Chapter 4

Signed Graphs

4.1 Signed Graphs and Signed Laplacians

Intuitively, in a weighted graph, an edge with a positive weight denotes similarity or proximity of its endpoints.

For many reasons, it is desirable to allow edges labeled with *negative weights*, the intuition being that *a negative weight indicates dissimilarity or distance*.

Weighted graphs for which the weight matrix is a symmetric matrix in which negative and positive entries are allowed are called *signed graphs*.

Such graphs (with weights $(-1, 0, +1)$) were introduced as early as 1953 by Harary [9], to model social relations involving disliking, indifference, and liking.

The problem of clustering the nodes of a signed graph arises naturally as a generalization of the clustering problem for weighted graphs.

From our perspective, we would like to know whether clustering using normalized cuts can be extended to signed graphs.

Given a signed graph $G = (V, W)$ (where W is a symmetric matrix with zero diagonal entries), the *underlying graph* of G is the graph with node set V and set of (undirected) edges $E = \{\{v_i, v_j\} \mid w_{ij} \neq 0\}$.

The first obstacle is that the degree matrix may now contain zero or negative entries.

As a consequence, the Laplacian L may no longer be positive semidefinite, and worse, $D^{-1/2}$ may not exist.

A simple remedy is to use the absolute values of the weights in the degree matrix!

This idea applied to signed graph with weights $(-1, 0, 1)$ occurs in Hou [10]. Kolluri, Shewchuk and O'Brien [11] take the natural step of using absolute values of weights in the degree matrix in their original work on surface reconstruction from noisy point clouds.

Kunegis et al. [12] appear to be the first to make a systematic study of spectral methods applied to signed graphs.

In fact, many results in this section originate from Kunegis et al. [12].

However, it should be noted that only 2-clustering is considered in the above papers.

As we will see, the trick of using absolute values of weights in the degree matrix allows the whole machinery that we have presented to be used to attack the problem of clustering signed graphs using normalized cuts.

This requires a modification of the notion of normalized cut.

This new notion it is quite reasonable, as we will see shortly.

If (V, W) is a signed graph, where W is an $m \times m$ symmetric matrix with zero diagonal entries and with the other entries $w_{ij} \in \mathbb{R}$ arbitrary, for any node $v_i \in V$, the *signed degree* of v_i is defined as

$$\overline{d}_i = \overline{d}(v_i) = \sum_{j=1}^{m} |w_{ij}|,$$

and the *signed degree matrix* $\overline{D}$ as

$$\overline{D} = \mathrm{diag}(\overline{d}(v_1), \ldots, \overline{d}(v_m)).$$

For any subset A of the set of nodes V, let

$$\mathrm{vol}(A) = \sum_{v_i \in A} \overline{d}_i = \sum_{v_i \in A} \sum_{j=1}^{m} |w_{ij}|.$$

For any two subsets A and B of V, define $\mathrm{links}^+(A,B)$, $\mathrm{links}^-(A,B)$, and $\mathrm{cut}(A,\overline{A})$ by

$$\begin{aligned}
\mathrm{links}^+(A,B) &= \sum_{\substack{v_i\in A, v_j\in B\\ w_{ij}>0}} w_{ij}\\
\mathrm{links}^-(A,B) &= \sum_{\substack{v_i\in A, v_j\in B\\ w_{ij}<0}} -w_{ij}\\
\mathrm{cut}(A,\overline{A}) &= \sum_{\substack{v_i\in A, v_j\in \overline{A}\\ w_{ij}\neq 0}} |w_{ij}|.
\end{aligned}$$

Note that $\mathrm{links}^+(A,B) = \mathrm{links}^+(B,A)$, $\mathrm{links}^-(A,B) = \mathrm{links}^-(B,A)$, and

$$\mathrm{cut}(A,\overline{A}) = \mathrm{links}^+(A,\overline{A}) + \mathrm{links}^-(A,\overline{A}).$$

Then, the *signed Laplacian* $\overline{L}$ is defined by

$$\overline{L} = \overline{D} - W,$$

and its normalized version $\overline{L}_{\mathrm{sym}}$ by

$$\overline{L}_{\mathrm{sym}} = \overline{D}^{-1/2}\,\overline{L}\,\overline{D}^{-1/2} = I - \overline{D}^{-1/2} W \overline{D}^{-1/2}.$$

For a graph without isolated vertices, we have $\overline{d}(v_i) > 0$ for $i = 1, \ldots, m$, so $\overline{D}^{-1/2}$ is well defined.

The signed Laplacian is symmetric positive semidefinite. As for the Laplacian of a weight matrix (with nonnegative entries), this can be shown in two ways.

The first method consists in defining a notion of incidence matrix for a signed graph, and appears in Hou [10].

