline{\rho})} = \bigoplus_{k=1}^{n_{\overline{\rho}}} \mathbb{C} m_{kj}^{(\overline{\rho})}.$$

The above fact is worth recording as a proposition.

Proposition 4.9. *The left regular representation* $\mathbf{R}\colon G \to \mathbf{U}(\mathrm{L}^2(G))$ *of a compact (metrizable) group* G *in* $\mathrm{L}^2(G)$ *contains every irreducible unitary representation* M_ρ *of* G*, and each one is contained* n_ρ *times, where* n_ρ *is the dimension of the space of the representation.*

If V is a *finite-dimensional unitary representation*, then the trace of the linear map $V(s)$ plays a crucial role.

In fact, it determines this representation up to equivalence.

Proposition 4.10. *Let G be a (metrizable) compact group. For any unitary representation $V : G \to \mathbf{U}(H)$ of G in a finite-dimensional hermitian space H of dimension d, assume that for every $\rho \in R$, the irreducible representation M_ρ is contained d_ρ times in V, so that*

$$d = \sum_{\rho \in R} d_\rho n_\rho,$$

where $d_\rho \neq 0$ for only finitely many $\rho \in R$. Then we have

$$\mathrm{tr}(V(s)) = \sum_{\rho \in R} d_\rho \chi_\rho(s), \qquad \text{for all } s \in G.$$

Theorem 4.11. *Let G be a (metrizable) compact group. Two unitary representations $V_1\colon G \to \mathbf{U}(H_2)$ and $V_2\colon G \to \mathbf{U}(H_2)$ of G in finite-dimensional hermitian spaces H_1 and H_2 of dimensions d_1 and d_2 are equivalent if and only if*

$$\mathrm{tr}(V_1(s)) = \mathrm{tr}(V_2(s)) \qquad \text{for all } s \in G.$$

In particular, if V_1 and V_2 are equivalent, then $d_1 = d_2$. Moreover, if V_1 and V_2 are any two equivalent irreducible unitary representations, then

$$\mathrm{tr}(V_1(s)) = \mathrm{tr}(V_2(s)) = \chi_\rho(s), \quad s \in G,$$

where M_ρ is the irreducible representation from Theorem 4.8 to which V_2 and V_2 are equivalent.

Theorem 4.11 suggests the following (standard) definition.

Definition 4.10. Let G be a (metrizable) compact group. For any unitary representation $V\colon G \to \mathbf{U}(H)$ of G in a finite-dimensional hermitian space H of dimension d, we define the *character χ_V of the representation V* as the map $\chi_V\colon G \to \mathbb{C}$ given by

$$\chi_V(s) = \mathrm{tr}(V(s)) \qquad \text{for all } s \in G.$$

The characters of a finite-dimensional unitary representation are central functions, and in view of Proposition 4.10, they have many of the properties of the characters χ_ρ.

By definition, the character χ_ρ of the compact group G is identical to the character χ_{M_ρ} of the special representation M_ρ, which is irreducible by Peter–Weyl II, and by Theorem 4.11, it is also the character of *all* equivalent irreducible unitary representations of G equivalent to M_ρ.

Thus the set $(\chi_\rho)_{\rho \in R}$ is the set of characters of *all* irreducible unitary representations of G.

If we have some complete set of irreducible unitary representations for G, we can determine the characters of G.

If the group G is finite, then there are finitely many irreducible representations up to equivalence, so this method can be used practically.

Example 4.2. Let G be a finite group and assume that $\{\rho_1, \ldots, \rho_r\}$ is a complete set of irreducible unitary representations $\rho_i \colon G \to \mathbf{U}(W_i)$ of G (where r is the number of conjugacy classes of G) so that $R = \{\rho_1, \ldots, \rho_r\}$, write $n_i = \dim(W_i)$, and let $\chi_1, \ldots, \chi_r$ be the characters of G (which are equal to the characters of the ρ_i).

If $U \colon G \to \mathbf{U}(E)$ is any unitary representation of G (where E is finite-dimensional), then by Peter–Weyl II, we have a direct sum

$$E = E_{i_1} \oplus \cdots \oplus E_{i_h} \tag{$\dagger_1$}$$

for some subset $\{\rho_{i_1}, \ldots, \rho_{i_h}\}$ of R ($h \leq r$), and each E_{i_j} ($1 \leq j \leq h$) is a direct sum

$$E_{i_j} = E_{i_j}^1 \oplus \cdots \oplus E_{i_j}^{d_j} \quad (d_j \geq 1) \tag{$\dagger_2$}$$

such that for $k = 1, \ldots, d_j$, each representation $U \colon G \to \mathbf{U}(E_{i_j}^k)$ is equivalent to the irreducible representation $\rho_{i_j} \colon G \to \mathbf{U}(W_{i_j})$.

Each subspace E_{i_j} is the projection of E by the projection $\pi^U_{i_j}$ given by

$$\pi^U_{i_j}(x) = \frac{n_{i_j}}{|G|} \sum_{s \in G} \overline{\chi_{i_j}}(s) U(s)(x) \quad x \in E. \qquad (\dagger_3)$$

The E_{i_j} in $(\dagger_1)$ are uniquely determined by U (in terms of the projections $\pi^U_{i_j}$), but the splitting of E_{i_j} as a direct sum as above in $(\dagger_2)$ is not.

The decomposition of $U \colon G \to \mathbf{U}(E)$ into the h unitary representations $U \colon G \to \mathbf{U}(E_{i_j})$ $(1 \leq j \leq h)$ is called the *canonical decomposition of U*.

For finite groups, these results can be obtained more directly; see Serre [37] (Section 2.6, in particular, Theorem 8).

Each representation $U\colon G \to \mathbf{U}(E_{i_j})$ $(1 \leq j \leq h)$ contains the irreducible representation $\rho_{i_j}\colon G \to \mathbf{U}(W_{i_j})$ d_j times, so it is not irreducible unless $d_j = 1$.

It is actually possible to obtain a specific decomposition of each E_{i_j} into some subspaces $E_{i_j}^k$ as in $(\dagger_2)$ given by projections expressed in terms of matrix representations for the irreducible representations $\rho_{i_j}\colon G \to \mathbf{U}(W_{i_j})$;

See Serre [37] (Section 2.7).

Operations on (finite-dimensional) vector space induce operations on finite-dimensional, not necessarily unitary, representations, which in turn induce operations on their characters.

Given two finite-dimensional representations $U_1 \colon G \to \mathbf{U}(H_2)$ and $U_2 \colon G \to \mathbf{U}(H_2)$, with $d_1 = \dim(H_1)$ and $d_2 = \dim(H_2)$, we already defined their *direct sum* as the representation $U_1 \oplus U_2$ of G in $H_1 \oplus H_2$ given by

$$(U_1 \oplus U_2)(s)(x_1 + x_2) = U_1(s)(x_1) + U_2(s)(x_2), \quad s \in G,\ x_1 \in H_1,\ x_2 \in H_2.$$

The tensor product of representations is also useful because it can be used to characterize the irreducible finite-dimensional representations of products of compact groups.

4.5 Tensor Products of Finite-Dimensional Representations

If H_1 and H_2 are two finite-dimensional vector spaces, following Serre, the tensor product of H_1 and H_2 can be defined in a way that avoids the rather abstract universal mapping property.

Definition 4.11. If H_1 and H_2 are two finite-dimensional (real or complex) vector spaces, a *tensor product* $H_1 \otimes H_2$ of H_1 and H_2 is a (real or complex) vector space together with a map $\iota_\otimes \colon H_1 \times H_1 \to H_1 \otimes H_2$ such that the following two conditions hold:

(1) The map $\iota_\otimes \colon H_1 \times H_1 \to H_1 \otimes H_2$ is bilinear. For any $u \in H_1$ and any $v \in H_2$, we denote $\iota_\otimes(u, v)$ by $u \otimes v$.

