Chapter 2

Group Representations

2.1 Finite-Dimensional Group Representations

For simplicity, we begin with finite-dimensional representations.

Definition 2.1. Given a locally compact group G and a normed vector space V of dimension n, a *continuous linear representation of G in V of dimension* (or *degree*) n is a group homomorphism $\rho\colon G \to \mathbf{GL}(V)$, where $\mathbf{GL}(V)$ denotes the group of invertible linear maps from V to itself, such that the following condition holds:

(C) The map $g \mapsto \rho(g)(u)$ is continuous for every $u \in V$.

The space V, called the *representation space*, may be a real or a complex vector space.

If V has a Hermitian (resp. Euclidean) inner product $\langle -, - \rangle$, we say that $\rho\colon G \to \mathbf{GL}(V)$ is a *continuous unitary representation* if

(U) Every linear map $\rho(g)$ is an *isometry*, that is,

$$\langle \rho(g)(u), \rho(g)(v) \rangle = \langle u, v \rangle,$$
$$\text{for all } g \in G \text{ and all } u, v \in V.$$

A unitary representation is denoted $\rho\colon G \to \mathbf{U}(V)$.

Thus, a continuous linear representation of G is a map $\rho\colon G \to \mathbf{GL}(V)$ satisfying Condition (C) as well as the properties:

$$\begin{aligned} \rho(gh) &= \rho(g)\rho(h) \\ \rho(g^{-1}) &= \rho(g)^{-1} \\ \rho(1) &= \mathrm{id}_V \end{aligned}$$

for all $g, h \in G$.

If ρ is a unitary representation, then we also have

$$(\rho(g))^{-1} = (\rho(g))^*.$$

If G is a finite group, the continuity requirement is omitted.

To avoid confusion when representations involving different groups arise we denote the space of the representation ρ by V_ρ, and so we denote a representation as $\rho\colon G \to \mathbf{GL}(V_\rho)$.

To reduce the amount of parentheses we often write $\rho_g(u)$ instead of $\rho(g)(u)$.

The representation such that $\rho(g) = \mathrm{id}_V$ for all $g \in G$ is called the *trivial representation*.

It should be noted that because V is finite-dimensional, the condition that for every $u \in V$, the map $g \mapsto \rho(g)(u)$ is continuous, is actually equivalent to the fact that the map $g \mapsto \rho(g)$ from G to $\mathcal{L}(V)$ equipped with the operator norm induced by any norm on V, is continuous.

Since the space V of a representation $\rho\colon G \to \mathbf{GL}(V)$ is finite-dimensional, say n, it is often convenient to pick a basis $(e_1, \ldots, e_n)$ of V, and then every invertible linear map $\rho(g) \in \mathbf{GL}(V)$ is represented by an $n \times n$ matrix that we denote [1]

$$M_\rho(g) = (\rho_{ij}(g)).$$

We obtain a continuous map $M_\rho\colon G \to \mathbf{GL}(n, \mathbb{C})$ assigning an invertible $n \times n$ complex matrix $M_\rho(g) = (\rho_{ij}(g))$ to $g \in G$ satisfying the properties

$$\begin{aligned} M_\rho(gh) &= M_\rho(g)M_\rho(h) \\ M_\rho(g^{-1}) &= (M_\rho(g))^{-1} \\ M_\rho(1) &= I_n \end{aligned}$$

for all $g, h \in G$.

[1] To be perfectly rigorous the matrix M_ρ should be indexed by the basis $\mathcal{E} = (e_1, \ldots, e_n)$, say as $M_\rho^{\mathcal{E}}$, but this is just too much decoration.

The continuity of M_ρ is equivalent to the fact that the n^2 functions $g \mapsto \rho_{ij}(g)$ are continuous. If ρ is a unitary representation, then we also have

$$(M_\rho(g))^{-1} = (M_\rho(g))^*.$$

If G is finite we drop the continuity requirement. Conversely we have the notion of representation in matrix form.

Definition 2.2. Given a locally compact group G a *continuous linear representation of G of dimension (or degree) n in matrix form* is a mapping $M_\rho \colon G \to \mathbf{GL}(n, \mathbb{C})$ assigning an invertible $n \times n$ complex matrix $M_\rho(g) = (\rho_{ij}(g))$ to $g \in G$ satisfying the properties

$$\begin{aligned} M_\rho(gh) &= M_\rho(g)M_\rho(h) \\ M_\rho(g^{-1}) &= (M_\rho(g))^{-1} \\ M_\rho(1) &= I_n \end{aligned}$$

for all $g, h \in G$, and such that the n^2 functions $g \mapsto \rho_{ij}(g)$ are continuous.

If M_ρ is a unitary representation, then we also have

$$(M_\rho(g))^{-1} = (M_\rho(g))^*.$$

In this case M_ρ is a homomorphism $M_\rho\colon G \to \mathbf{U}(n)$. If G is finite we drop the continuity requirement.

A representation in matrix form $M_\rho\colon G \to \mathbf{GL}(n, \mathbb{C})$ (resp. $M_\rho\colon G \to \mathbf{U}(n)$) defines the representation $\rho\colon G \to \mathbf{GL}(\mathbb{C}^n)$ (resp. $\rho\colon G \to \mathbf{U}(\mathbb{C}^n)$) given by

$$(\rho(g))(z) = M_\rho(g)z, \quad z \in \mathbb{C}^n,\ g \in G.$$

We also often identify a matrix representation with the representation associated with it. The same issue arises in linear algebra and we hope that the reader is already familiar with it and will not be confused.

Given any basis $(e_1, \ldots, e_n)$ of V, we may think of the scalar functions $g \mapsto \rho_{ij}(g)$ as *special functions* on G.

As explained in Dieudonné [4] (see also Vilenkin [16]), essentially all special functions (Legendre polynomials, ultraspherical polynomials, Bessel functions *etc.*) arise in this way by choosing some suitable G and V.

Remark: In Chapter **??** we will consider the situation where G is a group not equipped with any topology, and V is a vector space, possibly infinite-dimensional, not equipped with any norm.

Then a *linear representation of G in V* is simply a homomorphism $\rho\colon G \to \mathbf{GL}(V)$, which amounts to dropping Condition (C) from Definition 2.1.

However, in this chapter and the next, all representations satisfy Condition (C).

Example 2.1. Consider the group $\mathfrak{S}_3$ of permutations on the set $\{1, 2, 3\}$. There are $3! = 6$ permutations

$$\pi_1 = (1, 2, 3), \;\; \pi_2 = (1, 3, 2), \;\; \pi_3 = (2, 1, 3),$$
$$\pi_4 = (2, 3, 1), \;\; \pi_5 = (3, 1, 2), \;\; \pi_6 = (3, 2, 1).$$

The first permutation $\pi_1 = (1, 2, 3)$ is the identity; the permutations

$$\pi_2 = (1, 3, 2), \;\; \pi_3 = (2, 1, 3), \;\; \pi_6 = (3, 2, 1)$$

are transpositions and thus have negative signature, and the permutations

$$\pi_4 = (2, 3, 1), \;\; \pi_5 = (3, 1, 2)$$

are cyclic permutations and thus have positive signature. We obtain a representation $\rho_1 \colon \mathfrak{S}_3 \to \mathbf{GL}(\mathbb{C}^3)$ as follows.