Definition 4.1. Given a signed graph $G = (V, W)$, with $V = \{v_1, \ldots, v_m\}$, if $\{e_1, \ldots, e_n\}$ are the edges of the underlying graph of G (recall that $\{v_i, v_j\}$ is an edge of this graph iff $w_{ij} \neq 0$), for any oriented graph G^σ obtained by giving an orientation to the underlying graph of G, the *incidence matrix* B^σ of G^σ is the $m \times n$ matrix whose entries $b_{i\,j}$ are given by

$$b_{i\,j} = \begin{cases} +\sqrt{w_{ij}} & \text{if } w_{ij} > 0 \text{ and } s(e_j) = v_i \\ -\sqrt{w_{ij}} & \text{if } w_{ij} > 0 \text{ and } t(e_j) = v_i \\ \sqrt{-w_{ij}} & \text{if } w_{ij} < 0 \text{ and } (s(e_j) = v_i \text{ or } t(e_j) = v_i) \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have the following proposition whose proof is easily adapted from the proof of Proposition 1.2.

Proposition 4.1. *Given any signed graph $G = (V, W)$ with $V = \{v_1, \ldots, v_m\}$, if B^σ is the incidence matrix of any oriented graph G^σ obtained from the underlying graph of G and $\overline{D}$ is the signed degree matrix of W, then*

$$B^\sigma (B^\sigma)^\top = \overline{D} - W = \overline{L}.$$

Consequently, $B^\sigma (B^\sigma)^\top$ is independent of the orientation of the underlying graph of G and $\overline{L} = \overline{D} - W$ is symmetric and positive semidefinite; that is, the eigenvalues of $\overline{L} = \overline{D} - W$ are real and nonnegative.

Another way to prove that $\overline{L}$ is positive semidefinite is to evaluate the quadratic form $x^\top \overline{L} x$.

We will need this computation to figure out what is the new notion of normalized cut.

For any real $\lambda \in \mathbb{R}$, define $\mathrm{sgn}(\lambda)$ by

$$\mathrm{sgn}(\lambda) = \begin{cases} +1 & \text{if } \lambda > 0 \\ -1 & \text{if } \lambda < 0 \\ 0 & \text{if } \lambda = 0. \end{cases}$$

Proposition 4.2. *For any $m \times m$ symmetric matrix $W = (w_{ij})$, if we let $\overline{L} = \overline{D} - W$ where $\overline{D}$ is the signed degree matrix associated with W, then we have*

$$x^{\top}\overline{L}x = \frac{1}{2}\sum_{i,j=1}^{m} |w_{ij}|(x_i - \mathrm{sgn}(w_{ij})x_j)^2 \quad \text{for all } x \in \mathbb{R}^m.$$

Consequently, $\overline{L}$ is positive semidefinite.

4.2 Signed Normalized Cuts

As in Section 3.3, given a partition of V into K clusters $(A_1, \ldots, A_K)$, if we represent the jth block of this partition by a vector X^j such that

$$X_i^j = \begin{cases} a_j & \text{if } v_i \in A_j \\ 0 & \text{if } v_i \notin A_j, \end{cases}$$

for some $a_j \neq 0$, then we have the following result.

Proposition 4.3. *For any vector X^j representing the jth block of a partition $(A_1, \ldots, A_K)$ of V, we have*

$$(X^j)^\top \overline{L} X^j = a_j^2(\mathrm{cut}(A_j, \overline{A_j}) + 2\mathrm{links}^-(A_j, A_j)).$$

Since with the revised definition of $\mathrm{vol}(A_j)$, we also have

$$(X^j)^\top \overline{D} X^j = a_j^2 \sum_{v_i \in A_j} \overline{d}_i = a_j^2 \mathrm{vol}(A_j),$$

we deduce that

$$\frac{(X^j)^\top \overline{L} X^j}{(X^j)^\top \overline{D} X^j} = \frac{\mathrm{cut}(A_j, \overline{A_j}) + 2\mathrm{links}^-(A_j, A_j)}{\mathrm{vol}(A_j)}.$$

The calculations of the previous paragraph suggest the following definition.

Definition 4.2. The *signed normalized cut* $\mathrm{sNcut}(A_1, \ldots, A_K)$ of the partition $(A_1, \ldots, A_K)$ is defined as

$$\mathrm{sNcut}(A_1, \ldots, A_K) = \sum_{j=1}^{K} \frac{\mathrm{cut}(A_j, \overline{A_j})}{\mathrm{vol}(A_j)} + 2\sum_{j=1}^{K} \frac{\mathrm{links}^-(A_j, A_j)}{\mathrm{vol}(A_j)}.$$

Remark: Kunegis et al. [12] deal with a different notion of cut, namely ratio cut (in which $\mathrm{vol}(A)$ is replaced by the size $|A|$ of A), and only for two clusters.

In this case, by a clever choice of indicator vector, they obtain a notion of signed cut that only takes into account the positive edges between A and $\overline{A}$, and the negative edges among nodes in A and nodes in $\overline{A}$.