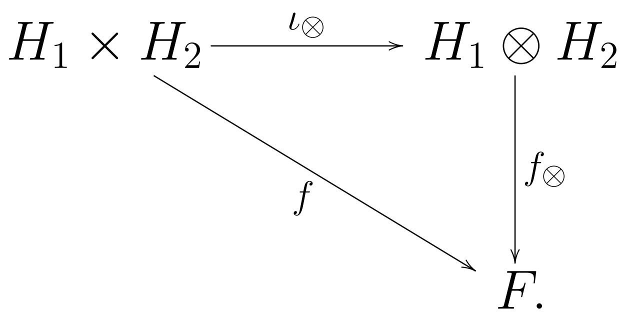
(2) For any basis $(u_1, \ldots, u_m)$ of H_1 and any basis $(v_1, \ldots, v_n)$ of H_2, the $m \times n$ vectors $u_i \otimes v_j$ form a basis of $H_1 \otimes H_2$.

By standard methods of linear algebra it can shown that such a space $H_1 \otimes H_2$ exists and is unique up to isomorphism; for example, see Gallier and Quaintance [23] (Chapter 2).

The tensor product $H_1 \otimes H_2$ has the following *universal mapping property*: for every vector space F and every *bilinear map* $f\colon H_1 \times H_2 \to F$, there is a *unique linear map* $f_\otimes\colon H_1 \otimes H_2 \to F$ such that

$$f = f_\otimes \circ \iota_\otimes,$$

as illustrated in the following diagram:



Given two linear maps $f\colon E \to E'$ and $g\colon F \to F'$, by the universal mapping property there is a unique linear map

$$f \otimes g\colon E \otimes F \to E' \otimes F'$$

such that

$$(f \otimes g)(u \otimes v) = f(u) \otimes g(v), \quad \text{for all } u \in E \text{ and all } v \in F.$$

For proofs of the above facts (in a more general framework) see Gallier and Quaintance [23] (Chapter 2).

In terms of matrices, given a basis $(u_1, \ldots, u_{d_1})$ of H_1 and a basis $(v_1, \ldots, v_{d_2})$ of H_2, assume f_1 is represented by the matrix A_1 and f_2 is represented by the matrix A_2.

Then with respect to the basis $(u_i \otimes v_j)_{1\leq i\leq d_1, 1\leq j\leq d_2}$, the linear map $f_1 \otimes f_2$ is defined by a $(d_1 d_2) \times (d_1 d_2)$ matrix; as a block matrix, it is the $d_1 \times d_1$ matrix of $d_2 \times d_2$ blocks where *the (i, j) block is the matrix* *$(A_1)_{ij} A_2$ $(1 \leq i, j \leq d_1)$*.

This matrix is called the *Kronecker product* of A_1 and A_2.

Given a complex vector space H, recall that $\overline{H}$ is the complex vector space with the same additive operation $+$ but with multiplication by a scalar defined by

$$(\lambda, u) \mapsto \overline{\lambda} u, \quad u \in H, \lambda \in \mathbb{C}.$$

Then a map $f\colon H \to \mathbb{C}$ is *semilinear* iff $f\colon \overline{H} \to \mathbb{C}$ is linear, which means that

$$\begin{aligned} f(u+v) &= f(u) + f(v) \\ f(\lambda u) &= \overline{\lambda} u, \end{aligned}$$

for all $u, v \in H$ and all $\lambda \in \mathbb{C}$.

Observe that a map $\varphi\colon H \times H \to \mathbb{C}$ is *sesquilinear*, which means linear in its first argument and semilinear in its second argument, iff $\varphi\colon H \times \overline{H} \to \mathbb{C}$ is bilinear.

We define a hermitian inner product on the tensor product $H_1 \otimes H_2$ of two finite-dimensional complex vector spaces H_1 and H_2 each equipped with a hermitian inner product $\langle -, - \rangle_i$ ($i = 1, 2$) following Bourbaki [4] (Chapter 9, Section 1.9), which considers a more general situation.

Using the universal mapping property the following definition can be justified.

Definition 4.12. If $(H_1, \langle -, - \rangle_1)$ and $(H_2, \langle -, - \rangle_2)$ are two finite-dimensional complex vector spaces each equipped with a hermitian inner product $\langle -, - \rangle_i$ $(i = 1, 2)$, there is a unique hermitian inner product $\langle -, - \rangle_\otimes \colon (H_1 \otimes H_2) \times (H_1 \otimes H_2) \to \mathbb{C}$ on $H_1 \otimes H_2$ satisfying the equation

$$\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle_\otimes = \langle u_1, u_2 \rangle_1 \langle v_1, v_2 \rangle_2 \qquad (\langle\ \rangle_\otimes)$$

for all $u_1, u_2 \in H_1$ and all $v_1, v_2 \in H_2$.

Observe that if $(u_1, \ldots, u_{d_1})$ is an orthonormal basis of H_1 and $(v_1, \ldots, v_{d_2})$ is an orthonormal basis of H_2, then $(u_i \otimes v_j)_{1 \leq i \leq d_1, 1 \leq j \leq d_2}$ is an orthonormal basis of $H_1 \otimes H_2$ with respect to the inner product $\langle -, - \rangle_\otimes$.

If $f_1\colon H_1 \to H_1$ and $f_2\colon H_2 \to H_2$ are unitary linear maps, then it can be shown that
$f_1 \otimes f_2\colon H_1 \otimes H_2 \to H_1 \otimes H_2$ is unitary for the hermitian inner product $\langle -, - \rangle_\otimes$ on $H_1 \otimes H_2$.

As a consequence of all this we can make the following definition.

Definition 4.13. Given two finite-dimensional unitary representations $U_1 \colon G \to \mathbf{U}(H_1)$ and $U_2 \colon G \to \mathbf{U}(H_2)$ of the locally compact (metrizable, separable) group G, we define the *tensor product* $U_1 \otimes U_2$ of U_1 and U_2 as the unitary representation $U_1 \otimes U_2 \colon G \to \mathbf{U}(H_1 \otimes H_2)$ of G in $H_1 \otimes H_2$ (with the hermitian inner product $\langle -, - \rangle_{\otimes}$ on $H_1 \otimes H_2$ defined in ($\langle\ \rangle_{\otimes}$)) given by

$$(U_1 \otimes U_2)(s) = U_1(s) \otimes U_2(s), \quad s \in G,$$

where $U_1(s) \otimes U_2(s)$ is the tensor product linear map given by

$$(U_1(s) \otimes U_2(s))(x_1 \otimes x_2) = U(s)(x_1) \otimes U(s)(x_2), \quad \text{for all } x_1 \in H_1, x_2 \in H_2.$$

In terms of matrices, the linear map $U_1(s) \otimes U_2(s)$ is defined by a $(d_1 d_2) \times (d_1 d_2)$ matrix, namely the Kronecker product of $U_1(s)$ and $U_2(s)$.

As a block matrix, it is the $d_1 \times d_1$ matrix of $d_2 \times d_2$ blocks where the (i, j) block is the matrix $U_1(s)_{ij} U_2(s)$ $(1 \leq i, j \leq d_1)$.

It is well known that

$$
\begin{aligned}
\mathrm{tr}(U_1(s) \oplus U_2(s)) &= \mathrm{tr}(U_1(s)) + \mathrm{tr}(U_2(s)) \\
\mathrm{tr}(U_1(s) \otimes U_2(s)) &= \mathrm{tr}(U_1(s)) \mathrm{tr}(U_2(s)).
\end{aligned}
$$

Let us now assume that G is compact until Definition 4.14.

If $U_1 = M_{\rho'}$ and $U_2 = M_{\rho''}$ are two irreducible representations of G, then since $\chi_{\rho'}\chi_{\rho''} = \mathrm{tr}(M_{\rho'} \otimes M_{\rho''})$ and $M_{\rho'} \otimes M_{\rho''}$ is finite-dimensional, Proposition 4.10 implies that

$$\chi_{\rho'}\chi_{\rho''} = \sum_{\rho \in R} c^{\rho}_{\rho',\rho''}\chi_{\rho}, \tag{$\otimes$}$$

where $c^{\rho}_{\rho',\rho''} \geq 0$ is an integer, the number of times that the representations M_ρ is contained in $M_{\rho'} \otimes M_{\rho''}$ (this is d_ρ).

The integers $c^{\rho}_{\rho',\rho''}$ are often called *Clebsch–Gordan coefficients*.

The determination of the $c^{\rho}_{\rho',\rho''}$ is usually very difficult.

When $G = \mathbf{SU}(2)$, the irreducible representations can be completely determined and the $c^{\rho}_{\rho',\rho''}$ turn out to be either 1 or 0; see Chapter 5, Section **??**.