If (e_1, e_2, e_3) is the canonical basis of $\mathbb{C}^3$, then $\rho_1(\pi_i)$ is the linear map given by

$$\rho_1(\pi_i)(e_j) = e_{\pi_i(j)}, \quad 1 \leq i, j \leq 3.$$

In the basis (e_1, e_2, e_3), the linear maps $\rho_1(\pi_i)$ are represented by the 3×3 matrices $M_1, \ldots, M_6$ given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

This is an example of a permutation representation.

Here is another representation of the group $\mathfrak{S}_3$ in $\mathbb{C}^6$.

Example 2.2. This time we define the representation $\rho_{\mathbf{R}}\colon \mathfrak{S}_3 \to \mathbf{GL}(\mathbb{C}^6)$ as follows.

Let $(e_{\pi_1}, \ldots, e_{\pi_6})$ be the canonical basis of $\mathbb{C}^6$ indexed by the permutations π_i $(1 \leq i \leq 6)$, and set

$$\rho_{\mathbf{R}}(\pi_i)(e_{\pi_j}) = e_{\pi_i \circ \pi_j}, \quad 1 \leq i, j \leq 6.$$

Note that the 6×6 matrix representing $\rho_{\mathbf{R}}(\pi_i)$ in the basis $(e_{\pi_1}, \ldots, e_{\pi_6})$ consists of the permutation of the columns of the identity matrix I_6 whose indices are given by the ith row of the multiplication table of the group $\mathfrak{S}_3$.

This multiplication table is given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 6 & 3 & 4 \\ 3 & 4 & 1 & 2 & 6 & 5 \\ 4 & 3 & 6 & 5 & 1 & 2 \\ 5 & 6 & 2 & 1 & 4 & 3 \\ 6 & 5 & 4 & 3 & 2 & 1, \end{pmatrix},$$

where we denote π_i simply by i and where the (i, j) entry represents $\pi_i \circ \pi_j$.

We obtain the following 6 matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The representation $\rho_{\mathbf{R}}$ is called the *regular representation* of $\mathfrak{S}_3$.

Example 2.3. For an example involving an infinite group, we describe a class of representations of $G = \mathbf{SL}(2, \mathbb{C})$, the group of complex matrices with determinant $+1$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1.$$

Recall that $\mathcal{P}_m^{\mathbb{C}}(2)$ denotes the vector space of *complex homogeneous polynomials of degree m* in two variables (z_1, z_2).

A complex homogeneous polynomials of degree m in two variables (z_1, z_2) is an expression of the form

$$P(z_1, z_2) = \sum_{i=0}^{m} c_i z_1^i z_2^{m-i},$$

with $c_i \in \mathbb{C}$.

For every matrix $A \in \mathbf{SL}(2, \mathbb{C})$, with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

for every homogeneous polynomial $P \in \mathcal{P}_m^{\mathbb{C}}(2)$, we define $U_m(A)(P(z_1, z_2))$ by

$$U_m(A)(P(z_1, z_2)) = P(dz_1 - bz_2, -cz_1 + az_2).$$

If we think of the homogeneous polynomial $Q(z_1, z_2)$ as a function $P\binom{z_1}{z_2}$ of the vector $\binom{z_1}{z_2}$, then

$$U_m(A)\left(P\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = PA^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

This is a left regular representation, as discussed later in Definition 2.6.

The expression above makes it clear that

$$U_m(AB) = U_m(A)U_m(B)$$

for any two matrices $A, B \in \mathbf{SL}(2, \mathbb{C})$, so U_m is indeed a representation of $\mathbf{SL}(2, \mathbb{C})$ into $\mathcal{P}_m^{\mathbb{C}}(2)$.

The representations U_m also yield representations of the subgroup $\mathbf{SU}(2)$ of $\mathbf{SL}(2, \mathbb{C})$.

Recall that the group $\mathbf{SU}(2)$ consists of all 2×2 complex matrices

$$S = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad \alpha\overline{\alpha} + \beta\overline{\beta} = 1.$$

As above, the representation $U_m \colon \mathbf{SU}(2) \to \mathbf{GL}(\mathcal{P}_m^{\mathbb{C}}(2))$ is given by

$$U_m(S)(P(z_1, z_2)) = P(\overline{\alpha} z_1 - \beta z_2, \overline{\beta} z_1 + \alpha z_2).$$

It can be shown that $\mathbf{SL}(2,\mathbb{C})$ has *no* nontrivial *unitary* finite-dimensional representations!

This is because $\mathbf{SL}(2,\mathbb{C})$ is a connected simple noncompact Lie group with finite center; see Dieudonné [5] (Section 21.6, Problem 5).

Example 2.4. We define the representation $\rho_9 \colon \mathbf{SO}(3) \to \mathbf{GL}(\mathrm{M}_3(\mathbb{C}))$ as follows: for any 3×3 complex matrix $A \in \mathrm{M}_3(\mathbb{C})$, for any $Q \in \mathbf{SO}(3)$,

$$\rho_9(Q)(A) = QAQ^\top.$$

This is a representation in the vector space $\mathrm{M}_3(\mathbb{C})$, which has dimension 9.

To obtain a version of ρ_9 as a matrix representation M_{ρ_9} we need to pick a basis of $\mathrm{M}_3(\mathbb{C})$.

Let us choose the canonical basis of nine matrices $E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}, E_{31}, E_{32}, E_{33}$, where E_{ij} contains 1 as the (i, j) entry and 0 otherwise.

A matrix $M \in \mathrm{M}_3(\mathbb{C})$ is then written as the column vector

$$\mathrm{vec}(A) = (a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}).$$

It follows that over this basis, the matrix $M_{\rho_9}(Q)$ representing the linear map $\rho_9(Q)$ is given by

$$M_{\rho_9}(Q)(\mathrm{vec}(A)) = \mathrm{vec}(QAQ^\top).$$

However, it is a fact of linear algebra that for any $m \times m$ matrix A, any $n \times n$ matrix B, and $m \times n$ matrix Z, we have the identity

$$\mathrm{vec}(AZB) = (B^\top \otimes A)\mathrm{vec}(Z),$$

where $\otimes$ denotes the *Kronecker product* of matrices.

Therefore we deduce that

$$M_{\rho_9}(Q)(\mathrm{vec}(A)) = \mathrm{vec}(QAQ^\top) = (Q \otimes Q)\mathrm{vec}(A),$$

that is,

$$M_{\rho_9}(Q) = Q \otimes Q,$$

a 9×9-matrix. The definition of the representation ρ_9 as acting on the vector space $\mathrm{M}_3(\mathbb{C})$ is a lot more economical than its matrix version M_{ρ_9} acting on $\mathbb{C}^9$.

The representation ρ_9 is *reducible*. Indeed observe that both the subspace of symmetric matrices and the subspace of skew-symmetric matrices are invariant since $(QAQ^\top)^\top = QA^\top Q^\top$.

The subspace of symmetric matrices A with $\mathrm{tr}(A) = 0$ is also invariant.

Definition 2.3. Given any two representations $\rho_1 \colon G \to \mathbf{GL}(V_1)$ and $\rho_2 \colon G \to \mathbf{GL}(V_2)$, a *G-map* (or *morphism of representations*) $\varphi \colon \rho_1 \to \rho_2$ is a linear map $\varphi \colon V_1 \to V_2$ which is *equivariant*, which means that the following diagram commutes for every $g \in G$:

$$\begin{array}{ccc} V_1 & \xrightarrow{\rho_1(g)} & V_1 \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ V_2 & \xrightarrow[\rho_2(g)]{} & V_2, \end{array}$$

i.e.

$$\varphi \circ \rho_1(g) = \rho_2(g) \circ \varphi, \qquad g \in G.$$

The space of all G-maps between two representations as above is denoted $\mathrm{Hom}_G(\rho_1, \rho_2)$.