This trick does not seem to generalize to more than two clusters, and this is why we use our representation for partitions. Our definition of a signed normalized cut appears to be novel.

Based on previous computations, we have

$$\mathrm{sNcut}(A_1, \ldots, A_K) = \sum_{j=1}^{K} \frac{(X^j)^\top \overline{L} X^j}{(X^j)^\top \overline{D} X^j}.$$

where X is the $N \times K$ matrix whose jth column is X^j.

Therefore, this is the same problem as in Chapter 3, with L replaced by $\overline{L}$ and D replaced by $\overline{D}$.

Observe that minimizing $\mathrm{sNcut}(A_1, \ldots, A_K)$ amounts to minimizing the number of positive and negative edges between clusters, and also minimizing the number of negative edges within clusters.

This second minimization captures the intuition that *nodes connected by a negative edge should not be together* (they do not "like" each other; they should be far from each other).

The K-clustering problem for signed graphs is related but not equivalent to another problem known as *correlation clustering*.

In correlation clustering, in our terminology and notation, given a graph $G = (V, W)$ with positively and negatively weighted edges, one seeks a clustering of V that minimizes the sum $\mathrm{links}^-(A_j, A_j)$ of the absolute values of the negative weights of the edges within each cluster A_j, and minimizes the sum $\mathrm{links}^+(A_j, \overline{A_j})$ of the positive weights of the edges between distinct clusters.

In contrast to K-clustering, the number K of clusters is not given in advance, and there is no normalization with respect to size of volume.

Furthermore, in correlation clustering, only the contribution $\mathrm{links}^+(A_j, \overline{A}_j)$ of positively weighted edges is minimized, but our method only allows us to minimize $\mathrm{cut}(A_j, \overline{A}_j)$, which also takes into account negatively weighted edges between distinct clusters.

Correlation clustering was first introduced and studied for complete graphs by Bansal, Blum and Chawla [1].

They prove that this problem is NP-complete and give several approximation algorithms, including a PTAS for maximizing agreement.

Demaine and Immorlica [4] consider the same problem for arbitrary weighted graphs, and they give an $O(\log n)$-approximation algorithm based on linear programming.

Since correlation clustering does not assume that K is given and not not include nomalization by size or volume, it is not clear whether algorithms for correlation clustering can be applied to normalized K-clustering, and conversely.

4.3 Balanced Graphs

Since

$$\mathrm{sNcut}(A_1, \ldots, A_K) = \sum_{j=1}^{K} \frac{(X^j)^\top \overline{L} X^j}{(X^j)^\top \overline{D} X^j},$$

the whole machinery of Sections 3.3 and 3.4 can be applied with D replaced by $\overline{D}$ and L replaced by $\overline{L}$.

However, there is a new phenomenon, which is that $\overline{L}$ *blue may be positive definite.*

As a consequence, $\mathbf{1}$ is not always an eigenvector of $\overline{L}$.

As observed by Kunegis et al. [12], it is also possible to characterize for which signed graphs the Laplacian $\overline{L}$ is positive definite.

Such graphs are "cousins" of bipartite graphs and were introduced by Harary [9].

Since a graph is the union of its connected components, we restrict ourselves to connected graphs.

Definition 4.3. Given a signed graph $G = (V, W)$ with negative weights whose underlying graph is connected, we say that G is *balanced* if there is a partition of its set of nodes V into two blocks V_1 and V_2 such that all positive edges connect nodes within V_1 or V_2, and negative edges connect nodes between V_1 and V_2.

An example of a balanced graph is shown in Figure 4.1 on the left, in which positive edges are colored green and negative edges are colored red.

This graph admits the partition

$$(\{v_1, v_2, v_4, v_7, v_8\}, \{v_3, v_5, v_6, v_9\}).$$

On the other hand, the graph shown on the right contains the cycle $(v_2, v_3, v_6, v_5, v_4, v_2)$ with an odd number of negative edges (3), and thus is not balanced.

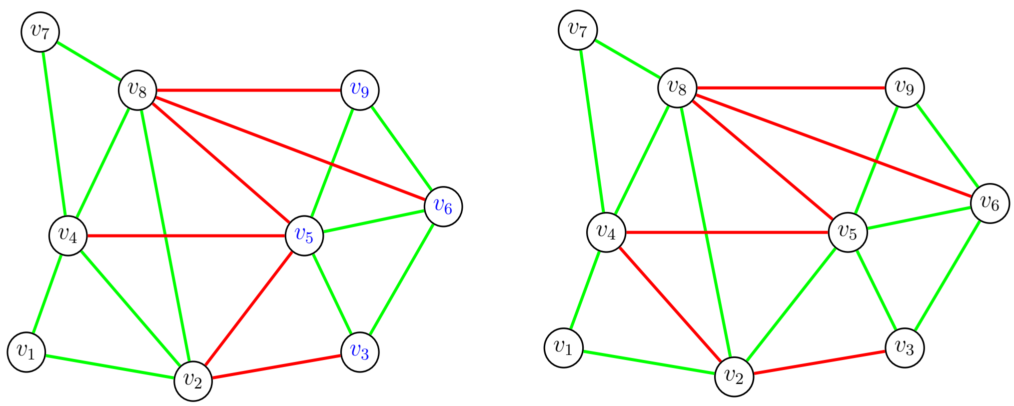


Figure 4.1: A balanced signed graph G_1 (left). An unbalanced signed graph G_2 (right).