They play an important role in physics.

Definition 4.13 is a special case of the notion of the tensor product of finite-dimensional unitary representations of two locally compact groups.

Definition 4.14. Given two finite-dimensional unitary representations $U_1 \colon G_1 \to \mathbf{U}(H_1)$ and $U_2 \colon G_2 \to \mathbf{U}(H_2)$ of the locally compact (metrizable, separable) groups G_1 and G_2, we define the *tensor product* $U_1 \otimes U_2$ of U_1 and U_2 as the unitary representation
$U_1 \otimes U_2 \colon G_1 \times G_2 \to \mathbf{U}(H_1 \otimes H_2)$ of $G_1 \times G_2$ in $H_1 \otimes H_2$ (with the hermitian inner product $\langle -, - \rangle_{\otimes}$ on $H_1 \otimes H_2$ defined in ($\langle\ \rangle_{\otimes}$)) given by

$$(U_1 \otimes U_2)(s_1, s_2) = U_1(s_1) \otimes U_2(s_2), \quad s_1 \in G,\ s_2 \in G_2$$

where $U_1(s_1) \otimes U_2(s_2)$ is the tensor product linear map given by

$$(U_1(s_1) \otimes U_2(s_2))(x_1 \otimes x_2) = U(s_1)(x_1) \otimes U(s_2)(x_2), \\ \text{for all } x_1 \in H_1, x_2 \in H_2.$$

As earlier, in terms of matrices, the linear map $U_1(s_1) \otimes U_2(s_2)$ is defined by a $(d_1 d_2) \times (d_1 d_2)$ matrix, namely the Kronecker product of $U_1(s_1)$ and $U_2(s_2)$.

As a block matrix, it is the $d_1 \times d_1$ matrix of $d_2 \times d_2$ blocks where the (i, j) block is the matrix $U_1(s_1)_{ij} U_2(s_2)$ $(1 \leq i, j \leq d_1)$.

This time, if $U_1 \colon G_1 \to \mathbf{U}(H_1)$ and $U_2 \colon G_2 \to \mathbf{U}(H_2)$ are irreducible, then the representation $U_1 \otimes U_2 \colon G_1 \times G_2 \to \mathbf{U}(H_1 \otimes H_2)$ is also irreducible.

If G_1 and G_2 are compact, this can be easily proven using the characters.

Proposition 4.12. *If G_1 and G_2 are two compact groups and if $U_1: G_1 \to \mathbf{U}(H_1)$ and $U_2: G_2 \to \mathbf{U}(H_2)$ are irreducible unitary representations, then the unitary representation $U_1 \otimes U_2: G_1 \times G_2 \to \mathbf{U}(H_1 \otimes H_2)$ is also irreducible.*

Remark: Observe that if $U_1: G \to \mathbf{U}(H_1)$ and $U_2: G \to \mathbf{U}(H_2)$ are two unitary finite-dimensional representation of G, we actually have two versions of tensor products, namely the first version which is a representation $U_1 \otimes U_2: G \to \mathbf{U}(H_1 \otimes H_2)$ of G, and the second version $U_1 \otimes U_2: G \times G \to \mathbf{U}(H_1 \otimes H_2)$ which is a representation of $G \times G$.

This confusion could be avoided by using a different notation for the two kinds of tensor products, but in most cases it is clear which one is used.

This also explains the apparent contradiction that if U_1 and U_2 are irreducible, then $U_1 \otimes U_2 \colon G \to \mathbf{U}(H_1 \otimes H_2)$ is not necessarily irreducible, while
$U_1 \otimes U_2 \colon G \times G \to \mathbf{U}(H_1 \otimes H_2)$ is irreducible.

Actually the converse of Proposition 4.12 holds.

Theorem 4.13. *Let G_1 and G_2 be two compact groups. The finite-dimensional unitary representations $U_1 \colon G_1 \to \mathbf{U}(H_1)$ and $U_2 \colon G_2 \to \mathbf{U}(H_2)$ are irreducible iff the the finite-dimensional unitary representation $U_1 \otimes U_2 \colon G_1 \times G_2 \to \mathbf{U}(H_1 \otimes H_2)$ is irreducible.*

Observe that the converse of Proposition 4.12 actually holds for any locally compact groups.

In fact, Theorem 4.13 holds for any locally compact (metrizable) groups.

This stronger version of Theorem 4.13 is proven in Folland [19] (Chapter 7, Theorem 7.20).

Folland also defines tensor products of Hilbert spaces and proves a version of Theorem 4.13 for unitary representations in Hilbert spaces.

If G_1 and G_2 are compact, then we have another very useful result.

Proposition 4.14. *If G_1 and G_2 are compact, then every finite-dimensional irreducible unitary representation $U\colon G_1 \times G_2 \to \mathbf{U}(H)$ is equivalent to the tensor product $U_1 \otimes U_2$ of two finite-dimensional irreducible unitary representations $U_1\colon G_1 \to \mathbf{U}(H_1)$ and $U_2\colon G_2 \to \mathbf{U}(H_2)$.*

Proposition 4.14 is proven in Bröcker and tom Dieck [5] (Chapter 2, Section 4, Proposition 4.14) and Folland [19] (Chapter 7, Theorem 7.25).

Example 4.3. Theorem 4.13 and Proposition 4.14 can be used to determine the irreducible representations of $\mathbf{O}(2m+1)$ in terms of the irreducible representations of $\mathbf{SO}(2m+1)$.

This is because if $Q \in \mathbf{O}(2m+1)$ and $\det(Q) = -1$, since $\det(-I_{2m+1}) = (-1)^{2m+1} = -1$, then $Q(-I) \in \mathbf{SO}(2m+1)$, and so *the direct product* $\mathbf{SO}(2m+1) \times \{I_{2m+1}, -I_{2m+1}\}$ *is isomorphic to* $\mathbf{O}(2m+1)$ under the isomorphism

$$(Q, X) \mapsto QX, Q \in \mathbf{SO}(2m+1), X \in \{I_{2m+1}, -I_{2m+1}\}.$$

The reason why the above map is a homomorphism is that Q and $-I_{2m+1}$ commute for all $Q \in \mathbf{SO}(2m+1)$.

It follows that the irreducible representations of $\mathbf{O}(2m+1)$ are the tensor product representations of irreducible representations of $\mathbf{SO}(2m+1)$ and irreducible representations of the finite abelian group $\{I_{2m+1}, -I_{2m+1}\} \simeq \mathbb{Z}/2\mathbb{Z}$, which are determined by their group of characters.

These are the trivial character ρ_0 given by $\rho_0(I_{2m+1}) = \rho_0(-I_{2m+1}) = 1$ and the character ρ_1 given by $\rho_1(I_{2m+1}) = 1$ and $\rho_1(-I_{2m+1}) = -1$.

Observe that ρ_1 is the determinant map.

When $m = 1$, the irreducible representations of $\mathbf{SO}(3)$ can be described in terms of harmonic polynomials (see Section 5.2, Proposition 5.3), so we have a complete description of the irreducible representation of $\mathbf{O}(3)$.

The irreducible representations of $\mathbf{O}(3)$ are of the form $\mathbf{R}_n \otimes \rho_k$, with $k \in \{0, 1\}$ and $n \in \mathbb{N}$, or more explicitly

$$(\mathbf{R}_n \otimes \rho_k)(Q, X) = \rho_k(X)\mathbf{R}_n(Q),$$
$$Q \in \mathbf{SO}(3), X \in \{I_{2m+1}, -I_{2m+1}\},\ k \in \{0, 1\},\ n \in \mathbb{N}.$$

The case of $\mathbf{O}(2m)$ is more delicate.

The problem is that $-I_{2m}$ is no longer a reflection since $\det(-I_{2m}) = (-1)^{2m} = +1$.

We need to use a hyperplane reflection, such as the $(2m) \times (2m)$-matrix $J = \mathrm{diag}(-1, 1, \ldots, 1)$. If $Q \in \mathbf{O}(2m)$, then $QJ \in \mathbf{SO}(2m)$.

However, J does *not* commute with all matrices in $\mathbf{SO}(2m)$, so this time we have an isomorphism between the *semi-direct* product $\mathbf{SO}(2m) \rtimes \{I_{2m}, J\}$ and $\mathbf{O}(2m)$; see Section **??** (note that $J^2 = I_{2m}$).

