Chapter 9

The Dual Space, Duality

9.1 The Dual Space E^* and Linear Forms

In Section 1.7 we defined linear forms, the dual space $E^* = \mathrm{Hom}(E, K)$ of a vector space E, and showed the existence of dual bases for vector spaces of finite dimension.

In this chapter, we take a deeper look at the connection between a space E and its dual space E^*.

As we will see shortly, every linear map $f\colon E \to F$ gives rise to a linear map $f^\top\colon F^* \to E^*$, and it turns out that in a suitable basis, the matrix of $f^\top$ is the *transpose* of the matrix of f.

Thus, the notion of dual space provides a conceptual explanation of the phenomena associated with transposition.

But it does more, because it allows us to view subspaces as solutions of sets of linear equations and vice-versa.

Consider the following set of two "linear equations" in $\mathbb{R}^3$,

$$\begin{aligned} x - y + z &= 0 \\ x - y - z &= 0, \end{aligned}$$

and let us find out what is their set V of common solutions $(x, y, z) \in \mathbb{R}^3$.

By subtracting the second equation from the first, we get $2z = 0$, and by adding the two equations, we find that $2(x - y) = 0$, so the set V of solutions is given by

$$\begin{aligned} y &= x \\ z &= 0. \end{aligned}$$

This is a one dimensional subspace of $\mathbb{R}^3$. Geometrically, this is the line of equation $y = x$ in the plane $z = 0$.

Now, why did we say that the above equations are linear?

This is because, as functions of (x, y, z), both maps $f_1\colon (x, y, z) \mapsto x - y + z$ and $f_2\colon (x, y, z) \mapsto x - y - z$ are linear.

The set of all such linear functions from $\mathbb{R}^3$ to $\mathbb{R}$ is a vector space; we used this fact to form linear combinations of the "equations" f_1 and f_2.

Observe that the dimension of the subspace V is 1.

The ambient space has dimension $n = 3$ and there are two "independent" equations f_1, f_2, so it appears that the dimension $\dim(V)$ of the subspace V defined by m independent equations is

$$\dim(V) = n - m,$$

which is indeed a general fact (proved in Theorem 9.1).

More generally, in $\mathbb{R}^n$, a linear equation is determined by an n-tuple $(a_1, \ldots, a_n) \in \mathbb{R}^n$, and the solutions of this linear equation are given by the n-tuples $(x_1, \ldots, x_n) \in \mathbb{R}^n$ such that

$$a_1 x_1 + \cdots + a_n x_n = 0;$$

these solutions constitute the kernel of the linear map $(x_1, \ldots, x_n) \mapsto a_1 x_1 + \cdots + a_n x_n$.

The above considerations assume that we are working in the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$, but we can define "linear equations" independently of bases and in any dimension, by viewing them as elements of the vector space $\mathrm{Hom}(E, K)$ of linear maps from E to the field K.

Definition 9.1. Given a vector space E, the vector space $\mathrm{Hom}(E, K)$ of linear maps from E to K is called the *dual space (or dual)* of E. The space $\mathrm{Hom}(E, K)$ is also denoted by E^*, and the linear maps in E^* are called *the linear forms*, or *covectors*. The dual space E^{**} of the space E^* is called the *bidual* of E.

As a matter of notation, linear forms $f \colon E \to K$ will also be denoted by starred symbol, such as u^*, x^*, etc.

Given a vector space E and any basis $(u_i)_{i\in I}$ for E, we can associate to each u_i a linear form $u_i^* \in E^*$, and the u_i^* have some remarkable properties.

Definition 9.2. Given a vector space E and any basis $(u_i)_{i\in I}$ for E, by Proposition 1.14, for every $i \in I$, there is a unique linear form u_i^* such that

$$u_i^*(u_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

for every $j \in I$. The linear form u_i^* is called the *coordinate form* of index i w.r.t. the basis $(u_i)_{i\in I}$.

The reason for the terminology *coordinate form* was explained in Section 1.7.

We proved in Theorem 1.17 that if $(u_1, \ldots, u_n)$ is a basis of E, then $(u_1^*, \ldots, u_n^*)$ is a basis of E^* called the *dual basis*.

If $(u_1, \ldots, u_n)$ is a basis of $\mathbb{R}^n$ (more generally K^n), it is possible to find explicitly the dual basis $(u_1^*, \ldots, u_n^*)$, where each u_i^* is represented by a row vector.

For example, consider the columns of the Bézier matrix

$$B_4 = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The form u_1^* is represented by a row vector $(\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4)$ such that

$$\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4\end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}.$$

This implies that u_1^* is the first row of the inverse of B_4.

Since

$$B_4^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & 1/3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the linear forms $(u_1^*, u_2^*, u_3^*, u_4^*)$ correspond to the rows of B_4^{-1}.

In particular, u_1^* is represented by $(1\ 1\ 1\ 1)$.

The above method works for any n. Given any basis $(u_1, \ldots, u_n)$ of $\mathbb{R}^n$, if P is the $n \times n$ matrix whose jth column is u_j, then the dual form u_i^* is given by the ith row of the matrix P^{-1}.

When E is of finite dimension n and $(u_1, \ldots, u_n)$ is a basis of E, we noted that the family $(u_1^*, \ldots, u_n^*)$ is a basis of the dual space E^*,

Let us see how the coordinates of a linear form $\varphi^* \in E^*$ over the basis $(u_1^*, \ldots, u_n^*)$ vary under a change of basis.

Let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be two bases of E, and let $P = (a_{i\,j})$ be the change of basis matrix from $(u_1, \ldots, u_n)$ to $(v_1, \ldots, v_n)$, so that

$$v_j = \sum_{i=1}^{n} a_{i\,j} u_i.$$

If

$$\varphi^* = \sum_{i=1}^{n} \varphi_i u_i^* = \sum_{i=1}^{n} \varphi_i' v_i^*,$$

after some algebra, we get

$$\varphi_j' = \sum_{i=1}^{n} a_{i\,j} \varphi_i.$$

Comparing with the change of basis

$$v_j = \sum_{i=1}^{n} a_{i\,j} u_i,$$

we note that this time, the coordinates (φ_i) of the linear form φ^* change in the *same direction* as the change of basis.

For this reason, we say that the coordinates of linear forms are *covariant*.

By abuse of language, it is often said that linear forms are *covariant*, which explains why the term *covector* is also used for a linear form.