Observe that if we delete all positive edges in a balanced graph, then the resulting graph is bipartite.

Then, it is not surprising that connected balanced graphs can be characterized as signed graphs in which every cycle has an even number of negative edges.

This is analogous to the characterization of a connected bipartite graph as a graph in which every cycle has even length.

The following proposition was first proved by Harary [9].

Proposition 4.4. *If $G = (V, W)$ is a connected signed graph with negative weights, then G is balanced iff every cycle contains an even number of negative edges.*

We can also detect whether a connected signed graph is balanced in terms of the kernel of the transpose of any of its incidence matrices.

Proposition 4.5. *If $G = (V, W)$ is a connected signed graph with negative weights and with m nodes, for any orientation of its underlying graph, let B be the corresponding incidence matrix. The underlying graph of G is balanced iff* $\mathrm{rank}(B) = m - 1$. *Furthermore, if G is balanced, then there is a vector u with $u_i \in \{-1, 1\}$ such that $B^\top u = 0$, and the sets of nodes $V_1 = \{v_i \mid u_i = -1\}$ and $V_2 = \{v_i \mid u_i = +1\}$ form a partition of V for which G is balanced.*

Remark: A simple modification of the proof of Proposition 4.5 shows that if there are c_1 components containing only positive edges, c_2 components that are balanced graphs, and c_3 components that are not balanced (and contain some negative edge), then

$$c_1 + c_2 = m - \mathrm{rank}(B).$$

Since by Proposition 4.1 we have $\overline{L} = BB^\top$ for any incidence matrix B associated with an orientation of the underlying graph of G, we obtain the following important result (which is proved differently in Kunegis et al. [12]).

Theorem 4.6. *The signed Laplacian $\overline{L}$ of a connected signed graph G is positive definite iff G is not balanced (possesses some cycle with an odd number of negative edges).*

If $G = (V, W)$ is a balanced graph, then there is a partition (V_1, V_2) of V such that for every edge $\{v_i, v_j\}$, if $w_{ij} > 0$, then $v_i, v_j \in V_1$ or $v_i, v_j \in V_2$, and if $w_{ij} < 0$, then $v_i \in V_1$ and $v_j \in V_2$.

It follows that if we define the vector x such that $x_i = +1$ iff $v_i \in V_1$ and $x_i = -1$ iff $v_i \in V_2$, then for every edge $\{v_i, v_j\}$ we have

$$\mathrm{sgn}(w_{ij}) = x_i x_j.$$

We call x a *bipartition* of V.

The signed Laplacian of the balanced graph G_1 is given by

$$\overline{L}_1 = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 5 & 1 & -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 5 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 6 & -1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 1 & -1 & 6 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 3 \end{pmatrix}$$

Using `Matlab`, we find that its eigenvalues are

$$0,\ 1.4790,\ 1.7513,\ 2.7883,\ 4.3570,\ 4.8815,\ 6.2158,\\ 7.2159,\ 7.3112.$$

The eigenvector corresponding to the eigenvalue 0 is

$$(0.3333,\ 0.3333,\ -0.3333,\ 0.3333,\ -0.3333,\\ -0.3333,\ 0.3333,\ 0.3333,\ -0.3333)$$

It gives us the bipartition

$$(\{v_1, v_2, v_4, v_7, v_8\}, \{v_3, v_5, v_6, v_9\}),$$

as guaranteed by Proposition 4.5.

The signed Laplacian of the unbalanced graph G_2 is given by

$$\overline{L}_2 = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 5 & 1 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 5 & 1 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 6 & -1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 1 & -1 & 6 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 3 \end{pmatrix}$$

The eigenvalues of $\overline{L}_2$ are

$$0.5175,\ 1.5016,\ 1.7029,\ 2.7058,\ 3.7284,\ 4.9604,\ 5.6026,\ 7.0888,\ 8.1921.$$

The matrix $\overline{L}_2$ is indeed positive definite (since G_2 is unbalanced).

Hou [10] gives bounds on the smallest eigenvalue of an unbalanced graph. The lower bound involves a measure of how unbalanced the graph is (see Theorem 3.4 in Hou [10]).

Following Kunegis et al., we can prove the following result showing that the eigenvalues and the eigenvectors of $\overline{L}$ and its unsigned counterpart $\mathcal{L}$ are strongly related.