Two representations $\rho_1 \colon G \to \mathbf{GL}(V_1)$ and $\rho_2 \colon G \to \mathbf{GL}(V_2)$ are *equivalent* iff $\varphi \colon V_1 \to V_2$ is an invertible linear map (which implies that $\dim V_1 = \dim V_2$).

In matrix form, the representations $\rho_1 \colon G \to \mathbf{GL}(n, \mathbb{C})$ and $\rho_2 \colon G \to \mathbf{GL}(n, \mathbb{C})$ are equivalent iff there is some invertible $n \times n$ matrix P so that

$$\rho_2(g) = P\rho_1(g)P^{-1}, \qquad g \in G.$$

If $W \subseteq V$ is a subspace of V, then in some cases, a representation $\rho\colon G \to \mathbf{GL}(V)$ yields a representation $\rho\colon G \to \mathbf{GL}(W)$.

This is interesting because under certain conditions on G (*e.g.*, G compact) every representation may be decomposed into a "sum" of so-called *irreducible representations* (defined below), and thus the study of all representations of G boils down to the study of irreducible representations of G;

Definition 2.4. Let $\rho\colon G \to \mathbf{GL}(V)$ be a representation of G. If $W \subseteq V$ is a subspace of V, then we say that W is *invariant* (or *stable*) under ρ iff $\rho(g)(w) \in W$, for all $g \in G$ and all $w \in W$.

If W is invariant under ρ, then we have a homomorphism, $\rho\colon G \to \mathbf{GL}(W)$, called a *subrepresentation* of G.

A representation $\rho\colon G \to \mathbf{GL}(V)$ with $V \neq (0)$ is *irreducible* iff it only has the two subrepresentations $\rho\colon G \to \mathbf{GL}(W)$ corresponding to $W = (0)$ or $W = V$.

Example 2.5. The representation $\rho_1 \colon \mathfrak{S}_3 \to \mathbf{GL}(\mathbb{C}^3)$ of Example 1.2 is reducible.

Indeed, the one-dimensional subspace V_1 spanned by $e_1+e_2+e_3$ is invariant under ρ_1 since each $\rho_1(\pi_i)$ permutes the indices $1, 2, 3$.

The corresponding subrepresentation of $\mathfrak{S}_3$ in V_1 is equivalent to the irreducible trivial representation in $\mathbb{C}$, given by $\rho_{\mathrm{triv}}(\pi_i) = 1$ $(1 \leq i \leq 6)$.

The orthogonal complement V_2 of V_1 is the plane of equation

$$x_1 + x_2 + x_3 = 0,$$

which has $(e_1 - e_2, e_2 - e_3)$ as a basis.

It is easy to see that the subspace V_2 is also invariant under ρ_1. It is instructive to find an equivalent representation of ρ_1 in the basis (v_1, v_2, v_3) given by

$$\begin{aligned} v_1 &= (1/3)(e_1 + e_2 + e_3) \\ v_2 &= (1/3)(e_1 - e_2) \\ v_3 &= (1/3)(e_2 - e_3). \end{aligned}$$

The change of basis matrix P from the basis (e_1, e_2, e_3) to the basis (v_1, v_2, v_3) is

$$P = \begin{pmatrix} 1/3 & 1/3 & 0 \\ 1/3 & -1/3 & 1/3 \\ 1/3 & 0 & -1/3 \end{pmatrix},$$

whose inverse is

$$P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Using the linear map φ from $\mathbb{C}^3$ to itself given by P^{-1} (which transforms the coordinates of a vector in $\mathbb{C}^3$ over the basis (e_1, e_2, e_3) to the coordinates of this vector over the basis (v_1, v_2, v_3)), we obtain the equivalent representation ρ_1' given by

$$\rho_1'(\pi_i) = \varphi \rho_1(\pi_i) \varphi^{-1},$$

and over the basis (v_1, v_2, v_3), the matrices representing the linear maps $\rho_1'(\pi_i)$ are the matrices $P^{-1} M_i P$ shown below:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Some of the above matrices are not unitary.

We can fix this by choosing an orthonormal basis (w_1, w_2, w_3) with $w_1 = (1/\sqrt{3})v_1$, a basis of V_1, and (w_2, w_3), a basis of V_2.

For example we can pick

$$\begin{aligned} w_1 &= (1/\sqrt{3})(e_1 + e_2 + e_3) \\ w_2 &= (1/\sqrt{2})(e_1 - e_2) \\ w_3 &= (1/\sqrt{6})(e_1 + e_2 - 2e_3). \end{aligned}$$

The change of basis matrix Q from the basis (e_1, e_2, e_3) to the basis (w_1, w_2, w_3) is

$$Q = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}$$

and $Q^{-1} = Q^{\top}$.

We obtain an equivalent representation $\rho_1''(\pi_i)$ and over the basis (w_1, w_2, w_3), the unitary matrices representing the linear maps $\rho_1''(\pi_i)$ are the matrices $Q^{-1}M_iQ$ shown below:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix}.$$

It is now clear that the subspace V_1 spanned by w_1 and the subspace V_2 spanned by w_2 and w_3 are invariant.

It is not hard to show that the subrepresentation of ρ_1'' in V_2 is irreducible.

This representation is usually called the *standard representation* of $\mathfrak{S}_3$; see Fulton and Harris [9], Section 1.3,

Thus we have two irreducible representations of $\mathfrak{S}_3$, the second one being two-dimensional.

It turns out that $\mathfrak{S}_3$ only has one more irreducible representation.

How do we find it? The answer is, as a subrepresentation of the regular representation.

Recall the regular representation $\rho_{\mathbf{R}} \colon \mathfrak{S}_3 \to \mathbf{GL}(\mathbb{C}^6)$ of $\mathfrak{S}_3$ from Example 2.2.

The notion of regular representation can be defined for any finite group.

Definition 2.5. Let G be a finite group with $g = |G|$ elements. We define the *regular representation* $\rho_{\mathbf{R}} \colon G \to \mathbf{GL}(\mathbb{C}^g)$ as follows.

Let $(e_{s_1}, \ldots, e_{s_g})$ be the canonical basis of $\mathbb{C}^g$ indexed by the g elements of G and set

$$\rho_{\mathbf{R}}(s_i)(e_{s_j}) = e_{s_i s_j}, \quad 1 \leq i, j \leq g.$$

The following facts about irreducible finite-dimensional representations of a finite group G can be shown.

(1) Every irreducible finite-dimensional representation $\rho_i\colon G \to \mathbf{GL}(\mathbb{C}^{n_i})$ of the finite group G is equivalent to a subrepresentation of the regular representation $\rho_{\mathbf{R}}\colon G \to \mathbf{GL}(\mathbb{C}^g)$ of G in $\mathbb{C}^g$ (where $g = |G|$).

(2) Every irreducible representation $\rho_i\colon G \to \mathbf{GL}(\mathbb{C}^{n_i})$ occurs n_i times in the regular representation; see Proposition 4.6.

(3) If there are h irreducible representations $\rho_i\colon G \to \mathbf{GL}(\mathbb{C}^{n_i})$ (up to equivalence), then

$$n_1^2 + \cdots + n_h^2 = g;$$

see Section 4.3, Example **??**.