Unfortunately, the normal subgroup $\mathbf{SO}(2m)$ of $\mathbf{O}(2m)$ is *not* abelian for $m > 1$, which complicates matters.

For $m = 1$, the group $\mathbf{SO}(2)$ is abelian, so Mackey's little group method can be used to determine the irreducible representations of $\mathbf{O}(2)$; see Section **??**.

The irreducible representations of $\mathbf{U}(2)$ can also be determined using the following trick.

The trick is that $\mathbf{U}(2)$ is isomorphic to the quotient $(\mathbf{U}(1) \times \mathbf{SU}(2))/(\{(1, I_2), -(1, I_2)\})$.

But $\mathbf{U}(1) \simeq \mathbb{T}$ is a locally compact abelian group and its irreducible representations are determined by its characters χ_m, which are given by

$$e^{i\theta} \mapsto e^{im\theta},\ m \in \mathbb{Z},\ ,\theta \in \mathbb{R}/2\pi.$$

The irreducible representations of $\mathbf{SU}(2)$ are determined in Chapter 5; in particular, we have the irreducible representations U_n, with $n \in \mathbb{N}$; see Section 5.1.

The fact that we need to mod out by the subgroup $\{(1, I_2), -(1, I_2)\}$ implies that the irreducible representations of $\mathbf{U}(2)$ are of the form $\chi_m \otimes U_n$, with $m+n$ even. More explicitly,

$$(\chi_m \otimes U_n)(e^{i\theta}T) = e^{im\theta}U_n(T),\ \theta \in \mathbb{R}/2\pi,\ T \in \mathbf{SU}(2),$$

with $m \in \mathbb{Z}, n \in \mathbb{N}$, and $m+n$ even.

Details can be found in Bröcker and tom Dieck [5] (Chapter 2, Section 5, Page 87) and Folland [19] (Chapter 5, Section 5.4).

4.6 Contragedient Representations and Hom Representations

Later on in Chapter 7 we will need to consider the Hom representation defined by two representations $U_1\colon G \to \mathbf{U}(H_1)$ and $U_2\colon G \to \mathbf{U}(H_2)$.

In order to promote the isomorphism between the tensor product $E_1^* \otimes E_2$ and the space of linear maps $\mathrm{Hom}(E_1, E_2)$ (where E_1 and E_2 are finite-dimensional) to representations, given a representation $U\colon G \to \mathbf{U}(H)$ we first need to define a representation $\overline{U}\colon G \to \mathbf{U}(H^*)$ defined on the dual of H^*, namely the space of linear forms on H.

We begin by reviewing the duality between a finite-dimensional hermitian vector space E and its dual E^*; for a complete exposition see Gallier and Quaintance [24] (Chapter 13, Section 2).

For any $u \in E$, define the linear form $\varphi_u \in E^*$ by

$$\varphi_u(v) = \langle v, u \rangle, \quad v \in E.$$

Then the map $\flat$ from E to E^* given by $\flat(u) = \varphi_u$ is a semi-linear isomorphism (semi-linear means that $\flat(\lambda u) = \overline{\lambda}\, \flat(u)$, for $\lambda \in \mathbb{C}$).

Definition 4.15. Given a finite-dimensional hermitian space E, we give E^* the hermitian inner product induced by $\flat^{-1} \colon E^* \to E$, namely

$$\langle \varphi_1, \varphi_2 \rangle_{E^*} = \overline{\langle \flat^{-1}(\varphi_1), \flat^{-1}(\varphi_2) \rangle}, \quad \varphi_1, \varphi_2 \in E^*.$$

Definition 4.15 is a special case of a definition given in Bourbaki [4] (Chapter 9, Section 1.7, Definition 9) which considers a more general situation.

As observed in Bourbaki, without conjugation we obtain a left-sesquilinear form.

The conjugation on the right-hand side is necessary to make $\langle -, - \rangle_{E^*}$ a right-sesquilinear form, which means that it is linear in the first argument and semi-linear in the second argument since $\flat^{-1}$ is only semi-linear.

Observe that for all $u, v \in E$ we have

$$\langle \flat(u), \flat(v) \rangle_{E^*} = \overline{\langle \flat^{-1}(\flat(u)), \flat^{-1}(\flat(v)) \rangle} = \overline{\langle u, v \rangle}.$$

Also observe that if $f \colon E \to E$ is a linear map, then

$$(\varphi_u \circ f)(v) = \varphi_u(f(v)) = \langle f(v), u \rangle = \langle v, f^*(u) \rangle$$

where f^* is the adjoint of f, which shows that

$$\varphi_u \circ f = \varphi_{f^*(u)}. \qquad (\dagger_4)$$

If $(u_1, \ldots, u_n)$ is an orthonormal basis of E, then the definition of the hermitian inner product on E^* immediately implies that $(\varphi_{u_1}, \ldots, \varphi_{u_n})$ is an orthonormal basis of E^*. Also, we have

$$\varphi_{u_i}(u_j) = \langle u_j, u_i \rangle = \delta_{i\,j},$$

which shows that φ_{u_i} is the ith coordinate function over the basis $(u_1, \ldots, u_n)$.

Definition 4.16. Given any complex representation $U \colon G \to \mathbf{GL}(H)$ in a finite-dimensional vector space H, the *contragedient representation $\overline{U} \colon G \to \mathbf{GL}(H^*)$ of $U \colon G \to \mathbf{GL}(H)$* is given by

$$\overline{U}_g(\psi) = \psi \circ U_{g^{-1}}, \quad \psi \in H^*,\ g \in G.$$

Observe that $\overline{U}_g = (U_{g^{-1}})^\top$, the transpose of the linear map $U_{g^{-1}}$.

If H a finite-dimensional vector space with a hermitian inner product and U is a unitary representation $U\colon G \to \mathbf{U}(H)$, in terms of matrices, since U_g is a unitary matrix, $\overline{U}_g$ is the conjugate of the matrix U_g.

We need to check that $\overline{U}_g$ is a unitary map on H^*. This can be done by a tedious computation.

Therefore $\overline{U}\colon G \to \mathbf{U}(H^*)$ is indeed a unitary representation.

Remark: If H is a Hilbert space (of infinite dimension), then the dual of H is the space H' of continuous linear forms on H, and by the Riesz representation Theorem, $\flat\colon H \to H'$ is a bijection, so the above calculations go through and Definition 4.16 yields a unitary representation $\overline{U}\colon G \to \mathbf{U}(H')$.

We now review the relationship between $E^* \otimes F$ and $\mathrm{Hom}(E, F)$.

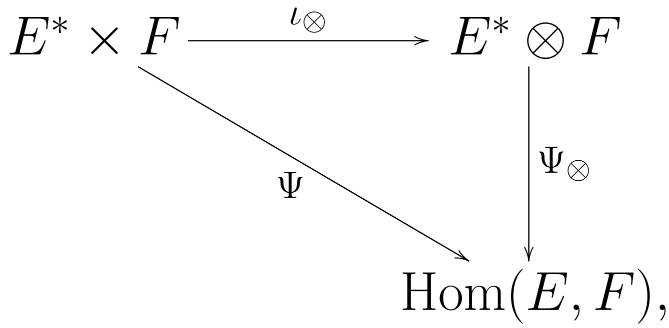
For a complete exposition see Gallier and Quaintance [23] (Chapter 2, Section 2.5).

Let E and F be two vector spaces and let $\Psi\colon E^* \times F \to \mathrm{Hom}(E, F)$ be the map defined such that

$$\Psi(u^*, f)(x) = u^*(x)f,$$

for all $u^* \in E^*$, $f \in F$, and $x \in E$.

This map is clearly bilinear, and thus it induces a linear map $\Psi_{\otimes}\colon E^* \otimes F \to \mathrm{Hom}(E, F)$ making the following diagram commute



such that

$$\Psi_{\otimes}(u^* \otimes f)(x) = u^*(x)f. \qquad (\dagger_5)$$

Then Proposition 2.7 in Gallier and Quaintance [23] tells us that

(1) The linear map $\Psi_\otimes\colon E^* \otimes F \to \mathrm{Hom}(E, F)$ is injective.