Observe that if $(e_1, \ldots, e_n)$ is a basis of the vector space E, then, as a linear map from E to K, every linear form $f \in E^*$ is represented by a $1 \times n$ matrix, that is, by a *row vector*

$$(\lambda_1 \; \cdots \; \lambda_n),$$

with respect to the basis $(e_1, \ldots, e_n)$ of E, and 1 of K, where $f(e_i) = \lambda_i$.

A vector $u = \sum_{i=1}^{n} u_i e_i \in E$ is represented by a $n \times 1$ matrix, that is, by a *column vector*

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix},$$

and the action of f on u, namely $f(u)$, is represented by the matrix product

$$\begin{pmatrix} \lambda_1 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \lambda_1 u_1 + \cdots + \lambda_n u_n.$$

On the other hand, with respect to the dual basis $(e_1^*, \ldots, e_n^*)$ of E^*, the linear form f is represented by the column vector

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}.$$

9.2 Pairing and Duality Between E and E^*

Given a linear form $u^* \in E^*$ and a vector $v \in E$, the result $u^*(v)$ of applying u^* to v is also denoted by

$$\langle u^*, v\rangle = u^*(v).$$

This defines a binary operation $\langle -, -\rangle \colon E^* \times E \to K$ satisfying the following properties:

$$\begin{aligned} \langle u_1^* + u_2^*, v\rangle &= \langle u_1^*, v\rangle + \langle u_2^*, v\rangle \\ \langle u^*, v_1 + v_2\rangle &= \langle u^*, v_1\rangle + \langle u^*, v_2\rangle \\ \langle \lambda u^*, v\rangle &= \lambda\langle u^*, v\rangle \\ \langle u^*, \lambda v\rangle &= \lambda\langle u^*, v\rangle. \end{aligned}$$

The above identities mean that $\langle -, -\rangle$ is a *bilinear map*, since it is linear in each argument.

It is often called the *canonical pairing* between E^* and E.

In view of the above identities, given any fixed vector $v \in E$, the map $\mathrm{eval}_v\colon E^* \to K$ (*evaluation at v*) defined such that

$$\mathrm{eval}_v(u^*) = \langle u^*, v\rangle = u^*(v) \quad \text{for every } u^* \in E^*$$

is a linear map from E^* to K, that is, eval_v is a linear form in E^{**}.

Again from the above identities, the map
$\mathrm{eval}_E\colon E \to E^{**}$, defined such that

$$\mathrm{eval}_E(v) = \mathrm{eval}_v \quad \text{for every } v \in E,$$

is a linear map.

We shall see that it is injective, and that it is an isomorphism when E has finite dimension.

We now formalize the notion of the set V^0 of linear equations vanishing on all vectors in a given subspace $V \subseteq E$, and the notion of the set U^0 of common solutions of a given set $U \subseteq E^*$ of linear equations.

The duality theorem (Theorem 9.1) shows that the dimensions of V and V^0, and the dimensions of U and U^0, are related in a crucial way.

It also shows that, in finite dimension, the maps $V \mapsto V^0$ and $U \mapsto U^0$ are inverse bijections from subspaces of E to subspaces of E^*.

Definition 9.3. Given a vector space E and its dual E^*, we say that a vector $v \in E$ and a linear form $u^* \in E^*$ are *orthogonal* iff $\langle u^*, v\rangle = 0$. Given a subspace V of E and a subspace U of E^*, we say that *V and U are orthogonal* iff $\langle u^*, v\rangle = 0$ for every $u^* \in U$ and every $v \in V$. Given a subset V of E (resp. a subset U of E^*), the *orthogonal V^0 of V* is the subspace V^0 of E^* defined such that

$$V^0 = \{u^* \in E^* \mid \langle u^*, v\rangle = 0, \text{ for every } v \in V\}$$

(resp. the *orthogonal U^0 of U* is the subspace U^0 of E defined such that

$$U^0 = \{v \in E \mid \langle u^*, v\rangle = 0, \text{ for every } u^* \in U\}).$$

The subspace $V^0 \subseteq E^*$ is also called the *annihilator* of V.

The subspace $U^0 \subseteq E$ annihilated by $U \subseteq E^*$ does not have a special name. It seems reasonable to call it the *linear subspace (or linear variety) defined by U*.

Informally, V^0 is the *set of linear equations that vanish on V*, and U^0 is *the set of common zeros of all linear equations in U*. We can also define V^0 by

$$V^0 = \{u^* \in E^* \mid V \subseteq \operatorname{Ker} u^*\}$$

and U^0 by

$$U^0 = \bigcap_{u^* \in U} \operatorname{Ker} u^*.$$

Observe that $E^0 = \{0\} = (0)$, and $\{0\}^0 = E^*$.

Furthermore, if $V_1 \subseteq V_2 \subseteq E$, then $V_2^0 \subseteq V_1^0 \subseteq E^*$, and if $U_1 \subseteq U_2 \subseteq E^*$, then $U_2^0 \subseteq U_1^0 \subseteq E$.

It can also be shown that that $V \subseteq V^{00}$ for every subspace V of E, and that $U \subseteq U^{00}$ for every subspace U of E^*.

We will see shortly that in finite dimension, we have

$$V = V^{00} \quad \text{and} \quad U = U^{00}.$$

Here are some examples. Let $E = \mathrm{M}_2(\mathbb{R})$, the space of real 2×2 matrices, and let V be the subspace of $\mathrm{M}_2(\mathbb{R})$ spanned by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We check immediately that the subspace V consists of all matrices of the form

$$\begin{pmatrix} b & a \\ a & c \end{pmatrix},$$

that is, all symmetric matrices.

The matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

in V satisfy the equation

$$a_{12} - a_{21} = 0,$$

and all scalar multiples of these equations, so V^0 is the subspace of E^* spanned by the linear form given by

$$u^*(a_{11}, a_{12}, a_{21}, a_{22}) = a_{12} - a_{21}.$$

By the duality theorem (Theorem 9.1) we have

$$\dim(V^0) = \dim(E) - \dim(V) = 4 - 3 = 1.$$

The above example generalizes to $E = \mathrm{M}_n(\mathbb{R})$ for any $n \geq 1$, but this time, consider the space U of linear forms asserting that a matrix A is symmetric; these are the linear forms spanned by the $n(n-1)/2$ equations

$$a_{ij} - a_{ji} = 0, \quad 1 \leq i < j \leq n;$$

Note there are no constraints on diagonal entries, and half of the equations

$$a_{ij} - a_{ji} = 0, \quad 1 \leq i \neq j \leq n$$

are redundant. It is easy to check that the equations (linear forms) for which $i < j$ are linearly independent.