(4) The number h of irreducible representations of G (up to equivalence) is equal to the number of conjugacy classes of G; see Section 4.3, Example **??**.

The proof of these standard facts of representation theory can be found in Serre [14], Fulton and Harris [9], Simon [15], Hall [13], or any book on representation theory.

There is also a notion of regular representation on a vector space of functions which occurs a lot in group equivariant deep learning.

Definition 2.6. Let G be a finite group with $g = |G|$ elements. The representation $\mathbf{R}$ given by

$$(\mathbf{R}_{s_i}(f))(s_k) = f(s_i^{-1}s_k), \quad f \in \mathbb{C}^G,\ 1 \leq i, k \leq g, \ (*_2)$$

is also called the *regular representation* of G in $\mathbb{C}^G$.

The representation of Definition 2.6 is a special case of the notion of regular representation defined in Definition **??** for locally compact groups.

To be very precise it is the *left regular representation* of G because it acts on the left on functions in $\mathbb{C}^G$.

At first glance the term $s_i^{-1}s_k$ may seem wrong, but it is necessary to use s_i^{-1} instead of s_i to insure that $\mathbf{R}$ is a left action on functions in $\mathbb{C}^G$. We already noticed this fact in Vol I, Section @@@8.2, Definition @@@8.7.

There is also a *right regular representation* defined by

$$(\mathbf{R}^r_{s_i}(f))(s_k) = f(s_k s_i), \quad f \in \mathbb{C}^G,\ 1 \leq i, k \leq g. \quad (*_3)$$

Representations as given by $(*_2)$ are said to be representations by *left shifts*, and representations as given by $(*_3)$ are said to be representations by *right shifts*.

Obviously the notion of left regular representation (and right regular representation) makes sense for any group G, finite or infinite, and any subspace $\mathcal{F}$ of the vector space all functions in $\mathbb{C}^G$, namely it is the representation $\mathbf{R}\colon G \to \mathbf{GL}(\mathcal{F})$ given by

$$(\mathbf{R}_s(f))(t) = f(s^{-1}t), \quad f \in \mathcal{F},\ s, t \in G. \qquad (*_4)$$

If G is an infinite locally compact groups, it is necessary to replace the vector space $\mathbb{C}^G$ of the representation by a space of functions defined on G, namely $\mathrm{L}^2_\lambda(G;\mathbb{C})$ (where λ is a left Haar measure on G).

If V has a hermitian inner product, then we can prove that any irreducible linear representation $\rho\colon G \to \mathbf{GL}(V)$ of a group G, finite or infinite, where ρ is not assumed to satisfy Condition (C), is equivalent to some (irreducible) subrepresentation $\widehat{\rho}\colon G \to \mathbf{GL}(\mathcal{F})$ of the left regular representation $\mathbf{R}\colon G \to \mathbf{GL}(\mathbb{C}^G)$.

The significance of this result is that for any group G, there is always some irreducible representation whose representation space has a cardinality at most the cardinality of $G^{\mathbb{C}}$.

We now return to the regular representation of Example 2.2.

Example 2.6. It is easy to see that the symmetric group has three conjugacy classes, $\{\pi_1\}$, $\{\pi_2, \pi_3, \pi_6\}$ and $\{\pi_4, \pi_5\}$, so it has three irreducible representations.

Going back to the regular representation $\rho_{\mathbf{R}} \colon \mathfrak{S}_3 \to \mathbf{GL}(\mathbb{C}^6)$, we see that the one-dimensional subspace V_1 spanned by $e_1 + e_2 + e_3 + e_4 + e_5 + e_6$ is invariant so the representation $\rho_{\mathbf{R}}$ is reducible.

The subrepresentation of $\rho_{\mathbf{R}}$ in V_1 is equivalent to the trivial representation, which is irreducible.

Although this is not obvious, there is another one-dimensional irreducible representation, which is the representation induced by the signature function ϵ on permutations.

Recall that for any permutation π, its signature $\epsilon(\pi)$ is $+1$ if π is the composition of an even number of transpositions, -1 if it is the composition of an odd number of transpositions.

The map $\epsilon\colon \mathfrak{S}_n \to \mathbb{C}$ is a homomorphism and it yields the irreducible representation $\rho_\epsilon\colon \mathfrak{S}_n \to \mathbf{U}(1)$ given by

$$(\rho_\epsilon(\pi))(z) = \epsilon(\pi)z, \quad z \in \mathbb{C}.$$

Then we see that the subspace V_2 spanned by the vector $e_1 - e_2 - e_3 + e_4 + e_5 - e_6$ (which corresponds to the signatures $+1, -1, -1, +1, +1, -1$ of the permutations $\pi_1, \ldots, \pi_6$) is invariant under $\rho_{\mathbf{R}}$, and the subrepresentation of $\rho_{\mathbf{R}}$ to V_2 is equivalent to the irreducible representation ρ_ϵ.

The orthogonal complement V_3 of $V_1 \oplus V_2$ is the intersection of the two hyperplanes in $\mathbb{C}^6$ given by the equations

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 &= 0 \\
x_1 - x_2 - x_3 + x_4 + x_5 - x_6 &= 0,
\end{aligned}
$$

a subspace of dimension 4.

By adding and subtracting these equations we see that the subspace V_3 is also defined by the equations

$$
\begin{aligned}
x_1 + x_4 + x_5 &= 0 \\
x_2 + x_3 + x_6 &= 0.
\end{aligned}
$$

We can prove directly that V_3 is invariant under $\rho_{\mathbf{R}}$, but since the representation $\rho_{\mathbf{R}}$ is actually unitary, we prefer using results form the next section.

An easy but crucial lemma about irreducible representations is *Schur's Lemma*.

Lemma 2.1. *(Schur's Lemma) Let* $\rho_1 \colon G \to \mathbf{GL}(V)$ *and* $\rho_2 \colon G \to \mathbf{GL}(W)$ *be any two real or complex finite-dimensional representations of a group* G*. If* ρ_1 *and* ρ_2 *are irreducible, then the following properties hold:*

(i) Every G*-map* $\varphi \colon \rho_1 \to \rho_2$ *is either the zero map or an isomorphism.*

(ii) If ρ_1 *is a complex representation, then every* G*-map* $\varphi \colon \rho_1 \to \rho_1$ *is of the form* $\varphi = \lambda \mathrm{id}$*, for some* $\lambda \in \mathbb{C}$*.*

Proof. (i) Observe that the kernel $\mathrm{Ker}\ \varphi \subseteq V$ of φ is *invariant under ρ_1*. Indeed, for every $v \in \mathrm{Ker}\ \varphi$ and every $g \in G$, we have

$$\varphi(\rho_1(g)(v)) = \rho_2(g)(\varphi(v)) = \rho_2(g)(0) = 0,$$

so $\rho_1(g)(v) \in \mathrm{Ker}\ \varphi$. Thus, $\rho_1 \colon G \to \mathbf{GL}(\mathrm{Ker}\ \varphi)$ is a subrepresentation of ρ_1, and as ρ_1 is irreducible, either $\mathrm{Ker}\ \varphi = (0)$ or $\mathrm{Ker}\ \varphi = V$.

In the second case, $\varphi = 0$. If $\mathrm{Ker}\ \varphi = (0)$, then φ is injective.