(2) If E or if F is finite-dimensional, then $\Psi_\otimes\colon E^* \otimes F \to \mathrm{Hom}(E, F)$ is an isomorphism.

If E and F are finite-dimensional and if each of them has a hermitian inner product, the isomorphism $\Psi_\otimes$ can be made more concrete by picking bases.

If $(u_1, \ldots, u_n)$ is an orthonormal basis of E and $(v_1, \ldots, v_m)$ is an orthonormal basis of F, then $(\varphi_{u_1}, \ldots, \varphi_{n_n})$ is an orthonormal basis of E^* (with the inner product on E^* induced by the inner product on E of Definition 4.15)).

Then the $m \times n$ tensors $\varphi_{u_j} \otimes v_i$ form a basis of $E^* \otimes F$, so any tensor $T \in E^* \otimes F$ can be expressed in terms of an $m \times n$ matrix $A = (a_{ij})$ $(a_{ij} \in \mathbb{C})$ as

$$T = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, \varphi_{u_j} \otimes v_i.$$

If we denote the linear map $\Psi_{\otimes}(\varphi_{u_j} \otimes v_i)$ from E to F as $\varphi_{u_j} v_i$, (this is the linear map such that $(\varphi_{u_j} v_i)(x) = \varphi_{u_j}(x) v_i$ for all $x \in E$), then the linear map $\Psi_{\otimes}(T)$ is expressed as

$$\Psi_{\otimes}(T) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, \varphi_{u_j} v_i.$$

Because $\Psi_{\otimes}$ is an isomorphism, the linear maps $\varphi_{u_j} v_i$ are linearly independent and form a basis of $\mathrm{Hom}(E, F)$.

The matrix representing the linear map $\Psi_{\otimes}(T)$ with respect to the bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_m)$ has for its j-column the coordinates of the vector

$$\begin{aligned}
\Psi_{\otimes}(T)(u_j) &= \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik}\, \varphi_{u_k}(u_j) v_i \\
&= \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik}\, \delta_{k\,j} v_i = \sum_{i=1}^{m} a_{ij} v_i,
\end{aligned}$$

and so it is also the matrix A.

We can use the isomorphism $\Psi_{\otimes}$ to transfer the hermitian inner product on $E^* \otimes F$ (see Definition 4.12 and Definition 4.15) to a hermitian inner product on $\mathrm{Hom}(E, F)$ so that $\Psi_{\otimes}$ becomes unitary.

Definition 4.17. Given two finite-dimensional hermitian spaces E and F, the inner product $\langle -, - \rangle_{\mathrm{Hom}}$ on $\mathrm{Hom}(E, F)$ is given by

$$\begin{aligned} \langle h_1, h_2 \rangle_{\mathrm{Hom}} = \langle \Psi_{\otimes}^{-1}(h_1), \Psi_{\otimes}^{-1}(h_2) \rangle_{E^* \otimes F}, \\ h_1, h_2 \in \mathrm{Hom}(E, F). \end{aligned}$$

Observe that

$$\begin{aligned}
&\langle \Psi_{\otimes}(u_1^* \otimes f_1), \Psi_{\otimes}(u_2^* \otimes f_2)\rangle_{\mathrm{Hom}} \\
&= \langle \Psi_{\otimes}^{-1}(\Psi_{\otimes}(u_1^* \otimes f_1)), \Psi_{\otimes}^{-1}(\Psi_{\otimes}(u_2^* \otimes f_2))\rangle_{E^* \otimes F} \\
&= \langle u_1^* \otimes f_1, u_2^* \otimes f_2\rangle_{E^* \otimes F} = \langle u_1^*, u_2^*\rangle_{E^*}\langle f_1, f_2\rangle_F,
\end{aligned}$$

so the inner product $\langle -, -\rangle_{\mathrm{Hom}}$ on $\mathrm{Hom}(E, F)$ is the inner product that makes the linear map $\Psi_{\otimes} \colon E^* \otimes F \to \mathrm{Hom}(E, F)$ an isometry.

In terms of the orthonormal bases, the tensors $\varphi_{u_j} \otimes v_i$ form an orthonormal basis of $E^* \otimes F$, so the hermitian inner product on $\mathrm{Hom}(E, F)$ is the one that makes the basis $(\varphi_{u_j} v_i)_{1 \leq i \leq m, 1 \leq j \leq n}$ orthonormal in $\mathrm{Hom}(E, F)$.

Then if $h_1 \colon E \to F$ and $h_2 \colon E \to F$ are two linear maps given by the matrices A and B with respect to the bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_m)$, a simple computation shows that the inner product of h_1 and h_2 is given by

$$\langle h_1, h_2 \rangle_{\mathrm{Hom}} = \mathrm{tr}(B^* A) = \mathrm{tr}(A^* B),$$

the Frobenius inner product of complex matrices!

We now define the Hom-*representation*, first for arbitrary vector spaces not necessarily equipped with an inner product.

Definition 4.18. Let $U_1\colon G \to \mathbf{GL}(H_1)$ and $U_2\colon G \to \mathbf{GL}(H_2)$ be two representations. The representation $\mathrm{Hom}(U_1, U_2)\colon G \to \mathbf{GL}(\mathrm{Hom}(H_1, H_2))$ is given by

$$[\mathrm{Hom}(U_1, U_2)(g)](f) = U_2(g) \circ f \circ U_1(g^{-1}), \qquad f \in \mathrm{Hom}(H_1, H_2),\ g \in G.$$

Working through the definitions, we prove the following result.

Proposition 4.15. *If* $U_1 \colon G \to \mathbf{GL}(H_1)$ *and* $U_2 \colon G \to \mathbf{GL}(H_2)$ *are two finite-dimensional representations, then the linear map* $\Psi_\otimes \colon H_1^* \otimes H_2 \to \mathrm{Hom}(H_1, H_2)$ *is an equivalence between the representations* $\overline{U_1} \otimes U_2$ *and* $\mathrm{Hom}(U_1, U_2)$*; that is, the diagram*

$$\begin{CD} H_1^* \otimes H_2 @>{(\overline{U_1} \otimes U_2)(g)}>> H_1^* \otimes H_2 \\ @V{\Psi_\otimes}VV @VV{\Psi_\otimes}V \\ \mathrm{Hom}(H_1, H_2) @>>{\mathrm{Hom}(U_1, U_2)(g)}> \mathrm{Hom}(H_1, H_2) \end{CD}$$

commutes for all $g \in G$.

If H_1 and H_2 are finite-dimensional and each one has a hermitian inner product so that $U_1: G \to \mathbf{U}(H_1)$ and $U_2: G \to \mathbf{U}(H_2)$ are unitary representations, then the representation of Definition 4.18 becomes a unitary representation $\mathrm{Hom}(U_1, U_2): G \to \mathbf{U}(\mathrm{Hom}(H_1, H_2))$ for the hermitian inner product on $\mathrm{Hom}(H_1, H_2)$ given in Definition 4.17 making $\Psi_{\otimes}$ unitary.

4.7 The Fourier Transform for Compact Groups

Definition 4.19. Let G be a compact group. For any function $f \in \mathrm{L}^1(G)$, the *Fourier transform* $\mathcal{F}(f)$ of f is the map with domain R given by

$$\mathcal{F}(f)(\rho) = \int f(t) M_\rho(t^{-1})\, d\lambda(t) = \int f(t)(M_\rho(t))^*\, d\lambda(t),$$

for all $\rho \in R$.

Remark: We can view $\mathcal{F}(f)(\rho)$ as being defined as a weak integral, or since we are in the finite-dimensional case, as the result of integrating term by term the matrix $f(t)(M_\rho(t))^*$.

Observe that $\mathcal{F}(f)(\rho) \in \mathrm{M}_{n_\rho}(\mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{n_\rho}, \mathbb{C}^{n_\rho})$.

The Fourier transform of Definition 4.19 is the natural generalization of the definition of the Fourier transform when G is an abelian compact group (Vol I, Definition @@@10.3),

$$\mathcal{F}(f)(\chi) = \int f(s)\overline{\chi(s)}\, d\lambda(s) = \int f(s)\chi(s^{-1})\, d\lambda(s);$$

the character $\chi\colon G \to \mathbf{U}(1)$ (a continuous homomorphism) is replaced by the irreducible representation M_ρ.