To be more precise, let U be the space of linear forms in E^* spanned by the linear forms

$$\begin{aligned} u^*_{ij}(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}) \\ = a_{ij} - a_{ji}, \quad 1 \leq i < j \leq n. \end{aligned}$$

The dimension of U is $n(n-1)/2$. Then, the set U^0 of common solutions of these equations is the space $\mathbf{S}(n)$ of symmetric matrices.

By the duality theorem (Theorem 9.1), this space has dimension

$$\frac{n(n+1)}{2} = n^2 - \frac{n(n-1)}{2}.$$

If $E = \mathrm{M}_n(\mathbb{R})$, consider the subspace U of linear forms in E^* spanned by the linear forms

$$
\begin{aligned}
u^*_{ij}(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}) \\
= a_{ij} + a_{ji}, \quad 1 \leq i < j \leq n \\
u^*_{ii}(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}) \\
= a_{ii}, \quad 1 \leq i \leq n.
\end{aligned}
$$

It is easy to see that these linear forms are linearly independent, so $\dim(U) = n(n+1)/2$.

The space U^0 of matrices $A \in \mathrm{M}_n(\mathbb{R})$ satifying all of the above equations is clearly the space $\mathbf{Skew}(n)$ of skew-symmetric matrices.

By the duality theorem (Theorem 9.1), the dimension of U^0 is

$$
\frac{n(n-1)}{2} = n^2 - \frac{n(n+1)}{2}.
$$

For yet another example with $E = \mathrm{M}_n(\mathbb{R})$, for any $A \in \mathrm{M}_n(\mathbb{R})$, consider the linear form in E^* given by

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn},$$

called the *trace* of A.

The subspace U^0 of E consisting of all matrices A such that $\mathrm{tr}(A) = 0$ is a space of dimension $n^2 - 1$.

The dimension equations

$$\begin{aligned} \dim(V) + \dim(V^0) &= \dim(E) \\ \dim(U) + \dim(U^0) &= \dim(E) \end{aligned}$$

are always true (if E is finite-dimensional). This is part of the duality theorem (Theorem 9.1).

In constrast with the previous examples, given a matrix $A \in \mathrm{M}_n(\mathbb{R})$, the equations asserting that $A^\top A = I$ are not linear constraints.

For example, for $n = 2$, we have

$$\begin{aligned} a_{11}^2 + a_{21}^2 &= 1 \\ a_{21}^2 + a_{22}^2 &= 1 \\ a_{11}a_{12} + a_{21}a_{22} &= 0. \end{aligned}$$

We have the following important duality theorem adapted from E. Artin.

9.3 The Duality Theorem

Theorem 9.1. *(Duality theorem) Let E be a vector space of dimension n. The following properties hold:*

(a) For every basis $(u_1, \ldots, u_n)$ of E, the family of coordinate forms $(u_1^*, \ldots, u_n^*)$ is a basis of E^*.

(b) For every subspace V of E, we have $V^{00} = V$.

(c) For every pair of subspaces V and W of E such that $E = V \oplus W$, with V of dimension m, for every basis $(u_1, \ldots, u_n)$ of E such that $(u_1, \ldots, u_m)$ is a basis of V and $(u_{m+1}, \ldots, u_n)$ is a basis of W, the family $(u_1^*, \ldots, u_m^*)$ is a basis of the orthogonal W^0 of W in E^*. Furthermore, we have $W^{00} = W$, and

$$\dim(W) + \dim(W^0) = \dim(E).$$

(d) For every subspace U of E^*, we have

$$\dim(U) + \dim(U^0) = \dim(E),$$

where U^0 is the orthogonal of U in E, and $U^{00} = U$.

Part (a) of Theorem 9.1 shows that

$$\dim(E) = \dim(E^*),$$

and if $(u_1, \ldots, u_n)$ is a basis of E, then $(u_1^*, \ldots, u_n^*)$ is a basis of the dual space E^* called the *dual basis* of $(u_1, \ldots, u_n)$.

Define the function $\mathcal{E}$ ($\mathcal{E}$ for *equations*) from subspaces of E to subspaces of E^* and the function $\mathcal{Z}$ ($\mathcal{Z}$ for *zeros*) from subspaces of E^* to subspaces of E by

$$\begin{aligned} \mathcal{E}(V) &= V^0, \quad V \subseteq E \\ \mathcal{Z}(U) &= U^0, \quad U \subseteq E^*. \end{aligned}$$

By part (c) and (d) of theorem 9.1,

$$\begin{aligned} (\mathcal{Z} \circ \mathcal{E})(V) &= V^{00} = V \\ (\mathcal{E} \circ \mathcal{Z})(U) &= U^{00} = U, \end{aligned}$$

so $\mathcal{Z} \circ \mathcal{E} = \mathrm{id}$ and $\mathcal{E} \circ \mathcal{Z} = \mathrm{id}$, and the maps $\mathcal{E}$ and $\mathcal{V}$ are inverse bijections.

These maps set up a *duality* between subspaces of E, and subspaces of E^*.

One should be careful that this bijection does not hold if E has infinite dimension. Some restrictions on the dimensions of U and V are needed.

Suppose that V is a subspace of $\mathbb{R}^n$ of dimension m and that $(v_1, \ldots, v_m)$ is a basis of V.

To find a basis of V^0, we first extend $(v_1, \ldots, v_m)$ to a basis $(v_1, \ldots, v_n)$ of $\mathbb{R}^n$, and then by part (c) of Theorem 9.1, we know that $(v^*_{m+1}, \ldots, v^*_n)$ is a basis of V^0.

Here is another example illustrating the power of Theorem 9.1.

Let $E = \mathrm{M}_n(\mathbb{R})$, and consider the equations asserting that the sum of the entries in every row of a matrix $A \in \mathrm{M}_n(\mathbb{R})$ is equal to the same number.

We have $n - 1$ equations

$$\sum_{j=1}^{n} (a_{ij} - a_{i+1j}) = 0, \quad 1 \leq i \leq n-1,$$

and it is easy to see that they are linearly independent.

Therefore, the space U of linear forms in E^* spanned by the above linear forms (equations) has dimension $n - 1$, and the space U^0 of matrices sastisfying all these equations has dimension $n^2 - n + 1$.

It is not so obvious to find a basis for this space.

We now discuss some applications of the duality theorem.

Problem 1 . Suppose that V is a subspace of $\mathbb{R}^n$ of dimension m and that $(v_1, \ldots, v_m)$ is a basis of V. The problem is to find a basis of V^0.