However, $\varphi(V) \subseteq W$ is *invariant under ρ_2*, since for every $v \in V$ and every $g \in G$,

$$\rho_2(g)(\varphi(v)) = \varphi(\rho_1(g)(v)) \in \varphi(V),$$

and as $\varphi(V) \neq (0)$ (as $V \neq (0)$ since ρ_1 is irreducible) and ρ_2 is irreducible, we must have $\varphi(V) = W$; that is, φ is an isomorphism.

The proof also works for infinite-dimensional spaces.

(ii) Since V is a complex vector space of finite dimension, the linear map φ has some *eigenvalue* $\lambda \in \mathbb{C}$.

Let $E_\lambda \subseteq V$ be the eigenspace associated with λ.

The subspace E_λ is *invariant under* ρ_1, since for every $u \in E_\lambda$ and every $g \in G$, we have

$$\varphi(\rho_1(g)(u)) = \rho_1(g)(\varphi(u)) = \rho_1(g)(\lambda u) = \lambda\rho_1(g)(u),$$

so $\rho_1 \colon G \to \mathbf{GL}(E_\lambda)$ is a subrepresentation of ρ_1, and as ρ_1 is irreducible and $E_\lambda \neq (0)$, we must have $E_\lambda = V$. $\square$

Part (i) of Schur's lemma also holds for infinite-dimensional representations as we noted in the proof.

An interesting corollary of Schur's Lemma is the following fact:

Proposition 2.2. *A complex irreducible finite-dimensional representation* $\rho\colon G \to \mathbf{GL}(V)$ *of a commutative group* G *is one-dimensional.*

2.2 Unitary Group Representations

We now generalize representations to allow the representing space to be a *complex Hilbert space* (typically separable).

Definition 2.7. Given a locally compact group G and a complex Hilbert space H, a *unitary representation of G in H* is a group homomorphism $U: G \to \mathbf{U}(H)$, where $\mathbf{U}(H)$ is the group of unitary operators on H, such that:

(C) The map $g \mapsto U(g)(u)$ is continuous for every $u \in H$.

(U) Every linear map $U(g)$ is an isometry; that is,

$$\langle U(g)(u), U(g)(v) \rangle = \langle u, v \rangle,$$

for all $g \in G$ and all $u, v \in H$. In particular $U(g)$ is continuous and

$$(U(g))^{-1} = (U(g))^{*} \qquad \text{for all } g \in G.$$

As in Definition 2.1, to avoid confusion when representations involving diffferent groups arise we denote the space of the representation U by H_U, and so we denote a representation as $U\colon G \to \mathbf{U}(H_U)$.

Remark: Sometimes, a unitary representation as in Definition 2.7 is called a *continuous* unitary representation.

Note that if H is infinite-dimensional, the map $g \mapsto U(g)$ is *not necessarily continuous*.

However, the left action $U^a\colon G \times H \to H$ associated with U given by

$$U^a(s, x) = U(s)(x), \qquad \text{for all } s \in G \text{ and all } x \in H$$

is continuous.

The notion of morphism of unitary representations and of equivalence is adapted as follows.

Definition 2.8. Given any two unitary representations $U_1 \colon G \to \mathbf{U}(H_1)$ and $U_2 \colon G \to \mathbf{U}(H_2)$, a *G-map* (or *morphism of representations*) $\varphi \colon U_1 \to U_2$ is a continuous linear map which is *equivariant*, which means that the following diagram commutes for every $g \in G$:

$$\begin{array}{ccc} H_1 & \xrightarrow{U_1(g)} & H_1 \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ H_2 & \xrightarrow[U_2(g)]{} & H_2, \end{array}$$

i.e.

$$\varphi \circ U_1(g) = U_2(g) \circ \varphi, \qquad g \in G.$$

The space of all G-maps between two representations as above is denoted $\mathrm{Hom}_G(U_1, U_2)$.

A G-map is also called an *intertwining operator*.

Two unitary representations $U_1\colon G \to \mathbf{U}(H_1)$ and $U_2\colon G \to \mathbf{U}(H_2)$ are *equivalent* iff $\varphi\colon H_1 \to H_2$ is an invertible linear isometry whose inverse is also continuous; thus

$$U_2(g) = \varphi \circ U_1(g) \circ \varphi^{-1},$$

for all $g \in G$.

When $U_1 = U_2$, the space of G-maps $\mathrm{Hom}_G(U, U)$ is a unital subalgebra of $\mathcal{L}(H)$ denoted by $\mathcal{C}(U)$ and is called the *commutant* or *centralizer* of U.

Observe that

$$\mathcal{C}(U) = \{\varphi \in \mathcal{L}(H) \mid \varphi \circ U(g) = U(g) \circ \varphi \quad \text{for all } g \in G\}.$$

Given a unitary representation $U \colon G \to \mathbf{U}(H)$, the definition of an *invariant subspace* $W \subseteq H$ is the same as in Definition 2.4.

If $W \subseteq H$ is invariant under U, we say that the subrepresentation $U \colon G \to \mathbf{U}(W)$ is *closed* if W is closed in H.

As in the case of unitary representations of algebras, the notion of *subrepresentation* is only well defined for closed invariant subspaces of H.

However, by Proposition 2.4, since the closure $\overline{W}$ of an invariant subspace W is closed, the notion of subrepresentation of G in $\overline{W}$ is well defined.

In the definition of an *irreducible* unitary representation $U\colon G \to \mathbf{U}(H)$ $(H \neq (0))$, we require that the only *closed* subrepresentations $U\colon G \to \mathbf{U}(W)$ of the representation $U\colon G \to \mathbf{U}(H)$ correspond to $W = (0)$ or $W = H$.

As for representations of algebras, we can define topologically cyclic representations and cyclic vectors.

Definition 2.9. Let $U\colon G \to \mathbf{U}(H)$ be a unitary representation of G in H. A vector $x_0 \in H$ is called a *totalizer*, or *totalizing vector*, or *cyclic vector* for the representation U if the subspace of H spanned by the set $\{U(s)(x_0) \mid s \in G\}$ is dense in H.

Equivalently if $\mathcal{M}_{x_0}$ denotes the closure of the set $\{U(s)(x_0) \mid s \in G\}$, called the *cyclic subspace* generated by x_0, which is invariant under U, then x_0 is a totalizer (a cyclic vector) if $\mathcal{M}_{x_0} = H$.

A representation which admits a totalizer is said to be *topologically cyclic*.

The importance of totalizers stems from the following result.

Proposition 2.3. *Let $U: G \to \mathbf{U}(H)$ be a unitary representation of G in H. Then H is the Hilbert sum of a sequence $(H_\alpha)_{\alpha\in\Lambda}$ of closed subspaces $H_\alpha \neq (0)$ of H invariant under U, and such that the restriction of U to each H_α is topologically cyclic. If H is separable, the family Λ is countable (possibly finite).*

Proposition 2.4. *Let $U: G \to \mathbf{U}(H)$ be a unitary representation of G in H.*

(1) If the subspace E of H is invariant under U, then its closure $\overline{E}$ is also invariant under U.

(2) Let E be a closed subspace of H invariant under U. If $E^\perp$ is the orthogonal complement of E in H, then $E^\perp$ is invariant under U.

If $U_1(s)$ and $U_2(s)$ are the restrictions of $U(s)$ to E and $E^\perp$, then $H = E \oplus E^\perp$ (the algebraic direct sum), and the representation U is the Hilbert sum of the representations U_1 and U_2.