Given a symmetric signed matrix W, we define the unsigned matrix $\mathcal{W}$ such that $\mathcal{W}_{ij} = |w_{ij}|$ $(1 \leq i, j \leq m)$. We let $\mathcal{L}$ be the Laplacian associated with $\mathcal{W}$.

Note that

$$\mathcal{L} = \overline{D} - \mathcal{W}.$$

The following proposition is shown in Kunegis et al. [12]).

Proposition 4.7. *Let $G = (V, W)$ be a signed graph and let $\mathcal{W}$ be the unsigned matrix associated with W. If G is balanced, and x is a bipartition of V, then for any diagonalization $\overline{L} = P\Lambda P^\top$ of $\overline{L}$, where P is an orthogonal matrix of eigenvectors of $\overline{L}$, if we define the matrix $\mathcal{P}$ so that*

$$\mathcal{P}_i = x_i P_i,$$

where $\mathcal{P}_i$ is the ith row of $\mathcal{P}$ and P_i is the ith row of P, then $\mathcal{P}$ is orthogonal and

$$\mathcal{L} = \mathcal{P}\Lambda\mathcal{P}^\top$$

is a diagonalization of $\mathcal{L}$. In particular, $\overline{L}$ and $\mathcal{L}$ have the same eigenvalues with the same multiplicities.

4.4 K-Way Clustering of Signed Graphs

Using the signed Laplacians $\overline{L}$ and $\overline{L}_{\mathrm{sym}}$, we can define the optimization problems as in Section 3.3 and solve them as in Section 3.4, except that we drop the constraint

$$X(X^\top X)^{-1}X^\top \mathbf{1} = \mathbf{1},$$

since $\mathbf{1}$ is not necessarily an eigenvector of $\overline{L}$.

By Proposition A.3, the sum of the K smallest eigenvalues of $\overline{L}_{\mathrm{sym}}$ is a lower bound for $\mathrm{tr}(Y^\top \overline{L}_{\mathrm{sym}} Y)$, and the minimum of problem $(**_2)$ is achieved by any K unit eigenvectors $(u_1, \ldots, u_k)$ associated with the smallest eigenvalues

$$0 \leq \nu_1 \leq \nu_2 \leq \ldots \leq \nu_K$$

of $\overline{L}_{\mathrm{sym}}$.

The difference with unsigned graphs is that ν_1 may be strictly positive.

Here is the result of applying this method to various examples.

First, we apply our algorithm to find three clusters for the balanced graph G_1.

The graph G_1 as outputted by the algorithm is shown in Figure 4.2 and the three clusters are shown in Figure 4.3.

As desired, these clusters do not contain negative edges.

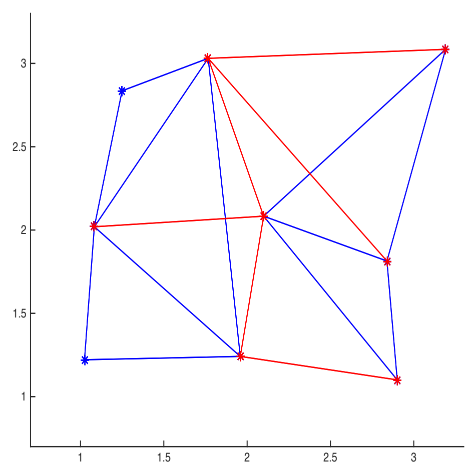


Figure 4.2: The balanced graph G_1.

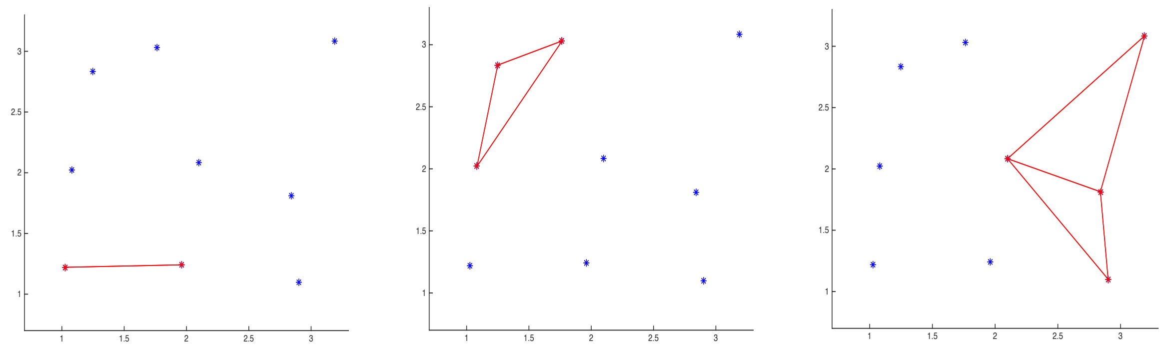


Figure 4.3: Three blocks of a normalized cut for the graph associated with G_1.