We first extend $(v_1, \ldots, v_m)$ to a basis $(v_1, \ldots, v_n)$ of $\mathbb{R}^n$, and then by part (c) of Theorem 9.1, we know that $(v_{m+1}^*, \ldots, v_n^*)$ is a basis of V^0.

Example 9.1. For example, suppose that V is the subspace of $\mathbb{R}^4$ spanned by the two linearly independent vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix},$$

the first two vectors of the Haar basis in $\mathbb{R}^4$.

The four columns of the Haar matrix

$$W = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}$$

form a basis of $\mathbb{R}^4$, and the inverse of W is given by

$$\begin{aligned} W^{-1} &= \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix}. \end{aligned}$$

Since the dual basis $(v_1^*, v_2^*, v_3^*, v_4^*)$ is given by the rows of W^{-1}, the last two rows of W^{-1},

$$\begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix},$$

form a basis of V^0. We also obtain a basis by rescaling by the factor $1/2$, so the linear forms given by the row vectors

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

form a basis of V^0, the space of linear forms (linear equations) that vanish on the subspace V.

The method that we described to find V^0 requires first extending a basis of V and then inverting a matrix, but there is a more direct method.

Indeed, let A be the $n \times m$ matrix whose columns are the basis vectors $(v_1, \ldots, v_m)$ of V. Then a linear form u represented by a row vector belongs to V^0 iff $uv_i = 0$ for $i = 1, \ldots, m$ iff

$$uA = 0$$

iff

$$A^\top u^\top = 0.$$

Therefore, all we need to do is to find a basis of the nullspace of $A^\top$. This can be done quite effectively using the reduction of a matrix to reduced row echelon form (rref); see Section 6.9.

Example 9.2. For example, if we reconsider the previous example, $A^\top u^\top = 0$ becomes

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the rref of $A^\top$ is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

the above system is equivalent to

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_1 + u_2 \\ u_3 + u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the free variables are associated with u_2 and u_4. Thus to determine a basis for the kernel of $A^\top$, we set $u_2 = 1, u_4 = 0$ and $u_2 = 0, u_4 = 1$ and obtain a basis for V^0 as

$$\begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 1 & -1 \end{pmatrix}.$$

Problem 2 . Let us now consider the problem of finding a basis of the hyperplane H in $\mathbb{R}^n$ defined by the equation

$$c_1x_1 + \cdots + c_nx_n = 0.$$

More precisely, if $u^*(x_1, \ldots, x_n)$ is the linear form in $(\mathbb{R}^n)^*$ given by $u^*(x_1, \ldots, x_n) = c_1x_1 + \cdots + c_nx_n$, then the hyperplane H is the kernel of u^*.

Of course we assume that some c_j is nonzero, in which case the linear form u^* spans a one-dimensional subspace U of $(\mathbb{R}^n)^*$, and $U^0 = H$ has dimension $n - 1$.

Since u^* is not the linear form which is identically zero, there is a *smallest positive index* $j \leq n$ such that $c_j \neq 0$, so our linear form is really

$$u^*(x_1, \ldots, x_n) = c_jx_j + \cdots + c_nx_n.$$

We claim that the following $n-1$ vectors (in $\mathbb{R}^n$) form a basis of H:

$$
\begin{array}{c}
\\
1 \\ 2 \\ \vdots \\ j-1 \\ j \\ j+1 \\ j+2 \\ \vdots \\ n
\end{array}
\begin{array}{c}
\begin{array}{cccccccc}
1 & 2 & \ldots & j-1 & j & j+1 & \ldots & n-1
\end{array} \\
\begin{pmatrix}
1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & -c_{j+1}/c_j & -c_{j+2}/c_j & \ldots & -c_n/c_j \\
0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\end{array}.
$$

Observe that the $(n-1)\times(n-1)$ matrix obtained by deleting row j is the identity matrix, so the columns of the above matrix are linearly independent.

A simple calculation also shows that the linear form $u^*(x_1, \ldots, x_n) = c_j x_j + \cdots + c_n x_n$ vanishes on every column of the above matrix.

For a concrete example in $\mathbb{R}^6$, if

$$u^*(x_1, \ldots, x_6) = x_3 + 2x_4 + 3x_5 + 4x_6,$$

we obtain the basis for the hyperplane H of equation

$$x_3 + 2x_4 + 3x_5 + 4x_6 = 0$$

given by the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -3 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Problem 3 . Conversely, given a hyperplane H in $\mathbb{R}^n$ given as the span of $n-1$ linearly vectors $(u_1, \ldots, u_{n-1})$, it is possible using determinants to find a linear form $(\lambda_1, \ldots, \lambda_n)$ that vanishes on H.

In the case $n = 3$, we are looking for a row vector $(\lambda_1, \lambda_2, \lambda_3)$ such that if

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

are two linearly independent vectors, then

$$\begin{pmatrix} u_1 & u_2 & u_2 \\ v_1 & v_2 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and the cross-product $u \times v$ of u and v given by

$$u \times v = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$$

is a solution.

In other words, the equation of the plane spanned by u and v is

$$(u_2v_3 - u_3v_2)x + (u_3v_1 - u_1v_3)y + (u_1v_2 - u_2v_1)z = 0.$$

We will now pin down the relationship between a vector space E and its bidual E^{**}.

Proposition 9.2. *Let E be a vector space. The following properties hold:*

(a) The linear map $\mathrm{eval}_E\colon E \to E^{**}$ *defined such that*

$$\mathrm{eval}_E(v) = \mathrm{eval}_v, \quad \textit{for all } v \in E,$$

that is, $\mathrm{eval}_E(v)(u^*) = \langle u^*, v\rangle = u^*(v)$ *for every* $u^* \in E^*$*, is injective.*

(b) When E is of finite dimension n, the linear map $\mathrm{eval}_E\colon E \to E^{**}$ *is an isomorphism (called the canonical isomorphism).*

When E is of finite dimension and $(u_1, \ldots, u_n)$ is a basis of E, in view of the canonical isomorphism $\mathrm{eval}_E\colon E \to E^{**}$, the basis $(u_1^{**}, \ldots, u_n^{**})$ of the bidual is identified with $(u_1, \ldots, u_n)$.

Proposition 9.2 can be reformulated very fruitfully in terms of pairings.

Definition 9.4. Given two vector spaces E and F over K, a *pairing between E and F* is a bilinear map $\varphi\colon E \times F \to K$. Such a pairing is *nondegenerate* iff

(1) for every $u \in E$, if $\varphi(u, v) = 0$ for all $v \in F$, then $u = 0$, and

(2) for every $v \in F$, if $\varphi(u, v) = 0$ for all $u \in E$, then $v = 0$.

