Chapter 3

Fourier Analysis on a Locally Compact Abelian Group

3.1 Review of Fourier Analysis on $\mathbb{T}$

Definition 3.1. The *circle group* $\mathbb{T} = \mathbf{U}(1)$ is the group $\{z \in \mathbb{C} \mid |z| = 1\}$ of complex numbers of unit length under multiplication.

We give $\mathbb{T}$ the subspace topology induced by $\mathbb{C}$.

The circle group $\mathbb{T}$ is abelian (commutative). Geometrically, this is the *unit circle* $S = S^1$.

The map $\sigma\colon \mathbb{R} \to \mathbb{T}$ given by

$$\sigma(\theta) = e^{i\theta}$$

is clearly a surjective group homomorphism (with $\mathbb{R}$ under addition, and $\mathbb{T}$ under multiplication); see Figure 3.1.

Since $e^{i\theta} = 1$ iff $\theta = k2\pi$ with $k \in \mathbb{Z}$, we see that the kernel of σ is $2\pi\mathbb{Z}$, so by the first isomorphism theorem the *additive group $\mathbb{R}/(2\pi\mathbb{Z})$ is isomorphic to the multiplicative group $\mathbb{T}$*.

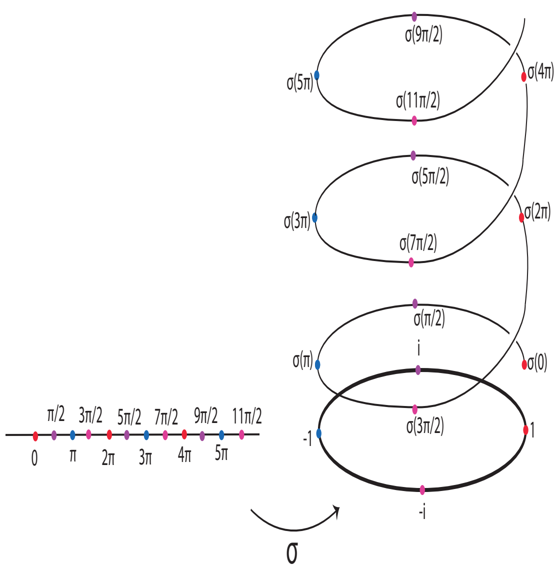

Figure 3.1: The map $\sigma\colon \mathbb{R} \to \mathbb{T}$ which "wraps" the line around the unit circle.

This isomorphism allows to view a complex number of unit length as $e^{i\theta}$, with θ defined modulo 2π, which is often more convenient than picking a representative of the equivalence class of $\theta \pmod{2\pi}$ in $[-\pi, \pi)$.

There is a bijection between the set of functions on $\mathbb{T}$ and the set of *periodic functions* on $\mathbb{C}$ of period 2π.

It is usuallly more convenient to deal with periodic functions. In this case we write $f(\theta)$ for $f(e^{i\theta})$.

The vector space $\mathrm{L}^1(\mathbb{T})$ is the space of functions $f\colon \mathbb{T} \to \mathbb{C}$ such that the integral

$$\int_{-\pi}^{\pi} f(\theta)\,\frac{d\theta}{2\pi}$$

is defined.

Remark *that can be ignored if you don't know any measure theory*.

We are using the σ-algebra generated by the Borel sets on $\mathbb{T}$ and the positive measure on it induced by the Lebesgue measure on $\mathbb{R}$. This measure is left and right invariant.

We define the *norm* $\|\ \|_{\mathrm{L}^1(\mathbb{T})}$ by

$$\|f\|_{\mathrm{L}^1(\mathbb{T})} = \int_{-\pi}^{\pi} |f(\theta)|\,\frac{d\theta}{2\pi}.$$

The space $\mathrm{L}^1(\mathbb{T})$ is a *Banach space* (a complete normed vector space).