By the way, for two clusters, the algorithm finds the bipartition of G_1, as desired.

Next, we apply our algorithm to find three clusters for the unbalanced graph G_2. The graph G_2 as outputted by the algorithm is shown in Figure 4.2 and the three clusters are shown in Figure 4.3.

As desired, these clusters do not contain negative edges.

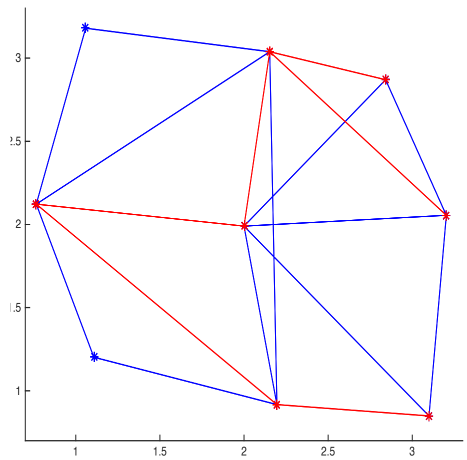


Figure 4.4: The unbalanced graph G_2.

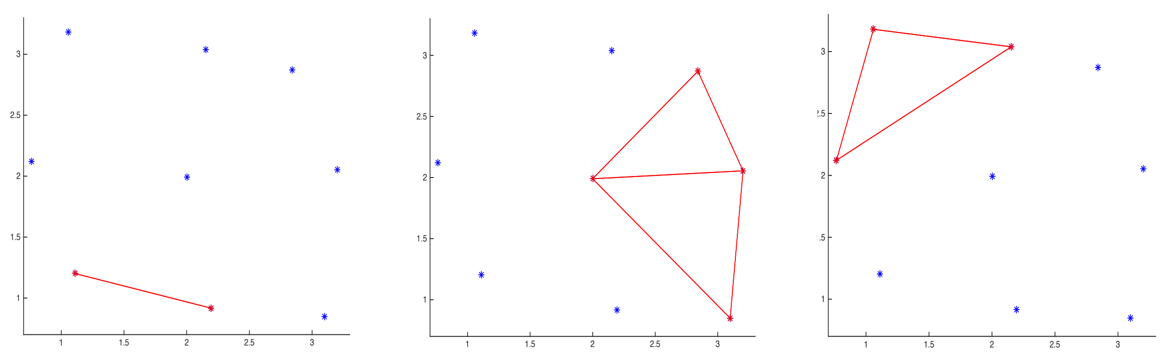


Figure 4.5: Three blocks of a normalized cut for the graph associated with G_2.

The algorithm finds the same clusters, but this is probably due to the fact that G_1 and G_2 only differ by the signs of two edges.

4.5 Signed Graph Drawing

Following Kunegis et al. [12], if our goal is to draw a signed graph $G = (V, W)$ with m nodes, a natural way to interpret negative weights is to assume that the end-points v_i and v_j of an edge with a negative weight should be placed far apart, which can be achieved if instead of assigning the point $\rho(v_j) \in \mathbb{R}^n$ to v_j, we assign the point $-\rho(v_j)$.

Then, if R is the $m \times n$ matrix of a graph drawing of G in $\mathbb{R}^n$, the energy function $\mathcal{E}(R)$ is redefined to be

$$\mathcal{E}(R) = \sum_{\{v_i, v_j\} \in E} |w_{ij}| \left\| \rho(v_i) - \operatorname{sgn}(w_{ij}) \rho(v_j) \right\|^2 .$$

We obtain the following version of Proposition 2.1.

Proposition 4.8. *Let $G = (V, W)$ be a signed graph, with $|V| = m$ and with W a $m \times m$ symmetric matrix, and let R be the matrix of a graph drawing ρ of G in $\mathbb{R}^n$ (a $m \times n$ matrix). Then, we have*

$$\mathcal{E}(R) = \mathrm{tr}(R^\top \overline{L} R).$$

Then, as in Chapter 2, we look for a graph drawing R that minimizes $\mathcal{E}(R) = \mathrm{tr}(R^\top \overline{L} R)$ subject to $R^\top R = I$.

The new ingredient is that $\overline{L}$ is positive definite iff G is not a balanced graph.

Also, in the case of a signed graph, $\mathbf{1}$ does not belong to the kernel of $\overline{L}$, so we do not get a balanced graph drawing.

If G is a signed balanced graph, then $\mathrm{Ker}\, L$ is nontrivial, and if G is connected, then $\mathrm{Ker}\, L$ is spanned by a vector whose components are either $+1$ or -1.

Thus, if we use the first n unit eigenvectors $(u_1, u_2, \ldots, u_n)$ associated with the n smallest eigenvalues $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ of $\overline{L}$, we obtain a drawing for which the nodes are partitionned into two sets living in two hyperplanes corresponding to the value of their first coordinate.

Let us call such a drawing a *bipartite drawing*.