A pairing $\varphi\colon E \times F \to K$ is often denoted by $\langle -, - \rangle\colon E \times F \to K$.

For example, the map $\langle -, - \rangle\colon E^* \times E \to K$ defined earlier is a nondegenerate pairing (use the proof of (a) in Proposition 9.2).

Given a pairing $\varphi\colon E \times F \to K$, we can define two maps $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ as follows:

For every $u \in E$, we define the linear form $l_\varphi(u)$ in F^* such that

$$l_\varphi(u)(y) = \varphi(u, y) \quad \text{for every } y \in F,$$

and for every $v \in F$, we define the linear form $r_\varphi(v)$ in E^* such that

$$r_\varphi(v)(x) = \varphi(x, v) \quad \text{for every } x \in E.$$

Proposition 9.3. *Given two vector spaces E and F over K, for every nondegenerate pairing $\varphi\colon E \times F \to K$ between E and F, the maps $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ are linear and injective. Furthermore, if E and F have finite dimension, then this dimension is the same and $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ are bijections.*

When E has finite dimension, the nondegenerate pairing $\langle -, - \rangle \colon E^* \times E \to K$ yields another proof of the existence of a natural isomorphism between E and E^{**}.

Interesting nondegenerate pairings arise in exterior algebra.

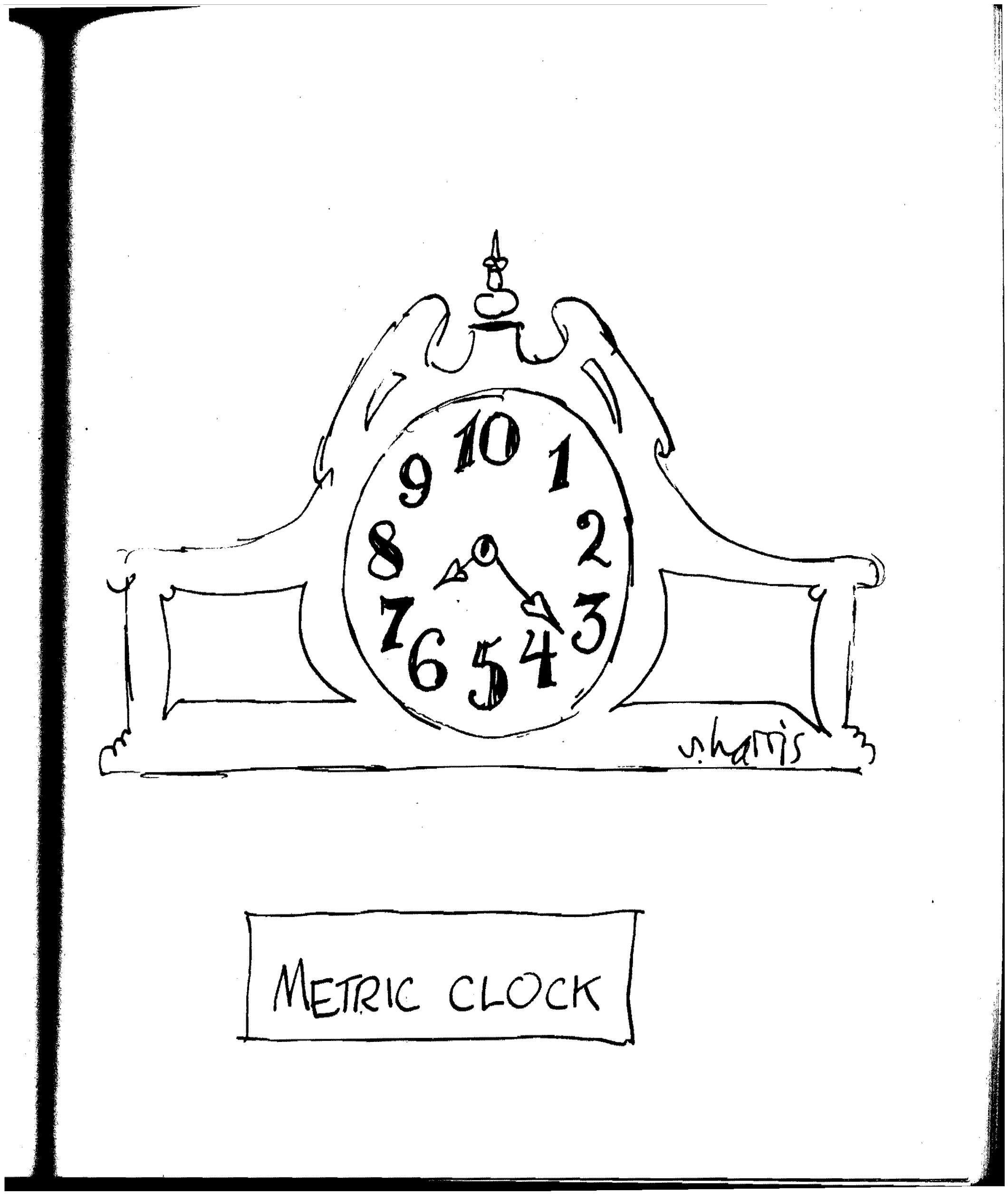


Figure 9.1: Metric Clock

9.4 Hyperplanes and Linear Forms

Actually, Proposition 9.4 below follows from parts (c) and (d) of Theorem 9.1, but we feel that it is also interesting to give a more direct proof.

Proposition 9.4. *Let E be a vector space. The following properties hold:*

(a) Given any nonnull linear form $f^* \in E^*$, its kernel $H = \mathrm{Ker}\, f^*$ is a hyperplane.

(b) For any hyperplane H in E, there is a (nonnull) linear form $f^* \in E^*$ such that $H = \mathrm{Ker}\, f^*$.

(c) Given any hyperplane H in E and any (nonnull) linear form $f^* \in E^*$ such that $H = \mathrm{Ker}\, f^*$, for every linear form $g^* \in E^*$, $H = \mathrm{Ker}\, g^*$ iff $g^* = \lambda f^*$ for some $\lambda \neq 0$ in K.

We leave as an exercise the fact that every subspace $V \neq E$ of a vector space E, is the intersection of all hyperplanes that contain V.

We now consider the notion of transpose of a linear map and of a matrix.

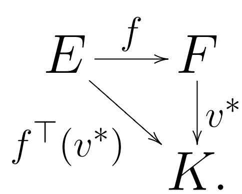
9.5 Transpose of a Linear Map and of a Matrix

Given a linear map $f\colon E \to F$, it is possible to define a map $f^\top\colon F^* \to E^*$ which has some interesting properties.