The vector space $\mathrm{L}^2(\mathbb{T})$ is the space of measurable functions $f\colon \mathbb{T} \to \mathbb{C}$ such that the function $t \mapsto |f(t)|^2$ belongs to $\mathrm{L}^1(\mathbb{T})$; that is, the integral

$$\int_{-\pi}^{\pi} |f(\theta)|^2 \, \frac{d\theta}{2\pi}$$

is defined.

We define an *inner product* $\langle -, - \rangle$ on $\mathrm{L}^2(\mathbb{T})$ by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)} \, \frac{d\theta}{2\pi}, \quad f, g \in \mathrm{L}^2(\mathbb{T}),$$

and the *norm* $\| \, \|_{\mathrm{L}^2(\mathbb{T})}$ by

$$\|f\|^2_{\mathrm{L}^2(\mathbb{T})} = \langle f, f \rangle = \int_{-\pi}^{\pi} |f(\theta)|^2 \, \frac{d\theta}{2\pi}.$$

Then $\mathrm{L}^2(\mathbb{T})$ with its norm $\|f\|_{\mathrm{L}^2(\mathbb{T})}$ is a *Banach space*, and with its inner product it is a *Hilbert space*.

Because $\mathbb{T}$ is *compact*, we have

$$\|f\|_{\mathrm{L}^1(\mathbb{T})} \leq \|f\|_{\mathrm{L}^2(\mathbb{T})}$$

and so

$$\mathrm{L}^2(\mathbb{T}) \subseteq \mathrm{L}^1(\mathbb{T}).$$

In general, this inclusion is *false* if the space is not compact.

Example 3.1. The periodic function $f\colon \mathbb{T} \to \mathbb{C}$ given by

$$f(\theta) = \begin{cases} \frac{1}{\sqrt{|\theta|}} & -\pi \leq \theta < 0,\ \ 0 < \theta \leq \pi, \\ 0 & \theta = 0 \end{cases}$$

belongs to $\mathrm{L}^1(\mathbb{T})$ but does not belong to $\mathrm{L}^2(\mathbb{T})$ because $|f|^2$ does not belong to $\mathrm{L}^1(\mathbb{T})$.

Recall that for any $p \geq 1$, the space $\ell^p(\mathbb{Z})$ is the set of sequences $x = (x_n)_{n\in\mathbb{Z}}$ with $x_n \in \mathbb{C}$ such that $\sum_{n\in\mathbb{Z}} |x_n|^p < \infty$.

Also, if $1 \leq p < q$, then

$$\ell^p(\mathbb{Z}) \subseteq \ell^q(\mathbb{Z}).$$

In particular,

$$\ell^1(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}),$$

but

$$\mathrm{L}^2(\mathbb{T}) \subseteq \mathrm{L}^1(\mathbb{T}).$$

Each space $\ell^p(\mathbb{Z})$ $(p \geq 1)$ is a normed vector space with the norm

$$\|(x_m)_{m\in\mathbb{Z}}\| = \left(\sum_{m\in\mathbb{Z}} |x_m|^p\right)^{1/p}.$$

The space $\ell^p(\mathbb{Z})$ $(p \geq 1)$ is a *Banach space* (it is complete).

For $p = 2$, the space $\ell^2(\mathbb{Z})$ is a *Hilbert space* with the inner product

$$\langle (x_m)_{m\in\mathbb{Z}}, (y_m)_{m\in\mathbb{Z}} \rangle = \sum_{m\in\mathbb{Z}} x_m \overline{y_m}$$

and norm

$$\|(x_m)_{m\in\mathbb{Z}}\| = \left(\sum_{m\in\mathbb{Z}} |x_m|^2\right)^{1/2}.$$

Definition 3.2. The *convolution* $f * g$ of two functions $f, g \in \mathrm{L}^1(\mathbb{T})$ is given by

$$(f * g)(\theta) = \int_{\mathbb{T}} f(\theta - \varphi) g(\varphi) \, \frac{d\varphi}{2\pi} = \int_{\mathbb{T}} f(\varphi) g(\theta - \varphi) \, \frac{d\varphi}{2\pi}.$$

It can be shown that $f * g \in \mathrm{L}^1(\mathbb{T})$.