Recall that for a locally compact abelian group G, the characters form a locally compact abelian group $\widehat{G}$ called the *Pontrjagin dual* of G.

If G is not abelian, the characters are replaced by representations, but they *no longer form a group*.

It is also not clear how to define a "good" topology on representations.

If $f \in \mathrm{L}^2(G)$, then we have the following version of *Fourier inversion*.

For any $f \in \mathrm{L}^2(G)$,

$$f(s) = \sum_{\rho \in R} n_\rho \operatorname{tr}\Big(\mathcal{F}(f)(\rho) M_\rho(s)\Big) s \in G. \tag{FI}$$

The Fourier transform $\mathcal{F}(f)$ is a function with domain R, the set of "equivalence classes" of irreducible representations of G, which plays the analog of $\widehat{G}$, to the space $\prod_{\rho \in R} \mathrm{M}_{n_\rho}(\mathbb{C})$, where $\mathrm{M}_{n_\rho}(\mathbb{C})$ is the algebra of $n_\rho \times n_\rho$ complex matrices.

Every element $F \in \prod_{\rho \in R} \mathrm{M}_{n_\rho}(\mathbb{C})$ is an R-indexed sequence $F = (F(\rho))_{\rho \in R}$ of $n_\rho \times n_\rho$ matrices $F(\rho)$.

Sequences in $\prod_{\rho\in R} \mathrm{M}_{n_\rho}(\mathbb{C})$ are added, multiplied, and multiplied by a scalar, componentwise.

Thus $\prod_{\rho\in R} \mathrm{M}_{n_\rho}(\mathbb{C})$ is a (complex) algebra.

Given $F \in \prod_{\rho\in R} \mathrm{M}_{n_\rho}(\mathbb{C})$, the adjoint F^* of F is defined componentwise by $F^* = (F_\rho^*)_{\rho\in R}$.

Definition 4.20. We define $\widehat{G}$ as $R(G)$ the set of indices of a complete set of unitary irreducible representations of G (see the comment just after Theorem 4.8).

Note the analogy to the situation where $G = \mathbb{T}$ and $\widehat{G} = \widehat{\mathbb{T}} = \mathbb{Z}$, except that, $\mathrm{L}^1(\widehat{\mathbf{T}}) = l^1(\mathbb{Z})$ consists of $\mathbb{Z}$-indexed sequences of complex numbers, but the $F(\rho)$ are matrices.

By analogy with the case $G = \mathbb{T}$ and $\widehat{\mathbb{T}} = \mathbb{Z}$, where the numbers $\mathcal{F}(f)(m) = \widehat{f}(m)$ are the Fourier coefficients of $f \in \mathrm{L}^1(\mathbb{T})$, the endomorphisms $\mathcal{F}(f)(\rho) \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{n_\rho}, \mathbb{C}^{n_\rho})$, represented by matrices in $\mathrm{M}_{n_\rho}(\mathbb{C})$, can be viewed as *generalized Fourier coefficients* of $f \in \mathrm{L}^1(G)$, where G is a compact group.

The equation (FI) is a kind of Fourier inversion formula.

Explicit examples of the Fourier transform and of the Fourier inversion formula (FI) will be given in Section 5.15 for the groups $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$.

We can define the *Fourier cotransform* $\overline{\mathcal{F}}$, defined on $\prod_{\rho\in R}\mathrm{M}_{n_\rho}(\mathbb{C})$, with input in G, by

$$\overline{\mathcal{F}}(F)(s) = \sum_{\rho\in R} n_\rho \operatorname{tr}(F(\rho)M_\rho(s)),\\ F \in \prod_{\rho\in R}\mathrm{M}_{n_\rho}(\mathbb{C}),\ s \in G. \tag{FC}$$

Of course, there is an issue of convergence with (FC).

The space $\prod_{\rho\in R}\mathrm{M}_{n_\rho}(\mathbb{C})$ is just too big, so following Hewitt and Ross [27] (Chapter VII, Section 28.24), we define normed subspaces $\mathrm{L}^p(\widehat{G})$ as follows.

First we need to define some norms on $n \times n$ matrices introduced by von Neumann.

4.8 von Neumann Norms and the Algebras $\mathrm{L}^p(\widehat{G})$

Definition 4.21. Let $A \in \mathrm{M}_n(\mathbb{C})$ be any complex $n \times n$ matrix, and let $(\sigma_1, \ldots, \sigma_n)$ be the sequence of nonnegative square roots of the eigenvalues of A^*A listed in any order (the positive square roots are the *singular values* of A).

For any p, $1 \leq p < \infty$, define the *von Neumann norm* $\|A\|_{\varphi_p}$ of A by

$$\|A\|_{\varphi_p} = \left(\sum_{k=1}^{n} \sigma_k^p \right)^{1/p},$$

and $\|A\|_{\varphi_\infty}$ by

$$\|A\|_{\varphi_\infty} = \max_{1 \leq k \leq n} \sigma_k.$$

It is not obvious that the functions defined in Definition 4.21 are matrix norms, but this is proven in Hewitt and Ross, see [27] (Appendix D, Theorem D40).

Since $(\sigma_1^2, \ldots, \sigma_n^2)$ are the eigenvalues of A^*A, we see that

$$\|A\|_{\varphi_2}^2 = \sum_{k=1}^{n} \sigma_k^2 = \mathrm{tr}(A^*A) = \|A\|_{\mathrm{HS}}^2 ,$$

where $\|A\|_{\mathrm{HS}}$ is a *Hilbert–Schmidt norm*, also known as *Frobenius norm*, of A (see Definition Vol I, @@@B.6).

We also have

$$\|A\|_{\varphi_1} = \sum_{k=1}^{n} \sigma_k,$$

and

$$\|A\|_{\varphi_\infty} = \max_{1\le k\le n} \sigma_k = \|A\|_2 ,$$

where $\|A\|_2$ is the *operator norm* induced by the 2-norm; see Vol I, Definition @@@B.7 and Proposition @@@B.8.

Next we use the norms of Definition 4.21 to define norms on $\prod_{\rho\in R}\mathrm{M}_{n_\rho}(\mathbb{C})$.

Definition 4.22. For any fixed sequence $(a_\rho)_{\rho\in R}$ of reals $a_\rho \geq 1$, for any $F \in \prod_{\rho\in R}\mathrm{M}_{n_\rho}(\mathbb{C})$, if $1 \leq p < \infty$, define $\|F\|_p$ by

$$\|F\|_p = \left(\sum_{\rho\in R} a_\rho \|F(\rho)\|_{\varphi_p}^p\right)^{1/p},$$

and for $p = \infty$, let

$$\|F\|_\infty = \sup_{\rho\in R}\|F(\rho)\|_{\varphi_\infty},$$

where $\|F(\rho)\|_{\varphi_p}$ is the von Neumann p-norm of the matrix $F(\rho)$.

Observe that for $p = 2$, we have

$$\|F\|_2 = \left(\sum_{\rho \in R} a_\rho \, \|F(\rho)\|_{\mathrm{HS}}^2 \right)^{1/2} = \left(\sum_{\rho \in R} a_\rho \, \mathrm{tr}\Big(F(\rho)^* F(\rho) \Big) \right)^{1/2}.$$

Following Hewitt and Ross [27] (Chapter VII, Section 28.24), we make the following definitions.

Definition 4.23. Denote $\prod_{\rho\in R}\mathrm{M}_{n_\rho}(\mathbb{C})$ by $\mathfrak{E}(\widehat{G})$. Pick a fixed sequence $(a_\rho)_{\rho\in R}$ of reals $a_\rho \geq 1$.

Let $\mathfrak{E}(\widehat{G})_{0,0}$ be the subspace of $\mathfrak{E}(\widehat{G})$ consisting of all sequences $F = (F(\rho))_{\rho\in R}$ such that the set $\{\rho \in R \mid F(\rho) \neq 0\}$ is finite, and let $\mathfrak{E}(\widehat{G})_0$ be the subspace of $\mathfrak{E}(\widehat{G})$ consisting of all sequences $F = (F(\rho))_{\rho\in R}$ such that the set $\{\rho \in R \mid \|F(\rho)\|_{\varphi_\infty} \geq \epsilon\}$ is finite for all $\epsilon > 0$.