One should realize that Property (2) of Proposition 2.4 *fails* for nonunitary representations. For example, the map

$$U : x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is a representation of $\mathbb{R}$ in $\mathbb{C}^2$, but the only nontrivial invariant subspace is the subspace spanned by $(1, 0)$, which is one-dimensional.

The problem is that because $\mathbb{R}$ is not compact, there is no way to define an inner product on $\mathbb{C}^2$ invariant under U.

However, using the Haar measure, Vol I, Theorem @@@8.36 shows that if H is a finite-dimensional hermitian space, then there is an inner-product on H for which the linear maps $U(s)$ are unitary.

Theorem 2.5. *(Complete Reducibility) Let $U: G \to \mathbf{GL}(H)$ be a linear representation of a compact group G in a Hermitian space H of dimension $n \geq 1$.*

There is a hermitian inner product $\langle -, - \rangle$ on H such that $U: G \to \mathbf{U}(H)$ is a unitary representation of G in the hermitian space $(H, \langle -, - \rangle)$. The representation U is the direct sum of a finite number of irreducible unitary representations.

Theorem 2.5 is very significant because it shows that the study of *arbitrary finite-dimensional* representations of a compact group G reduces to the study of the *irreducible unitary (finite-dimensional) representations* of G.

Example 2.7. The regular representation $\rho_{\mathbf{R}} \colon \mathfrak{S}_3 \to \mathbf{GL}(\mathbb{C}^6)$ of $\mathfrak{S}_3$ from Example 2.2 is obviously unitary.

Theorem 2.5 tells us that $\rho_{\mathbf{R}}$ is the direct sum of irreducible representations, and in Example 2.6 we already found two irreducible representations which are one-dimensional.

The discussion before Example 2.6 also shows that the standard representation (see Example 2.5) must occur in the representation $\rho_{\mathbf{R}}$, and if there are h irreducible representations, the equation $n_1^2 + \cdots + n_h^2 = g = 6$ implies that $1 + 1 + 2^2 + \cdots + n_h^2 = 6$, so $h = 3$ and the standard representation occurs twice.

Therefore the orthogonal complement V_3 of the direct sum $V_1 \oplus V_2$ given by the equations

$$x_1 + x_4 + x_5 = 0$$
$$x_2 + x_3 + x_6 = 0$$

must be the direct sum of 2 two-dimensional invariant subspaces.

With a little help from `Matlab` we find that the subspace V_1^3 spanned by the vectors

$$e_1 + e_2 - e_3 - e_4, \;\; e_3 + e_4 - e_5 - e_6$$

is invariant under $\rho_{\mathbf{R}}$, the subspace V_2^3 spanned by the vectors

$$e_1 - e_3 - e_4 + e_6, \;\; e_2 + e_4 - e_5 - e_6,$$

is also invariant under $\rho_{\mathbf{R}}$, both V_1^3 and V_2^3 are orthogonal to $V_1 \oplus V_2$, and

$$\mathbb{C}^6 = V_1 \oplus V_2 \oplus V_1^3 \oplus V_2^3.$$

To show that V_1^3 is invariant we observe that V_1^3 is also spanned by

$$e_1 + e_2 - e_3 - e_4,\;\; e_3 + e_4 - e_5 - e_6,\;\; e_1 + e_2 - e_5 - e_6,$$

and the action of $\rho_{\mathbf{R}}(\pi_i)$ is to permute these vectors, possibly flipping signs, and similarly V_2^3 is also spanned by

$$e_1 - e_3 - e_4 + e_6,\;\; e_2 + e_4 - e_5 - e_6,\;\; e_1 + e_2 - e_3 - e_5,$$

and the action of $\rho_{\mathbf{R}}(\pi_i)$ is also to permute these vectors, possibly flipping signs.

According to our previous discussion these two sub-representations of $\mathfrak{S}_3$ in V_1^3 and V_2^3 are equivalent to the standard representation given in Example 2.5.

Thus we identified explicitly the three irreducible representations of $\mathfrak{S}_3$ as subrepresentations of the regular representation.

Proposition 2.6. *Let* $U\colon G \to \mathbf{U}(H)$ *be a unitary representation of* G *in* H*. A closed subspace* E *of* H *is invariant under* U *iff* $P_E U(g) = U(g) P_E$ *for all* $g \in G$*, in other words,* $P_E \in \mathcal{C}(U) = \mathrm{Hom}_G(U, U)$*, where* $P_E\colon H \to E$ *is the orthogonal projection of* H *onto* E*.*

Proposition 2.6 it yields a method for proving that a unitary representation $U\colon G \to \mathbf{U}(H)$ is irreducible.

Indeed, *if* U *is reducible, then there is some nonzero* G*-map* $\varphi \in \mathrm{Hom}_G(U, U)$ *which is not invertible*.

Thus, *if every nonzero* G*-map in* $\mathrm{Hom}_G(U, U)$ *is invertible, then* U *must be irreducible*.

This technique is illustrated in the next example.

Example 2.8. Recall the representations $U_m\colon \mathbf{SU}(2) \to \mathbf{GL}(\mathcal{P}_m^{\mathbb{C}}(2))$ from Example 2.3, where $\mathcal{P}_m^{\mathbb{C}}(2)$ denotes the vector space of complex homogeneous polynomials

$$P(z_1, z_2) = \sum_{k=0}^{m} c_k z_1^k z_2^{m-k}$$

of degree m ($c_i \in \mathbb{C}$).

The $m+1$ monomials $P_k = z_1^k z_2^{m-k}$ ($0 \leq k \leq m$) form a basis of $\mathcal{P}_m^{\mathbb{C}}(2)$.

In the physics literature, it is customary to index homogeneous polynomials in terms of $\ell = m/2$, which is an integer when m is even but a half integer when m is odd.

In this context, the number $\ell = m/2$ is the *spin* of a particle.

In terms of $\ell = m/2$, a homogeneous polynomial is written as

$$P(z_1, z_2) = \sum_{k=-\ell}^{\ell} c_k z_1^{\ell-k} z_2^{\ell+k},$$

where it is assumed that $\ell + k = j$ where j takes the *integral* values $j = 0, 1, \ldots, 2\ell = m$, so that
$\ell - k = 2\ell - (\ell + k) = 2\ell - j$ takes the values
$2\ell, 2\ell - 1, \ldots, 0$.

Note that $k = j - \ell = j - m/2$ with $j = 0, 1, \ldots, 2\ell = m$, so k is an integer only if m is even.

If m is odd, say $m = 2h + 1$, then $\ell = h + \frac{1}{2}$ and k takes the $2\ell + 1 = m + 1$ values

$$-h - \frac{1}{2},\ -(h-1) - \frac{1}{2}, \ldots, -\frac{1}{2},\ \frac{1}{2},\ 1 + \frac{1}{2}, \ldots, h + \frac{1}{2},$$

and so $k \neq 0$.

If m is even, say $m = 2h$, then $\ell = h$ and k takes the $2\ell + 1 = m + 1$ values

$$-h,\ -(h-1), \ldots, -1,\ 0,\ 1, \ldots, h-1, h.$$

For example, if $\ell = \frac{3}{2}$, then k takes the four values

$$-\frac{3}{2},\ -\frac{1}{2},\ \frac{1}{2},\ \frac{3}{2},$$

and if $\ell = 2$, then k takes the five values

$$-2,\ -1,\ 0,\ 1,\ 2.$$

The representing space is then $\mathcal{P}^{\mathbb{C}}_{2\ell}(2)$ and it has dimension $2\ell + 1$.