Definition 9.5. Given a linear map $f\colon E \to F$, the *transpose* $f^\top\colon F^* \to E^*$ of f is the linear map defined such that

$$f^\top(v^*) = v^* \circ f,$$

for every $v^* \in F^*$, as shown in the diagram below:



Equivalently, the linear map $f^\top\colon F^* \to E^*$ is defined such that

$$\langle v^*, f(u)\rangle = \langle f^\top(v^*), u\rangle,$$

for all $u \in E$ and all $v^* \in F^*$.

It is easy to verify that the following properties hold:

$$
\begin{aligned}
(f+g)^\top &= f^\top + g^\top\\
(g\circ f)^\top &= f^\top \circ g^\top\\
\mathrm{id}_E^\top &= \mathrm{id}_{E^*}.
\end{aligned}
$$

Note the reversal of composition on the right-hand side of $(g\circ f)^\top = f^\top \circ g^\top$.

The equation $(g\circ f)^\top = f^\top \circ g^\top$ implies the following useful proposition.

Proposition 9.5. *If $f\colon E \to F$ is any linear map, then the following properties hold:*

(1) If f is injective, then $f^\top$ is surjective.

(2) If f is surjective, then $f^\top$ is injective.

We also have the following property showing the naturality of the eval map.

Proposition 9.6. *For any linear map $f \colon E \to F$, we have*

$$f^{\top\top} \circ \mathrm{eval}_E = \mathrm{eval}_F \circ f,$$

or equivalently, the following diagram commutes:

$$\begin{array}{ccc} E^{**} & \xrightarrow{f^{\top\top}} & F^{**} \\ {\scriptstyle \mathrm{eval}_E}\uparrow & & \uparrow{\scriptstyle \mathrm{eval}_F} \\ E & \xrightarrow[f]{} & F. \end{array}$$

If E and F are finite-dimensional, then eval_E and eval_F are isomorphisms, so Proposition 9.6 shows that

$$f^{\top\top} = \mathrm{eval}_F^{-1} \circ f \circ \mathrm{eval}_E. \tag{$*$}$$

The above equation is often interpreted as follows: if we identify E with its bidual E^{**} and F with its bidual F^{**}, then $f^{\top\top} = f$.

This is an abuse of notation; the rigorous statement is $(*)$.

The following proposition shows the relationship between orthogonality and transposition.

Proposition 9.7. *Given a linear map $f\colon E \to F$, for any subspace V of E, we have*

$$f(V)^0 = (f^\top)^{-1}(V^0) = \{w^* \in F^* \mid f^\top(w^*) \in V^0\}.$$

As a consequence,

$$\mathrm{Ker}\, f^\top = (\mathrm{Im}\, f)^0 \quad \textit{and} \quad \mathrm{Ker}\, f = (\mathrm{Im}\, f^\top)^0.$$

The following theorem shows the relationship between the rank of f and the rank of $f^\top$.

Theorem 9.8. *Given a linear map $f\colon E \to F$, the following properties hold.*

(a) The dual $(\operatorname{Im} f)^*$ *of* $\operatorname{Im} f$ *is isomorphic to* $\operatorname{Im} f^\top = f^\top(F^*)$*; that is,*

$$(\operatorname{Im} f)^* \approx \operatorname{Im} f^\top.$$

(b) If F is finite dimensional, then $\operatorname{rk}(f) = \operatorname{rk}(f^\top)$*.*

The following proposition can be shown, but it requires a generalization of the duality theorem.

Proposition 9.9. *If $f\colon E \to F$ is any linear map, then the following identities hold:*

$$\begin{aligned}
\operatorname{Im} f^\top &= (\operatorname{Ker}(f))^0 \\
\operatorname{Ker}(f^\top) &= (\operatorname{Im} f)^0 \\
\operatorname{Im} f &= (\operatorname{Ker}(f^\top))^0 \\
\operatorname{Ker}(f) &= (\operatorname{Im} f^\top)^0.
\end{aligned}$$

The following proposition shows the relationship between the matrix representing a linear map $f\colon E \to F$ and the matrix representing its transpose $f^\top\colon F^* \to E^*$.

Proposition 9.10. *Let E and F be two vector spaces, and let $(u_1, \ldots, u_n)$ be a basis for E, and $(v_1, \ldots, v_m)$ be a basis for F. Given any linear map $f\colon E \to F$, if $M(f)$ is the $m \times n$-matrix representing f w.r.t. the bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_m)$, the $n \times m$-matrix $M(f^\top)$ representing $f^\top\colon F^* \to E^*$ w.r.t. the dual bases $(v_1^*, \ldots, v_m^*)$ and $(u_1^*, \ldots, u_n^*)$ is the transpose $M(f)^\top$ of $M(f)$.*

We now can give a very short proof of the fact that the rank of a matrix is equal to the rank of its transpose.

Proposition 9.11. *Given a $m \times n$ matrix A over a field K, we have* $\mathrm{rk}(A) = \mathrm{rk}(A^\top)$.

Thus, given an $m \times n$-matrix A, the maximum number of linearly independent columns is equal to the maximum number of linearly independent rows.

Proposition 9.12. *Given any $m \times n$ matrix A over a field K (typically $K = \mathbb{R}$ or $K = \mathbb{C}$), the rank of A is the maximum natural number r such that there is an invertible $r \times r$ submatrix of A obtained by selecting r rows and r columns of A.*

For example, the 3×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

has rank 2 iff one of the three 2×2 matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix} \quad \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

is invertible. We saw in Chapter 5 that this is equivalent to the fact the determinant of one of the above matrices is nonzero.

This is not a very efficient way of finding the rank of a matrix. We will see that there are better ways using various decompositions such as LU, QR, or SVD.