For any p with $1 \leq p \leq \infty$, we define $\mathrm{L}^p(R) = \mathrm{L}^p(\widehat{G})$ as

$$\begin{aligned}\mathrm{L}^p(\widehat{G}) &= \left\{F \in \prod_{\rho\in R}\mathrm{M}_{n_\rho}(\mathbb{C}) \mid \|F\|_p < \infty\right\}\\ &= \left\{F \in \mathfrak{E}(\widehat{G}) \mid \|F\|_p < \infty\right\}.\end{aligned}$$

The following results are shown in Hewitt and Ross [27] (Theorem 28.25 and Theorem 28.26).

Proposition 4.16. *Let G be a compact group. For any fixed sequence $(a_\rho)_{\rho\in R}$ of reals $a_\rho \geq 1$, for any p such that $1 \leq p \leq \infty$, the space $\mathrm{L}^p(\widehat{G})$ is a Banach space. For any $F \in \mathrm{L}^p(\widehat{G})$, we have $F^* \in \mathrm{L}^p(\widehat{G})$ and $\|F^*\|_p = \|F\|_p$.*

The space $\mathrm{L}^\infty(\widehat{G})$ is a Banach algebra under componentwise multiplication, and $\|FF^*\|_\infty = \|F\|_\infty^2$ for any $F \in \mathrm{L}^\infty(\widehat{G})$.

The following result is also shown in Hewitt and Ross [27] (Theorem 28.27).

Proposition 4.17. *Let G be a compact group, and let $(a_\rho)_{\rho\in R}$ be any fixed sequence of reals $a_\rho \geq 1$. With the norm $\| \, \|_\infty$, the space $\mathfrak{E}(\widehat{G})_0$ is a closed two-sided ideal of $\mathrm{L}^\infty(\widehat{G})$. For any p such that $1 \leq p < \infty$, the space $\mathfrak{E}(\widehat{G})_{0,0}$ is a dense two-sided ideal of $\mathfrak{E}(\widehat{G})_0$, and a dense two-sided ideal of $\mathrm{L}^p(\widehat{G})$. Both $\mathfrak{E}(\widehat{G})_{0,0}$ and $\mathfrak{E}(\widehat{G})_0$ are closed under adjunction ($F \mapsto F^*$).*

It is also possible to define an inner product on $\mathrm{L}^p(\widehat{G})$ based on the following proposition shown in Hewitt and Ross [27] (Lemma 28.28).

Proposition 4.18. *Let G be a compact group, and let $(a_\rho)_{\rho\in R}$ be any fixed sequence of reals $a_\rho \geq 1$. For any p, $1 \leq p \leq \infty$, if q is defined such that $\frac{1}{p}+\frac{1}{q}=1$, then for all $E \in \mathrm{L}^p(\widehat{G})$ and all $F \in \mathrm{L}^q(\widehat{G})$, the following facts hold:*

(1) The number

$$\langle E, F\rangle = \sum_{\rho\in R} a_\rho \mathrm{tr}(F_\rho^* E_\rho)$$

is well defined (the series converges absolutely).

(2) We have

$$\langle F, E\rangle = \overline{\langle E, F\rangle}.$$

(3) (Hölder's inequality)

$$|\langle E, F\rangle| \leq \|E\|_p \, \|F\|_q \, .$$

Then we have the following theorem shown in Hewitt and Ross [27] (Theorem 28.30).

Theorem 4.19. *Let G be a compact group, and let $(a_\rho)_{\rho\in R}$ be any fixed sequence of reals $a_\rho \geq 1$. The space $\mathrm{L}^2(\widehat{G})$ is a Hilbert space with the inner product*

$$\langle E, F\rangle = \sum_{\rho\in R} a_\rho \mathrm{tr}(F_\rho^* E_\rho),$$

and we have

$$\|E\|_2^2 = \langle E, E\rangle.$$

We also have the following result shown in Hewitt and Ross [27] (Theorem 28.32).

Proposition 4.20. *Let G be a compact group, and let $(a_\rho)_{\rho\in R}$ be any fixed sequence of reals $a_\rho \geq 1$.*

(1) For any p such that $1 \leq p \leq \infty$, if q is such that $\frac{1}{p}+\frac{1}{q}=1$, for any $E \in \mathrm{L}^p(\widehat{G})$ and $F \in \mathrm{L}^q(\widehat{G})$, we have $EF \in \mathrm{L}^1(\widehat{G})$, and

$$\|EF\|_1 \leq \|E\|_p \|F\|_q .$$

(2) For any p, q such that $1 \leq p < q \leq \infty$, we have

$$\mathrm{L}^p(\widehat{G}) \subseteq \mathrm{L}^q(\widehat{G})$$

and for every $E \in \mathrm{L}^p(\widehat{G})$,

$$\|E\|_q \leq \|E\|_p .$$

(3) For any p such that $1 \leq p \leq \infty$, for all $E, F \in \mathrm{L}^p(\widehat{G})$, we have $EF \in \mathrm{L}^p(\widehat{G})$, and

$$\|EF\|_p \leq \|E\|_p \|F\|_p .$$

We now have the following results about the Fourier transform on a compact group, generalizing similar results about the Fourier transform on $\mathbb{T}$.

From now on, we assume that the sequence $(a_\rho)_{\rho\in R}$ *of reals* $a_\rho \geq 1$ *is the sequence of positive integers* $(n_\rho)_{\rho\in R}$.

Theorem 4.21. *Let* G *be a compact group.*

(1) If we define the multiplication on $\mathrm{L}^\infty(\widehat{G})$ *as* $(F_1 \cdot F_2)(\rho) = F_2(\rho)F_1(\rho)$, *then the map* $f \mapsto \mathcal{F}(f)$ *is a non norm-increasing injective involutive algebra homomorphism from* $\mathrm{L}^1(G)$ *into* $\mathrm{L}^\infty(\widehat{G})$. *In particular, for all* $f, g \in \mathrm{L}^1(G)$, *for all* $\rho \in R$, *we have*

$$(\mathcal{F}(f * g))(\rho) = \mathcal{F}(g)(\rho) \circ \mathcal{F}(f)(\rho).$$

(2) For every $\rho \in R$, *the map* $f \mapsto \mathcal{F}(f)(\rho)$ *is an algebra representation of* $\mathrm{L}^1(G)$ *in* $\mathbb{C}^{n_\rho}$.

Remark: Notice that the order of f and g is switched on the right-hand side.

This is the reason why, if we want $\mathcal{F}$ to be a homomorphism, that we have to switch the order of the arguments in the multiplication on $\mathrm{L}^\infty(\widehat{G})$.

Theorem 4.22. *Let G be a compact group. The map $f \mapsto \mathcal{F}(f)$ is a non norm-increasing involutive isomorphism of $\mathrm{L}^1(G)$ onto a dense subalgebra of $\mathfrak{C}_0(\widehat{G}) \subseteq \mathrm{L}^\infty(\widehat{G})$. In particular, the map $f \mapsto \mathcal{F}(f)$ is continuous.*

Theorem 4.22 (is a version of the Riemann–Lebesgue lemma for compact groups.

Indeed, since $\widehat{G} = R$ is discrete, by definition of $\mathfrak{C}_0(\widehat{G})$, we can view the vectors in $\mathfrak{C}_0(\widehat{G})$ as functions of $\rho \in R$ that tend to zero at infinity.

See Vol I, Proposition @@@10.18 in the case of abelian locally compact groups.

Theorem 4.23. *(Plancherel) Let G be a compact group. The map $f \mapsto \mathcal{F}(f)$ is an isometric isomorphism between the Hilbert space $\mathrm{L}^2(G)$ and the Hilbert space $\mathrm{L}^2(\widehat{G})$. In particular, the map $f \mapsto \mathcal{F}(f)$ is continuous. If we pick any orthonormal basis $(e_1^\rho, \ldots, e_{n_\rho}^\rho)$ in $\mathbb{C}^{n_\rho}$, then for every $f \in \mathrm{L}^2(G)$, we have*

$$f = \sum_{\rho \in R} n_\rho \sum_{j,k=1}^{n_\rho} \langle (\mathcal{F}(f)(\rho))(e_k^\rho), e_j^\rho \rangle u_{jk}^\rho,$$

where u_{jk}^ρ is the function on G given by

$$u_{jk}^\rho(s) = \langle M_\rho(s)(e_k^\rho), e_j^\rho \rangle, \qquad s \in G,\ 1 \leq j, k \leq n_\rho.$$

The functions u_{jk}^ρ are called the *coordinate functions for M_ρ and the basis $(e_1^\rho, \ldots, e_{n_\rho}^\rho)$*.