The physics notation makes it easier to make the connection between the matrix expression of the representations U_m (renamed as U_ℓ) and the special functions expressed in terms of Jacobi polynomials; see Vilenkin [16] (Chapter III, Sections 2 and 3).

For every matrix $S \in \mathbf{SU}(2)$, with

$$S = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad \alpha\overline{\alpha} + \beta\overline{\beta} = 1,$$

for every homogeneous polynomial $P \in \mathcal{P}^{\mathbb{C}}_m(2)$, $U_m(S)(P(z_1, z_2))$ is defined by

$$U_m(S)(P(z_1, z_2)) = P(\overline{\alpha} z_1 - \beta z_2, \overline{\beta} z_1 + \alpha z_2). \quad (U_m)$$

As defined, the representations U_m are not unitary, but since $\mathbf{SU}(2)$ is compact, we can apply Theorem 2.5 to find an invariant inner product on $\mathcal{P}^{\mathbb{C}}_m(2)$.

This can actually be done quite explicitly; we will come back to this point later.

Proposition 2.7. *The representations* $U_m\colon \mathbf{SU}(2) \to \mathbf{GL}(\mathcal{P}_m^{\mathbb{C}}(2))$ *are irreducible.*

Proof. To prove that the representations U_m are irreducible, it suffices to prove that every nonzero equivariant map A in $\mathrm{Hom}_{\mathbf{SU}(2)}(U_m, U_m)$ is invertible.

Actually, we will prove that $A = \lambda \mathrm{id}$, with $\lambda \in \mathbb{C}, \lambda \neq 0$.

A nice and rather short proof is given in Bröcker and tom Dieck [2], Chapter 2, Proposition 5.1.

The trick is to consider the matrices

$$r_x(\varphi) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \quad 0 < \varphi < \pi.$$

Plugging the matrix $r_x(\varphi)$ and $P = P_k = z_1^k z_2^{m-k}$ in Equation (U_m) yields

$$\begin{aligned} U_m(r_x(\varphi))(P_k) &= (e^{-i\varphi} z_1)^k (e^{i\varphi} z_2)^{m-k} \\ &= e^{i(m-2k)\varphi} z_1^k z_2^{m-k} = e^{i(m-2k)\varphi} P_k. \end{aligned}$$

Therefore, $(P_0, \ldots, P_m)$ is a basis (in fact, orthogonal) of eigenvectors of $U_m(r_x(\varphi))$ for the eigenvalues

$$(e^{im\varphi}, e^{i(m-2)\varphi}, \ldots, e^{-im\varphi}).$$

We can pick φ such that *these eigenvalues are all distinct*, for example $\varphi = 2\pi/m$.

Now if $A \in \mathrm{Hom}_{\mathbf{SU}(2)}(U_m, U_m)$ is equivariant, then

$$U_m(r_x(\varphi))A = AU_m(r_x(\varphi)),$$

so for $k = 0, \ldots, m$ we have

$$\begin{aligned} U_m(r_x(\varphi))AP_k &= AU_m(r_x(\varphi))P_k \\ &= Ae^{i(m-2k)\varphi}P_k = e^{i(m-2k)\varphi}AP_k. \end{aligned}$$

The above implies that either $AP_k = 0$ or AP_k is an eigenvector of $U_m(r_x(\varphi))$ for the eigenvalue $e^{i(m-2k)\varphi}$.

Since φ was chosen so that the eigenvalues $(e^{im\varphi}, \ldots, e^{i(m-2)\varphi}, \ldots, e^{-im\varphi})$ are all distinct, each eigenspace is one-dimensional, so $AP_k = c_k P_k$ for some $c_k \in \mathbb{C}$, $c_k \neq 0$.

In either case,

$$AP_k = c_k P_k$$

for some $c_k \in \mathbb{C}$.

We will now prove that $c_0 = c_1 = \cdots = c_m$.

This shows that $A = c_0 \mathrm{id}_{m+1}$, and since A is not the zero map, $c_0 \neq 0$, so A is invertible, as desired.

To prove that the c_k have the same value, we use the matrices

$$r_y(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad t \in \mathbb{R}.$$

Since A is equivariant,

$$AU_m(r_y(t)) = U_m(r_y(t))A,$$

so we need to compute $AU_m(r_y(t))P_m$ and $U_m(r_y(t))AP_m$.

Since $P_m = z_1^m$ and $AP_k = c_k P_k$, using Equation (U_m) we have

$$\begin{aligned} AU_m(r_y(t))P_m &= A(z_1 \cos t + z_2 \sin t)^m \\ &= A \sum_{k=1}^{m} \binom{m}{k} (\cos t)^k (\sin t)^{m-k} z_1^k z_2^{m-k} \\ &= \sum_{k=1}^{m} \binom{m}{k} (\cos t)^k (\sin t)^{m-k} AP_k \\ &= \sum_{k=1}^{m} \binom{m}{k} (\cos t)^k (\sin t)^{m-k} c_k P_k. \end{aligned}$$

We also have

$$\begin{aligned} U_m(r_y(t))AP_m &= U_m(r_y(t))c_mP_m = c_mU_m(r_y(t))P_m \\ &= c_m(z_1\cos t + z_2\sin t)^m \\ &= \sum_{k=1}^{m}\binom{m}{k}(\cos t)^k(\sin t)^{m-k}c_mP_k. \end{aligned}$$

Since $AU_m(r_y(t))P_m = U_m(r_y(t))AP_m$, comparing coefficients (since these equations hold for all $t \in \mathbb{R}$) we obtain

$$c_k = c_m, \quad 0 \le k \le m.$$

Therefore, on the basis $(P_0, \ldots, P_m)$ we have $AP_k = c_0P_k$, which means that $A = c_0\mathrm{id}_{m+1}$, as claimed. □

Therefore, the representations $U_m \colon \mathbf{SU}(2) \to \mathbf{GL}(\mathcal{P}^{\mathbb{C}}_m(2))$ are irreducible unitary representations of $\mathbf{SU}(2)$.

In fact, *they constitute all of them up to equivalence, but this is harder to prove*.

A good strategy is to use properties of the characters of compact groups; see Section 4.3.

Because there is a surjective homomorphism $\rho\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ whose kernel is $\{-I, I\}$, the irreducible representations of $\mathbf{SO}(3)$ can also be determined (up to equivalence).

Example 2.9. If $U : \mathbf{SO}(3) \to \mathbf{U}(H)$ is an irreducible unitary representation of $\mathbf{SO}(3)$, then $V = U \circ \rho$ is a unitary representation $V : \mathbf{SU}(2) \to \mathbf{U}(H)$ of $\mathbf{SU}(2)$ which must be irreducible, and $V(-I)$ is the identity.

Conversely, an irreducible unitary representation $V : \mathbf{SU}(2) \to \mathbf{U}(H)$ of $\mathbf{SU}(2)$ descends to an irreducible unitary representation $U : \mathbf{SO}(3) \to \mathbf{U}(H)$ iff $V(-I) = \mathrm{id}$.

Now by definition of U_m,

$$\begin{aligned} U_m(-I)(P_k) &= (-z_1)^k(-z_2)^{m-k} \\ &= (-1)^m z_1^k z_2^{m-k} = (-1)^m P_k. \end{aligned}$$

Therefore, $U_m(-I) = \mathrm{id}_{m+1}$ iff $(-1)^m = 1$ iff m is even. In summary we obtained the folowing result.