However, if G is connected, the vector u_2 does not belong to $\mathrm{Ker}\, \overline{L}$, so if $m \geq 3$, it must have at least three coordinates with distinct absolute values, and using $(u_2, \ldots, u_{n+1})$ we obtain a nonbipartite graph.

Then, the following version of Theorem 2.2 is easily shown.

Theorem 4.9. *Let $G = (V, W)$ be a signed graph with $|V| = m \geq 3$, assume that G has some negative edge and is connected, and let $\overline{L} = \overline{D} - W$ be the signed Laplacian of G.*

(1) If G is not balanced and if the eigenvalues of L are $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_m$, then the minimal energy of any orthogonal graph drawing of G in $\mathbb{R}^n$ is equal to $\lambda_1 + \cdots + \lambda_n$ The $m \times n$ matrix R consisting of any unit eigenvectors $u_1, \ldots, u_n$ associated with $\lambda_1 \leq \ldots \leq \lambda_n$ yields an orthogonal graph drawing of minimal energy.

(2) If G is balanced and if the eigenvalues of L are $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_m$, then the minimal energy of any orthogonal nonbipartite graph drawing of G in $\mathbb{R}^n$ is equal to $\lambda_2 + \cdots + \lambda_{n+1}$ (in particular, this implies that $n < m$). The $m \times n$ matrix R consisting of any unit eigenvectors $u_2, \ldots, u_{n+1}$ associated with $\lambda_2 \leq \ldots \leq \lambda_{n+1}$ yields an orthogonal nonbipartite graph drawing of minimal energy.

(3) If G is balanced, for $n = 2$, a graph drawing of G as a bipartite graph (with positive edges only withing the two blocks of vertices) is obtained from the $m \times 2$ matrix consisting of any two unit eigenvectors u_1 and u_2 associated with 0 and λ_2.

In all cases, the graph drawing R satisfies the condition $R^\top R = I$ (it is an orthogonal graph drawing).

Our first example is the signed graph $G4$ defined by the weight matrix given by the following `Matlab` program:

```
nn = 6; G3 = diag(ones(1,nn),1);
G3 = G3 + G3';
G3(1,nn+1) = 1; G3(nn+1,1) = 1;
G4 = -G3;
```

All edges of this graph are negative. The graph obtained by using $G3$ is shown on the left and the graph obtained by using the signed Laplacian of $G4$ is shown on the right in Figure 4.6.

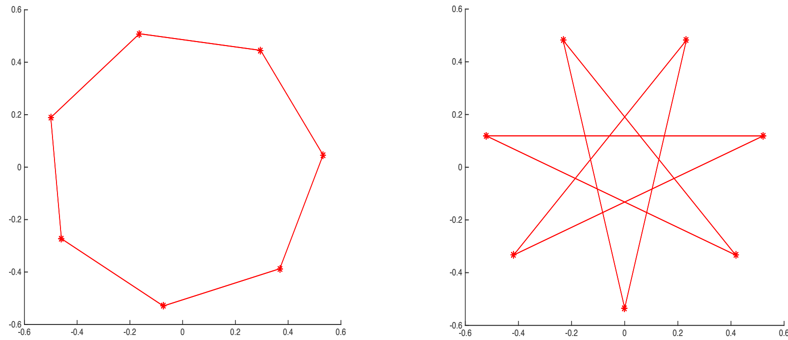


Figure 4.6: The signed graph $G4$.

The second example is the signed graph $G5$ obtained from $G3$ by making a single edge negative:

```
G5 = G3; G5(1,2) = -1; G5(2,1) = -1;
```

The graph obtained by using $G3$ is shown on the left and the graph obtained by using the signed Laplacian of $G5$ is shown on the right in Figure 4.7.

Positive edges are shown in blue and negative edges are shown in red.

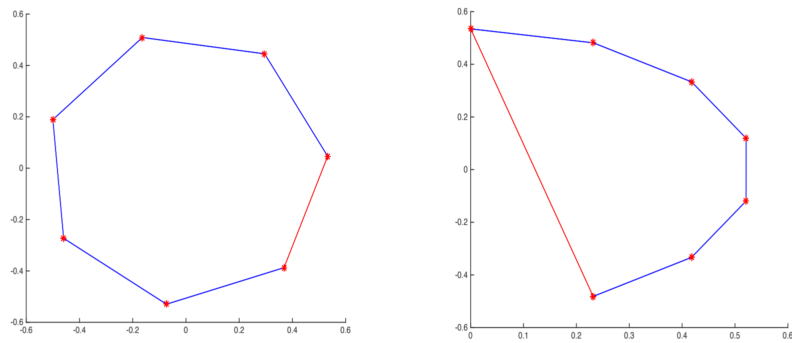


Figure 4.7: The signed graph $G5$.