Theorem 4.23 is proven in Hewitt and Ross [27] (Theorem 28.43).

We now return to the Fourier cotransform.

4.9 Fourier Inversion for Compact Groups

Definition 4.24. Let G be a compact group. For any $F \in \mathfrak{E}(\widehat{G}) = \prod_{\rho \in R} \mathrm{M}_{n_\rho}(\mathbb{C})$, the *Fourier cotransform* $\overline{\mathcal{F}}(F)$ of F is the function on G given by

$$\overline{\mathcal{F}}(F)(s) = \sum_{\rho \in R} n_\rho \operatorname{tr}(F(\rho) M_\rho(s)), \qquad s \in G.$$

In the above definition the infinite sum should be viewed as a formal expression. We will give below sufficient conditions that guarantee convergence.

Following Hewitt and Ross, it is natural to make the following definition (see [27], Definition 34.4).

Definition 4.25. Let G be a compact group. The subspace $\mathfrak{R}(G)$ of $\mathrm{L}^1(G)$ is defined by

$$\begin{aligned}\mathfrak{R}(G) &= \{f \in \mathrm{L}^1(G) \mid \|\mathcal{F}(f)\|_1 < \infty\} \\ &= \left\{ f \in \mathrm{L}^1(G) \;\middle|\; \sum_{\rho \in R} n_\rho \left\|\mathcal{F}(f)(\rho)\right\|_{\varphi_1} < \infty \right\}.\end{aligned}$$

The subspace $\mathfrak{R}(G)$ is called the space of *absolutely convergent Fourier series*. We define $\|f\|_{\varphi_1}$ by

$$\|f\|_{\varphi_1} = \|\mathcal{F}(f)\|_1 .$$

For any function $f \in \mathrm{L}^1(G)$, the formal expression

$$(\overline{\mathcal{F}}(\mathcal{F}(f)))(s) = \sum_{\rho \in R} n_\rho \operatorname{tr}(\mathcal{F}(f)(\rho) M_\rho(s))$$

is called the *Fourier series of f*.

Observe that Definition 4.25 is the generalization of the case $G = \mathbb{T}$ and $\widehat{\mathbb{T}} = \mathbb{Z}$ where for every $f \in \mathrm{L}^1(\mathbb{T})$ we define the Fourier series of f as the map

$$\theta \mapsto \sum_{m \in \mathbb{Z}} \widehat{f}(m) e^{im\theta} = (\overline{\mathcal{F}}(\widehat{f}))(\theta).$$

Here the character $\theta \mapsto e^{im\theta}$ is replaced by the irreducible representation M_ρ, and the trace function is needed to convert the matrix $\mathcal{F}(f)(\rho)M_\rho(s)$ to a number (and the dimensions n_ρ must be accounted for).

The following results are shown in Hewitt and Ross [27] (Theorem 34.5).

Theorem 4.24. *Let G be a compact group.*

(1) If $F \in \mathrm{L}^1(\widehat{G})$, then the map

$$s \mapsto \sum_{\rho \in R} n_\rho \left|\mathrm{tr}(F(\rho)M_\rho(s))\right|$$

is uniformly convergent.

(2) The map

$$s \mapsto (\overline{\mathcal{F}}(F))(s) = \sum_{\rho \in R} n_\rho \, \mathrm{tr}(F(\rho)M_\rho(s))$$

converges uniformly to a continuous function f. Furthermore, we have the Fourier inversion formula

$$(\overline{\mathcal{F}}(\mathcal{F}(f)))(s) = \sum_{\rho \in R} n_\rho \, \mathrm{tr}(\mathcal{F}(f)(\rho)M_\rho(s)), \qquad s \in G,$$

where $(\overline{\mathcal{F}}(\mathcal{F}(f)))(s)$ is the Fourier series of f, so $f \in \mathfrak{R}(G)$.

(3) We have

$$\|f\|_\infty \leq \|f\|_{\varphi_1} = \|\mathcal{F}(f)\|_1\,,$$

where $\|f\|_\infty$ is the sup norm on $\mathcal{C}(G;\mathbb{C})$.

The Fourier series of f is not necessarily convergent, but we have the following results; see Hewitt and Ross [27] (Corollary 34.6 and Corollary 34.7).

Theorem 4.25. *Let G be a compact group.*

(1) For any function $f \in \mathfrak{R}(G)$, the Fourier series of f converges uniformly and

$$f = (\overline{\mathcal{F}}(\mathcal{F}(f)))(s) = \sum_{\rho \in R} n_\rho \operatorname{tr}(\mathcal{F}(f)(\rho) M_\rho(s))$$

for almost all $s \in G$. We have

$$\|f\|_\infty \leq \|f\|_{\varphi_1}.$$

(2) The map $f \mapsto \mathcal{F}(f)$ is a norm-preserving linear isomorphism from $\mathfrak{R}(G)$ onto $\mathrm{L}^1(\widehat{G})$, so $\mathfrak{R}(G)$ is a Banach space.

For the record, in view of Theorem 4.23 and (FI), we have the following result (see also Hewitt and Ross [27], Chapter IX, Section 34.47(a)).

Theorem 4.26. *(Fourier inversion for* $\mathrm{L}^2(G)$*) Let* G *be a compact group. The Fourier cotransform* $\overline{\mathcal{F}}(F) \in \mathrm{L}^2(G)$ *of any* $F \in \mathrm{L}^2(\widehat{G})$ *converges as a series in the* L^2*-norm, and for every* $f \in \mathrm{L}^2(G)$*, we have*

$$f(s) = (\overline{\mathcal{F}}(\mathcal{F}(f)))(s) = \sum_{\rho \in R} n_\rho \operatorname{tr}\left(\mathcal{F}(f)(\rho) M_\rho(s)\right), s \in G$$

where the series converges in the L^2*-norm.*

Example 4.4. If G is a finite group, then $\widehat{G} = \{\rho_1, \ldots, \rho_r\}$, where r is the number of conjugacy classes of G.

If we give G the counting measure normalized so that G has measure 1, then the Fourier transform of any function $f \in \mathrm{L}^2(G)$ is given by

$$\mathcal{F}(f)(\rho) = \frac{1}{|G|} \sum_{s \in G} f(s)(M_\rho(s))^*,$$

where $M_{\rho_1}, \ldots, M_{\rho_r}$ are the irreducible representations of G (up to equivalence).

For every $F \in \mathrm{L}^2(\widehat{G})$, the Fourier cotransform of F is given by

$$\overline{\mathcal{F}}(F)(s) = \sum_{k=1}^{r} n_{\rho_k} \mathrm{tr}(F(\rho_k) M_{\rho_k}(s)), \qquad s \in G,$$

and the Fourier inversion formula is given by

$$f = \sum_{k=1}^{r} n_{\rho_k} \mathrm{tr}((\mathcal{F}(f))(\rho_k) M_{\rho_k}(s)).$$

The fact that $\mathcal{F}$ is an isometry is expressed by the equation

$$\begin{aligned}\langle f_1, f_2\rangle &= \frac{1}{|G|}\sum_{s\in G} f_1(s)\overline{f_2(s)}\\ &= \langle \mathcal{F}(f_1), \mathcal{F}(f_2)\rangle\\ &= \sum_{k=1}^{r} n_{\rho_k}\mathrm{tr}((\mathcal{F}(f_2))^*\mathcal{F}(f_1)),\end{aligned}$$

for all $f_1, f_2 \in \mathrm{L}^2(G)$.

For all $f, g \in \mathrm{L}^2(G)$, the convolution $f * g$ is given by

$$(f * g)(s) = \frac{1}{|G|} \sum_{s_1 s_2 = s} f(s_1) g(s_2) = \frac{1}{|G|} \sum_{t \in G} f(t) g(t^{-1} s),$$

and we can write explicitly the equation

$$(\mathcal{F}(f * g))(\rho) = (\mathcal{F}(g))(\rho) \circ (\mathcal{F}(f))(\rho).$$

We leave it to the diligence of the reader to check that it holds.