Because $\mathrm{L}^2(\mathbb{T}) \subseteq \mathrm{L}^1(\mathbb{T})$, it can be shown that if $f, g \in \mathrm{L}^2(\mathbb{T})$, then $f * g \in \mathrm{L}^2(\mathbb{T})$.

This is generally *false* if the space is not compact!

If $f \in \mathrm{L}^2(\mathbb{T})$, then the function f^* given by

$$f^*(t) = \overline{f(-t)}$$

is also in $\mathrm{L}^2(\mathbb{T})$. We obtain an involution $f \mapsto f^*$.

With convolution as multiplication this makes $\mathrm{L}^2(\mathbb{T})$ an *involutive Banach algebra*.

Definition 3.3. For any function $f \in \mathrm{L}^1(\mathbb{T})$, the *Fourier coefficients* c_m (or $\widehat{f}_m$) are defined by

$$c_m = \int_{-\pi}^{\pi} f(t) e^{-imt} \frac{dt}{2\pi}, \quad m \in \mathbb{Z}.$$

Theorem 3.1. *(Spectral Synthesis) Let $f \in \mathrm{L}^2(\mathbb{T})$. Then*

$$\lim_{n \mapsto \infty} \left\| f - \sum_{m=-n}^{m=n} c_m e^{im\theta} \right\|_2 = 0.$$

and we have the Parseval theorem:

$$\|f\|^2_{\mathrm{L}^2(\mathbb{T})} = \sum_{m=-\infty}^{m=\infty} |c_m|^2.$$

The above implies that $c = (c_m)_{m \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$.

Beware that if $f \in \mathrm{L}^2(\mathbb{T})$, then it is *generally false* that $c = (c_m)_{m\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ and thus the series $\sum_{m=-\infty}^{m=\infty} c_m e^{im\theta}$ may *not converge pointwise or uniformly*.

The expression

$$\sum_{m=-\infty}^{m=\infty} c_m e^{im\theta}$$

is called the *Fourier series of f*.

The *convergence of this series is an issue!*

If $f \in \mathrm{L}^2(\mathbb{T})$, the Fourier series *does not always converge pointwise*. However is does converge to f in the $\mathrm{L}^2(\mathbb{T})$-norm.

Lennart Carleson showed in 1966 that for any function $f \in \mathrm{L}^2(\mathbb{T})$, the partial sums of the Fourier series of f converge pointwise almost everywhere to f, putting a close to a problem that had been open for fifty years.

Remark: If $f \in \mathrm{L}^1(\mathbb{T})$ *and* if $c = (c_m)_{m \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$, then the Fourier series converges uniformly and

$$f(\theta) = \sum_{m=-\infty}^{m=\infty} c_m e^{im\theta}$$

almost everywhere, and if f is continuous, then they are equal.

3.2 Fourier Inversion on $\mathbb{T}$

Definition 3.4. Given any function $f \in L^1(\mathbb{T})$, the function $\mathcal{F}(f)\colon \mathbb{Z} \to \mathbb{C}$ given by $\mathcal{F}(f)(m) = c_m$, where c_m is the *Fourier coefficient*

$$c_m = \int_{-\pi}^{\pi} f(t) e^{-imt} \frac{dt}{2\pi},$$

is called the *Fourier transform* of f.

We identify the sequence $\mathcal{F}(f)$ with the sequence $(c_m)_{m\in\mathbb{Z}}$, which is also denoted by $\widehat{f}$.

Theorem 3.2. *(Plancherel) The map $\mathcal{F}\colon f \mapsto \widehat{f}$ is an isometric isomorphism of the Hilbert spaces $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$.*

In particular, $\mathcal{F}$ is continuous.