The third example is the signed graph $G6$ defined by the weight matrix given by the following **Matlab** program:

```
nn = 24; G6 = diag(ones(1,nn),1);
G6 = G6 + G6';
G6(1,nn+1) = 1; G6(nn+1,1) = 1;
G6(1,2) = -1; G6(2,1) = -1; G6(6,7) = -1;
G6(7,6) = -1; G6(11,12) = -1; G6(12,11) = -1;
G6(16,17) = -1; G6(17,16) = -1;
G6(21,22) = -1; G6(22,21) = -1;
```

The graph obtained by using absolute values in $G6$ is shown on the left and the graph obtained by using the signed Laplacian of $G6$ is shown on the right in Figure 4.8.

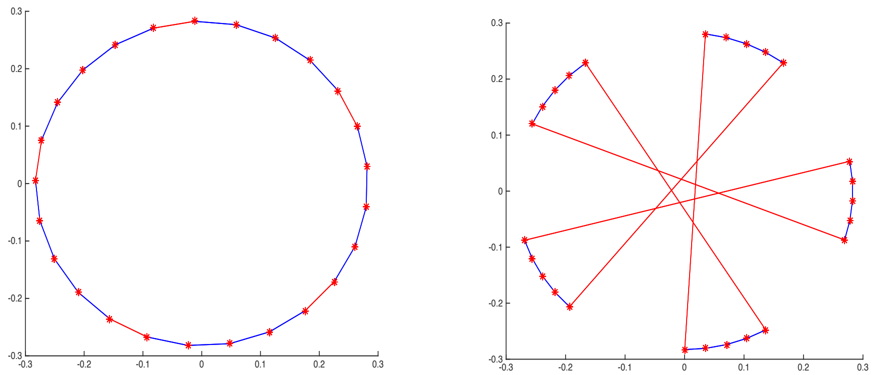


Figure 4.8: The signed graph $G6$.

The fourth example is the signed graph $G7$ defined by the weight matrix given by the following **Matlab** program:

```
nn = 26; G7 = diag(ones(1,nn),1);
G7 = G7 + G7'; G7(1,nn+1) = 1;
G7(nn+1,1) = 1; G7(1,2) = -1;
G7(2,1) = -1; G7(10,11) = -1;
G7(11,10) = -1; G7(19,20) = -1;
G7(20,19) = -1;
```

The graph obtained by using absolute values in $G7$ is shown on the left and the graph obtained by using the signed Laplacian of $G7$ is shown on the right in Figure 4.9.

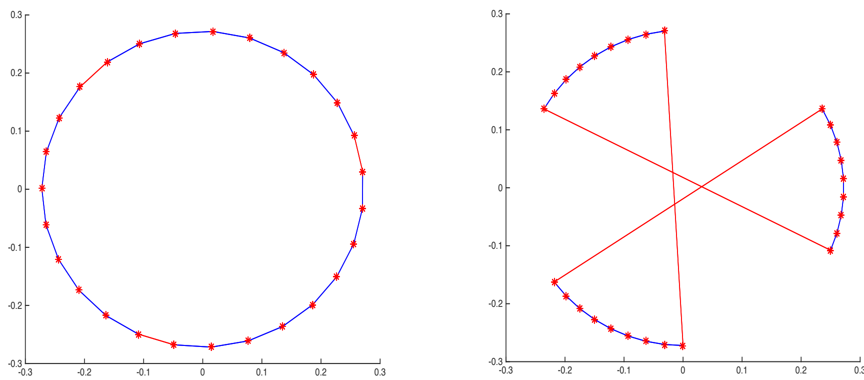
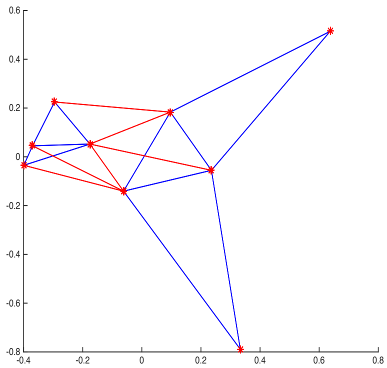


Figure 4.9: The signed graph $G7$.

These graphs are all unbalanced. As predicted, nodes linked by negative edges are far from each other.

Our last example is the balanced graph $G1$ from Figure 4.1.

The graph obtained by using absolute values in $G1$ is shown on the left and the bipartite graph obtained by using the signed Laplacian of $G1$ is shown on the right in Figure 4.10.



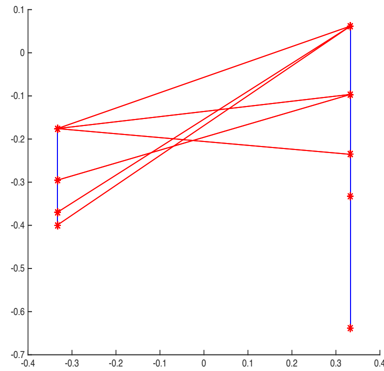


Figure 4.10: The balanced graph $G1$.