Proposition 2.8. *The unitary representations* $W_\ell \colon \mathbf{SO}(3) \to \mathbf{GL}(\mathcal{P}^{\mathbb{C}}_{2\ell}(2))$ *given by*

$$W_\ell(\rho_q) = U_{2\ell}(q) \quad q \in \mathbf{SU}(2), \quad \ell \geq 0$$

are irreducible. Observe that $\mathcal{P}^{\mathbb{C}}_{2\ell}(2)$ *has odd dimension* $2\ell + 1$.

We will prove later that *every irreducible unitary representation of $\mathbf{SU}(2)$ is equivalent to some representation U_m, and that every irreducible unitary representation of $\mathbf{SO}(3)$ is equivalent to some representation W_ℓ*; see Proposition **??**.

We will also present a more pleasant description of the irreducible unitary representation of $\mathbf{SO}(3)$ in terms of spaces of harmonic polynomials.

Remark: The representations
$U_m\colon \mathbf{SL}(2,\mathbb{C}) \to \mathbf{GL}(\mathcal{P}_m^{\mathbb{C}}(2))$ are not unitary, but they are irreducible.

There is a generalization of Schur's lemma to (complex) unitary representations, which says that if a unitary representation $U\colon G \to \mathbf{U}(H)$ is irreducible, then every G-map in $\mathrm{Hom}_G(U, U)$ is of the form $\alpha\, \mathrm{id}_H$, for some $\alpha \in \mathbb{C}$.

The proof requires much more machinery because a linear map on an infinite-dimensional vector space may not have eigenvectors!

It uses some results from the spectral theory of algebras, in particular, the complement to Theorem **??**.

Theorem 2.9. *(Schur's lemma for unitary representations) The following properties hold.*

(1) A (complex) unitary representation $U\colon G \to \mathbf{U}(H)$ is irreducible iff every G-map in $\mathcal{C}(U) = \mathrm{Hom}_G(U, U)$ is of the form $\alpha\, \mathrm{id}_H$, for some $\alpha \in \mathbb{C}$.

(2) Let $U_1\colon G \to \mathbf{U}(H_1)$ and $U_2\colon G \to \mathbf{U}(H_2)$ be two complex unitary representations. If U_1 and U_2 are equivalent, then $\mathrm{Hom}_G(U_1, U_2)$ is one-dimensional; otherwise we have $\mathrm{Hom}_G(U_1, U_2) = (0)$.

Proposition 2.10. *Every complex irreducible unitary representation* $U: G \to \mathbf{U}(H)$ *of a locally compact abelian group* G *in a Hilbert space* H *is one-dimensional.*

If the locally compact group G is abelian, then the following result shows that every irreducible unitary representation of G is uniquely defined by a *character* of G, as introduced in Vol I, Definition @@@10.1.

Proposition 2.11. *Let* G *be a locally compact abelian group. Every irreducible unitary representation* $U: G \to \mathbf{U}(1)$ *of* G *is of the form*

$$U(s)(z) = \chi(s)z, \quad \text{for all } s \in G \text{ and all } z \in \mathbb{C}$$

for a unique character $\chi \in \widehat{G}$.

Proof. If $U\colon G \to \mathbf{U}(1)$ is a unitary representation of G, then $U(s)$ is a unitary map of $\mathbb{C}$ for every $s \in G$, which means that there is a complex number of unit length, say $\chi(s) \in \mathbb{T}$, such that

$$U(s)(z) = \chi(s)z, \quad \text{for all } z \in \mathbb{C},$$

and for all $s_1, s_2 \in G$ we have

$$\chi(s_1 s_2)z = U(s_1 s_2)(z) = U(s_1)(U(s_2)(z)) = \chi(s_1)\chi(s_2)z$$

for all $z \in \mathbb{C}$,

which implies that

$$\chi(s_1 s_2) = \chi(s_1)\chi(s_2).$$

But then $\chi\colon G \to \mathbb{T}$ is a character of G, and so every unitary representation $U\colon G \to \mathbf{U}(1)$ of G is of the form

$$U(s)(z) = \chi(s)z, \quad \text{for all } s \in G \text{ and all } z \in \mathbb{C}$$

for a unique character $\chi \in \widehat{G}$. □

As an application of Theorem 2.5, Proposition 2.10 and Proposition 2.11, we describe *all finite-dimensional unitary representations of* $\mathbf{SO}(2) \simeq \mathbf{U}(1)$.

Here we use the isomorphism between $\mathbf{SO}(2)$ and $\mathbf{U}(1)$ given by

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto e^{i\theta},\ \theta \in [0, 2\pi).$$

Proposition 2.12. *Every finite-dimensional unitary representation* $U\colon \mathbf{SO}(2) \to \mathbf{U}(n)$ *of* $\mathbf{SO}(2) \simeq \mathbf{U}(1)$ $(n \geq 1)$ *is of the form*

$$U(e^{i\theta})(z) = \begin{pmatrix} e^{ik_1\theta} & 0 & \dots & 0 \\ 0 & e^{ik_2\theta} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & e^{ik_n\theta} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix},$$

$z \in \mathbb{C}^n$, $0 \leq \theta < 2\pi$, *for some* $k_1, \dots, k_n \in \mathbb{Z}$.

Proof. Since $\mathbf{SO}(2) \simeq \mathbf{U}(1)$ is compact and abelian, by Proposition 2.10, every irreducible unitary representation of $\mathbf{SO}(2) \simeq \mathbf{U}(1)$ is one-dimensional.

By Proposition 2.11, the irreducible unitary representations of $\mathbf{SO}(2) \simeq \mathbf{U}(1)$ are determined by the characters of $\mathbf{U}(1) = \mathbb{T}$.

By Vol I, Proposition @@@10.9(2), the characters of $\mathbf{U}(1) = \mathbb{T}$ are of the form

$$\chi_k(e^{i\theta}) = e^{ik\theta},$$

for some $k \in \mathbb{Z}$.

Since $\mathbf{SO}(2) \simeq \mathbf{U}(1)$ is compact, by Theorem 2.5, every finite-dimensional unitary representation
$U \colon \mathbf{SO}(2) \to \mathbf{U}(n)$ of $\mathbf{SO}(2)$ is the direct sum of n one-dimensional unitary representations $U_j \colon \mathbf{SO}(2) \to \mathbf{U}(1)$.

But each representation $U_j\colon \mathbf{SO}(2) \to \mathbf{U}(1)$ arises from a character of $\mathbf{U}(1)$, and so is of the form

$$U_j(e^{i\theta})(y) = e^{ik_j\theta}y, \quad y \in \mathbb{C},$$

for some $k_j \in \mathbb{Z}$.

The direct sum U of the representation $U_j\colon \mathbf{SO}(2) \to \mathbf{U}(1)$ acts on $\mathbb{C}^n$ as multiplication by a unitary matrix, namely

$$U(e^{i\theta}) = \begin{pmatrix} e^{ik_1\theta} & 0 & \dots & 0 \\ 0 & e^{ik_2\theta} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & e^{ik_n\theta} \end{pmatrix},$$

as claimed. $\square$

Remark: Let Q be the $n \times n$ matrix given by

$$Q = \begin{pmatrix} ik_1 & 0 & \dots & 0 \\ 0 & ik_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & ik_n \end{pmatrix}.$$

Observe that Q is skew-symmetric, so that $Q \in \mathfrak{u}(n)$, and we have

$$U(e^{i\theta}) = e^{\theta Q}.$$