Definition 3.5. Given a sequence $c = (c_m)_{m\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$, we define the *Fourier cotransform* $\overline{\mathcal{F}}(c)$ of c as the function $\overline{\mathcal{F}}(c)\colon \mathbb{T} \to \mathbb{C}$ defined on $\mathbb{T}$ given by

$$\overline{\mathcal{F}}(c)(\theta) = \sum_{m=-\infty}^{m=\infty} c_m e^{im\theta} = \sum_{m=-\infty}^{m=\infty} c_m (e^{i\theta})^m,$$

the *Fourier series* associated with c (with $\theta \in \mathbb{R}/2\pi\mathbb{Z}$).

Given a function $f \in \mathrm{L}^1(\mathbb{T})$, if $\widehat{f}$ is the Fourier transform of f, then the Fourier cotransform $\overline{\mathcal{F}}(\widehat{f}) = \sum_{m=-\infty}^{m=\infty} \widehat{f}_m e^{im\theta}$ of $\widehat{f}$ is called the *the Fourier series* of f.

Note that $e^{im\theta}$ is used instead of the term $e^{-im\theta}$ occurring in the Fourier transform.

Plancherel's theorem shows that the maps
$\mathcal{F}\colon \mathrm{L}^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ and $\overline{\mathcal{F}}\colon \ell^2(\mathbb{Z}) \to \mathrm{L}^2(\mathbb{T})$ are *mutual inverses*.

Example 3.2. Let $f\colon [-\pi,\pi] \to \mathbb{R}$ be the periodic function given by

$$f(\theta) = \theta, \qquad -\pi < \theta \leq \pi.$$

The graph of $f(\theta)$ is shown in Figure 3.2.

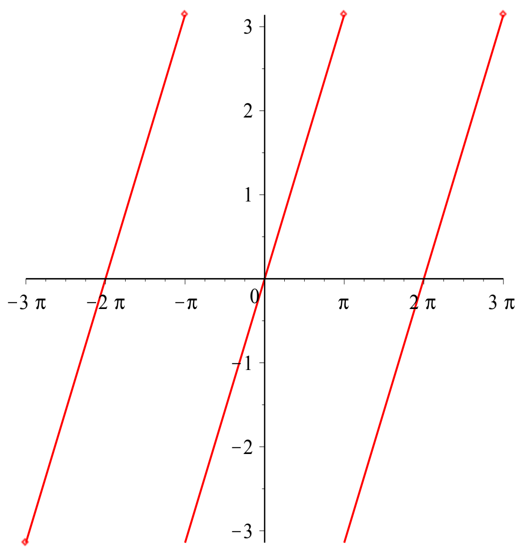

Figure 3.2: The graph of the periodic function $f(\theta) = \theta$, where $-\pi < \theta \leq \pi$.

The Fourier coefficients c_m are given by

$$c_0 = 0,$$

and for $m \neq 0$, by integrating by parts, we have

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta e^{-im\theta}\, d\theta = \frac{(-1)^{m+1}}{im}.$$

Hence the Fourier series for f is

$$\sum_{m\neq 0} \frac{(-1)^{m+1}}{im} e^{im\theta}.$$

We obtain the real Fourier series

$$2\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sin m\theta. \qquad (*)$$

The series belongs to $L^2(\mathbb{T})$ but it does not converge to f pointwise or uniformly.