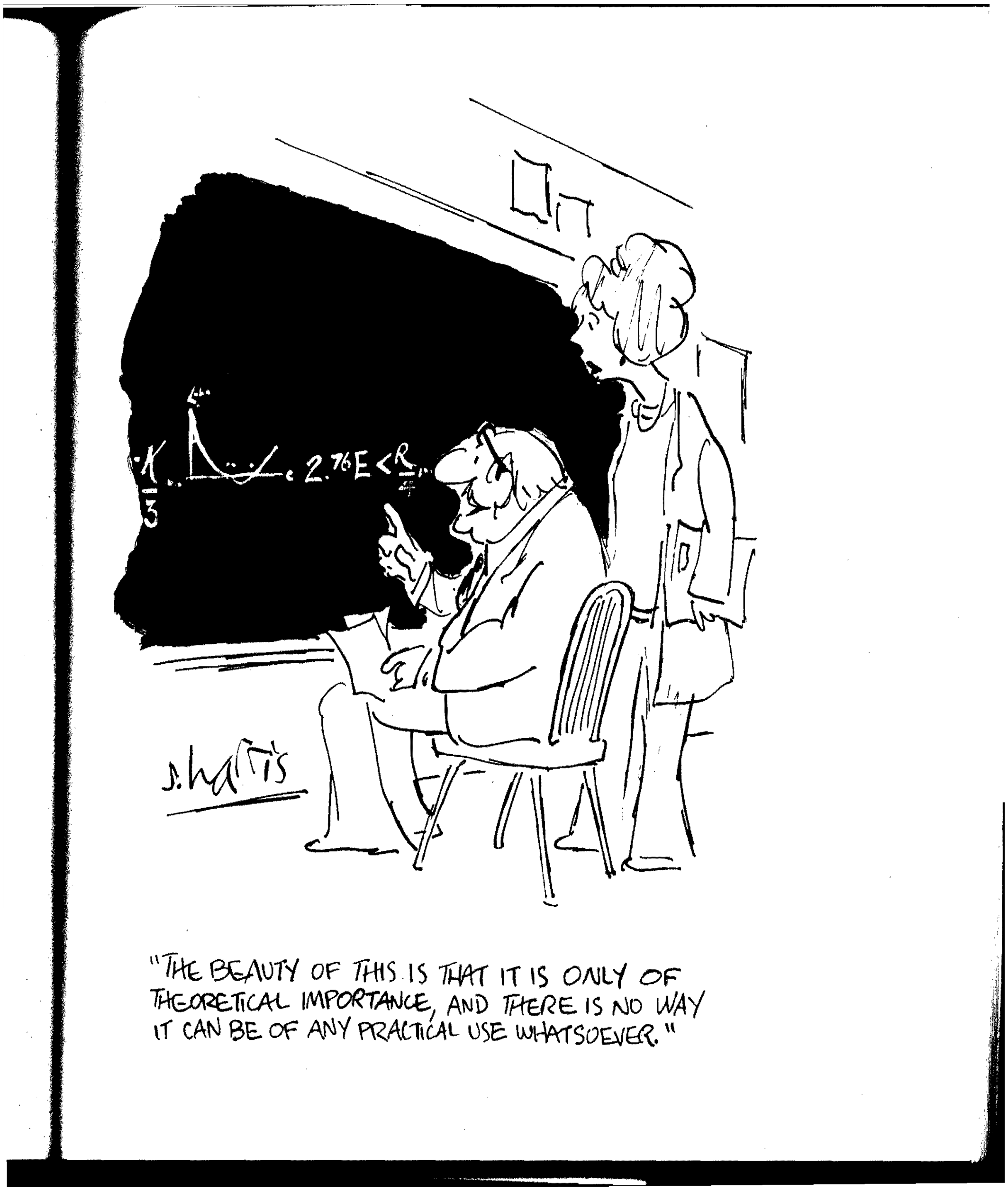


Figure 9.2: Beauty

9.6 The Four Fundamental Subspaces

Given a linear map $f\colon E \to F$ (where E and F are finite-dimensional), Proposition 9.7 revealed that the four spaces

$$\mathrm{Im}\, f,\ \mathrm{Im}\, f^{\top},\ \mathrm{Ker}\, f,\ \mathrm{Ker}\, f^{\top}$$

play a special role. They are often called the *fundamental subspaces* associated with f.

These spaces are related in an intimate manner, since Proposition 9.7 shows that

$$\begin{aligned} \mathrm{Ker}\, f &= (\mathrm{Im}\, f^{\top})^0 \\ \mathrm{Ker}\, f^{\top} &= (\mathrm{Im}\, f)^0, \end{aligned}$$

and Theorem 9.8 shows that

$$\mathrm{rk}(f) = \mathrm{rk}(f^{\top}).$$

It is instructive to translate these relations in terms of matrices (actually, certain linear algebra books make a big deal about this!).

If $\dim(E) = n$ and $\dim(F) = m$, given any basis $(u_1, \ldots, u_n)$ of E and a basis $(v_1, \ldots, v_m)$ of F, we know that f is represented by an $m \times n$ matrix $A = (a_{i\,j})$, where the jth column of A is equal to $f(u_j)$ over the basis $(v_1, \ldots, v_m)$.

Furthermore, the transpose map $f^\top$ is represented by the $n \times m$ matrix $A^\top$ (with respect to the dual bases).

Consequently, the four fundamental spaces

$$\operatorname{Im} f,\ \operatorname{Im} f^\top,\ \operatorname{Ker} f,\ \operatorname{Ker} f^\top$$

correspond to

(1) The *column space* of A, denoted by $\operatorname{Im} A$ or $\mathcal{R}(A)$; this is the subspace of $\mathbb{R}^m$ spanned by the columns of A, which corresponds to the image $\operatorname{Im} f$ of f.

(2) The *kernel* or *nullspace* of A, denoted by $\operatorname{Ker} A$ or $\mathcal{N}(A)$; this is the subspace of $\mathbb{R}^n$ consisting of all vectors $x \in \mathbb{R}^n$ such that $Ax = 0$.

(3) The *row space* of A, denoted by $\operatorname{Im} A^\top$ or $\mathcal{R}(A^\top)$; this is the subspace of $\mathbb{R}^n$ spanned by the rows of A, or equivalently, spanned by the columns of $A^\top$, which corresponds to the image $\operatorname{Im} f^\top$ of $f^\top$.

(4) The *left kernel* or *left nullspace* of A denoted by $\operatorname{Ker} A^\top$ or $\mathcal{N}(A^\top)$; this is the kernel (nullspace) of $A^\top$, the subspace of $\mathbb{R}^m$ consisting of all vectors $y \in \mathbb{R}^m$ such that $A^\top y = 0$, or equivalently, $y^\top A = 0$.

Recall that the dimension r of $\operatorname{Im} f$, which is also equal to the dimension of the column space $\operatorname{Im} A = \mathcal{R}(A)$, is the *rank* of A (and f).

Then, some our previous results can be reformulated as follows:

1. The column space $\mathcal{R}(A)$ of A has dimension r.
2. The nullspace $\mathcal{N}(A)$ of A has dimension $n - r$.
3. The row space $\mathcal{R}(A^\top)$ has dimension r.
4. The left nullspace $\mathcal{N}(A^\top)$ of A has dimension $m - r$.