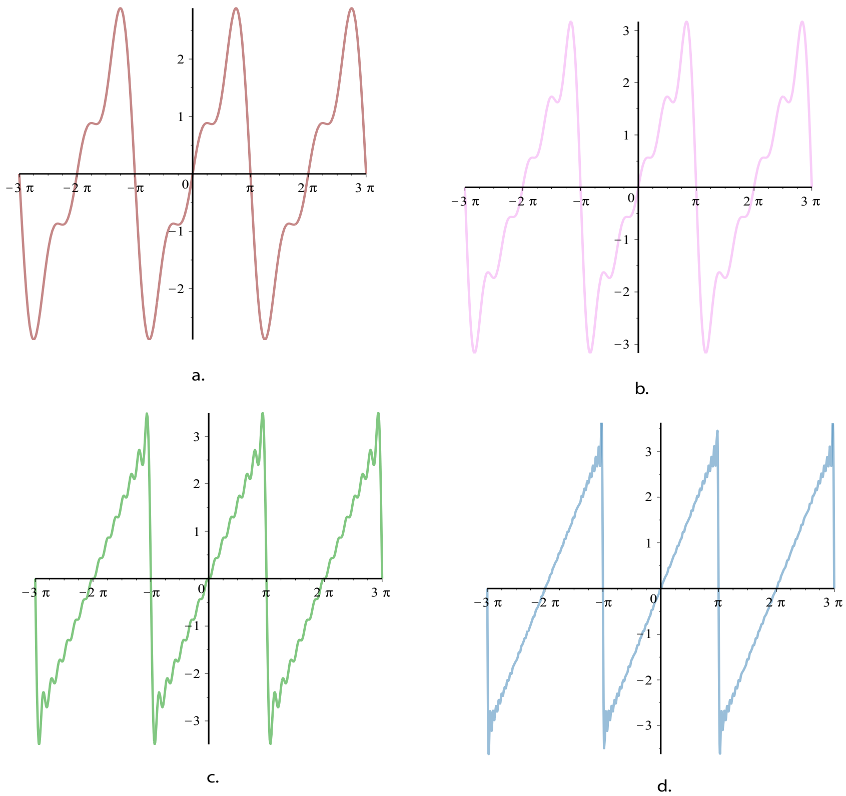

Figure 3.3: Let $S_M = 2\sum_{m=1}^{M} \frac{(-1)^{m+1}}{m} \sin m\theta$. Figure (a) is the graph of S_3; Figure (b) is the graph of S_5; Figure (c) is the graph of S_{14}, and Figure (d) is the graph of S_{40}.

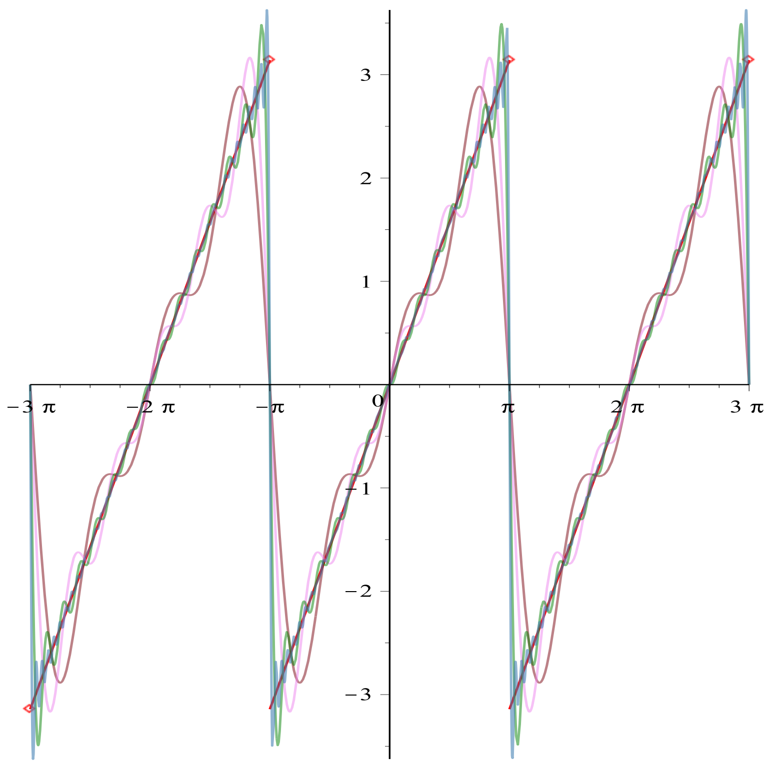

Figure 3.4: The partial sums S_3,S_5,S_{14},S_{40} approximating $f(\theta)$ of Example 3.2.

In fact, it is not obvious that the series

$$2\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\sin m\theta$$

converges pointwise. It does, with

$$2\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\sin m\theta=\begin{cases}\theta & \text{if } -\pi<\theta<\pi\\ 0 & \text{if } \theta=\pm\pi.\end{cases}$$

This series converges pointwise to the function f of Example 3.2, except for $\theta=(2k+1)\pi$ where $f((2k+1)\pi-)=\pi$ and $f((2k+1)\pi+)=-\pi$, according to a theorem of Dirichlet (see Section **??**).

As a consequence of the previous results the following facts hold:

(1) The Fourier coefficients

$$c_m = \int_{-\pi}^{\pi} f(t) e^{-imt} \frac{dt}{2\pi}, \quad m \in \mathbb{Z}.$$

are given by the inner products

$$c_m = \langle f, e^{im\theta} \rangle_{\mathrm{L}^2(\mathbb{T})}.$$

(2) The functions $e^{im\theta}$ form an orthonormal family:

$$\langle e^{im\theta}, e^{in\theta} \rangle_{\mathrm{L}^2(\mathbb{T})} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

(3) For every function $f \in \mathrm{L}^2(\mathbb{T})$, its Fourier series $\sum_{m=-\infty}^{m=\infty} c_m e^{im\theta}$ converges to f in the $\mathrm{L}^2(\mathbb{T})$-norm.

So

$$f = \sum_{m=-\infty}^{m=\infty} c_m e^{im\theta}$$

but the series *may not converge pointwise to f*.

The familly of functions $(e^{im\theta})_{m\in\mathbb{Z}}$ is kind of infinite orthonormal basis for $\mathrm{L}^2(\mathbb{T})$.

It is *not* a basis in the sense of linear algebra because a linear combination only has a *finite number* of nonzero coefficients.

But in general *infinitely many Fourier coefficients c_m are nonzero*.

However the linear subspace of $\mathrm{L}^2(\mathbb{T})$ (the set of (finite!) linear combinations of functions of the form $e^{im\theta}$) is dense in $\mathrm{L}^2(\mathbb{T})$.

Such a family of functions is called a *Hilbert basis*.

See Gallier and Quaintance, *Aspects of Harmonic Analysis and on Locally Compact Abelian Groups*, Appendix D, Section D2.

The maps $e^{i\theta} \mapsto e^{im\theta} = (e^{i\theta})^m$, for $m \in \mathbb{Z}$ and $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$, are continuous homomorphisms of the group $\mathbb{T} = \mathbf{U}(1)$ into itself.

In fact, it can be shown that *they are the only ones of this kind*.

They are called the *characters* of $\mathbb{T}$.

Obviously the set of characters of $\mathbb{T}$ is in bijection with $\mathbb{Z}$.

Thus the Fourier transform $\mathcal{F}(f)$ of a function $f \in \mathrm{L}^2(\mathbb{T})$, a sequence of complex numbers indexed by $\mathbb{Z}$, can be viewed as *a function of the characters of* $\mathbb{T}$.

The characters of $\mathbb{Z}$ are the group homomorphisms $\varphi\colon \mathbb{Z} \to \mathbb{T}$.

Since $\mathbb{Z}$ is generated by 1, a homomorphism satisfies the equation

$$\varphi(m) = (\varphi(1))^m, \qquad m \in \mathbb{Z},$$

so it is uniquely determined by picking $\varphi(1) = e^{i\theta} \in \mathbb{T}$ (with $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$), and is of the form

$$\varphi(m) = (e^{i\theta})^m = e^{im\theta}$$

for all $m \in \mathbb{Z}$.

Thus the set of characters of $\mathbb{Z}$ is in bijection with $\mathbb{T}$.

Then the Fourier cotransform $\overline{\mathcal{F}}(c)$ of a "function" $c \in \ell^2(\mathbb{Z})$ ($\overline{\mathcal{F}}(c)$ is the Fourier series associated with c) can also be viewed *as a function on the characters of* $\mathbb{Z}$, namely a function on $\mathbb{T}$.

This fact generalizes to an arbitrary *abelian locally compact group* and is the key to the definition of the Fourier transform on such a group.

3.3 The Fourier Transform and Cotransform on LCA Groups