The above statements constitute what Strang calls the *Fundamental Theorem of Linear Algebra, Part I* (see Strang [32]).

The two statements

$$\begin{aligned}\operatorname{Ker} f &= (\operatorname{Im} f^{\top})^0 \\ \operatorname{Ker} f^{\top} &= (\operatorname{Im} f)^0\end{aligned}$$

translate to

(1) The nullspace of A is the orthogonal of the row space of A.

(2) The left nullspace of A is the orthogonal of the column space of A.

The above statements constitute what Strang calls the *Fundamental Theorem of Linear Algebra, Part II* (see Strang [32]).

Since vectors are represented by column vectors and linear forms by row vectors (over a basis in E or F), a vector $x \in \mathbb{R}^n$ is orthogonal to a linear form y if

$$yx = 0.$$

Then, a vector $x \in \mathbb{R}^n$ is orthogonal to the row space of A iff x is orthogonal to every row of A, namely $Ax = 0$, which is equivalent to the fact that x belong to the nullspace of A.

Similarly, the column vector $y \in \mathbb{R}^m$ (representing a linear form over the dual basis of F^*) belongs to the nullspace of $A^\top$ iff $A^\top y = 0$, iff $y^\top A = 0$, which means that the linear form given by $y^\top$ (over the basis in F) is orthogonal to the column space of A.

Since (2) is equivalent to the fact that *the column space of A is equal to the orthogonal of the left nullspace of A*, we get the following criterion for the solvability of an equation of the form $Ax = b$:

The equation $Ax = b$ has a solution iff for all $y \in \mathbb{R}^m$, if $A^\top y = 0$, then $y^\top b = 0$.

Indeed, the condition on the right-hand side says that b is orthogonal to the left nullspace of A, that is, that b belongs to the column space of A.

This criterion can be cheaper to check that checking directly that b is spanned by the columns of A. For example, if we consider the system

$$\begin{aligned} x_1 - x_2 &= b_1 \\ x_2 - x_3 &= b_2 \\ x_3 - x_1 &= b_3 \end{aligned}$$

which, in matrix form, is written $Ax = b$ as below:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

we see that the rows of the matrix A add up to 0.

In fact, it is easy to convince ourselves that the left nullspace of A is spanned by $y = (1, 1, 1)$, and so the system is solvable iff $y^\top b = 0$, namely

$$b_1 + b_2 + b_3 = 0.$$

Note that the above criterion can also be stated negatively as follows:

The equation $Ax = b$ has no solution iff there is some $y \in \mathbb{R}^m$ such that $A^\top y = 0$ and $y^\top b \neq 0$.

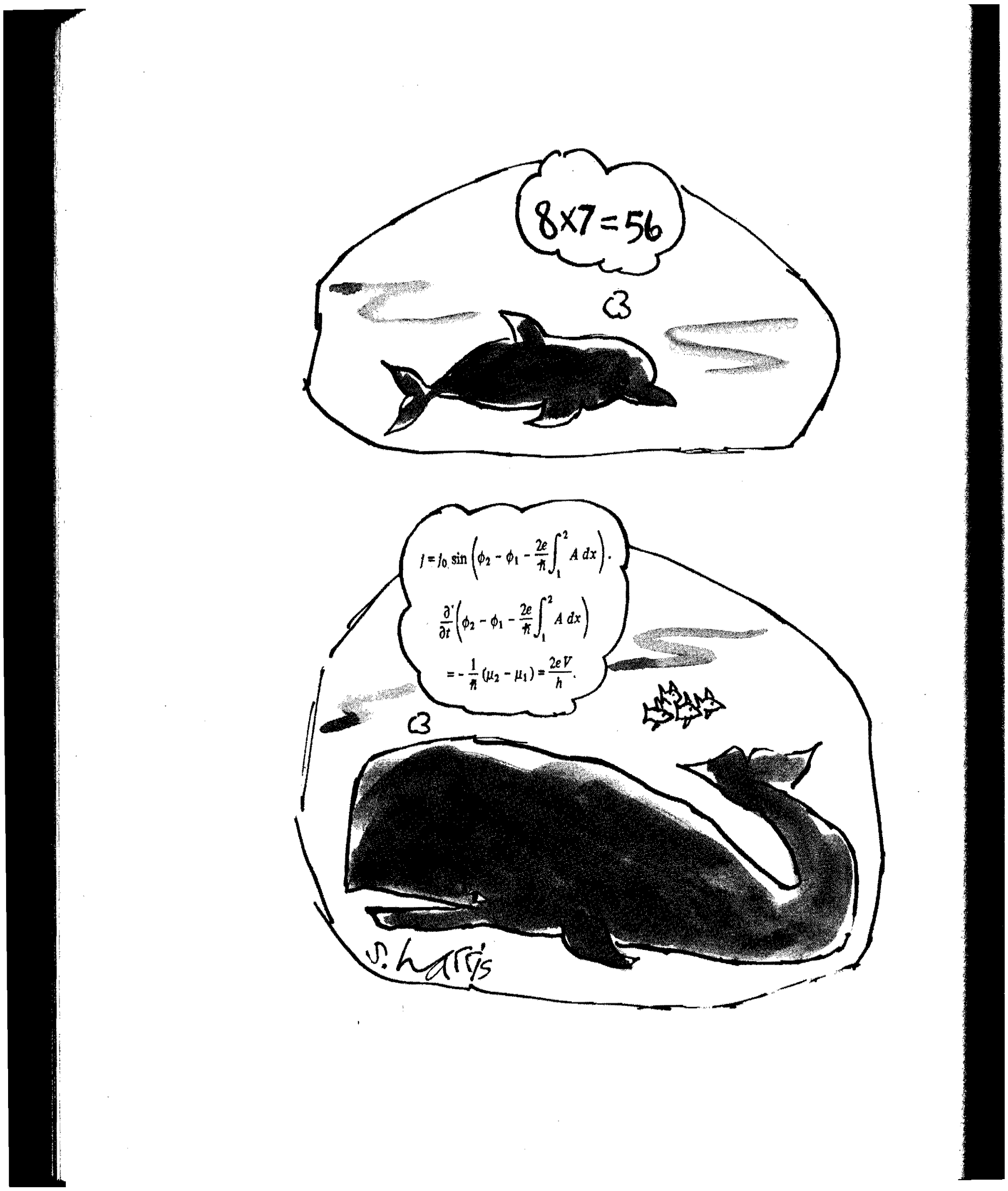


Figure 9.3: Brain Size?