CIS 515

Fundamentals of Linear Algebra and Optimization
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Project 2: Drawing Cubic Bézier Spline Curves
More Details

The purpose of this project is to design a `Matlab` program to plot a cubic Bézier spline curve given by a sequence of $N+1$ de Boor control points ($N \geq 4$).

The cases $N = 4, 5, 6$ require special handling. We also explain how to consolidate some of the cases to simplify the computations.

Recall that a *cubic Bézier spline* $F(t)$ (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is specified by a list of *de Boor control points* $(d_0, d_1, \ldots, d_N)$, with $N \geq 7$, and consists of $N-2$ Bézier cubic segments $C_1, \ldots, C_{N-2}$, such that if the control points of C_i are $(b_0^i, b_1^i, b_2^i, b_3^i)$, then they are determined by the following equations:

For C_1, we have

$$\begin{aligned}
b_0^1 &= d_0 \\
b_1^1 &= d_1 \\
b_2^1 &= \frac{1}{2}d_1 + \frac{1}{2}d_2 \\
b_3^1 &= \frac{1}{2}b_2^1 + \frac{1}{2}b_1^2 = \frac{1}{4}d_1 + \frac{7}{12}d_2 + \frac{1}{6}d_3.
\end{aligned}$$

The curve segment C_2 is given by

$$\begin{aligned}
b_0^2 &= \frac{1}{2}b_2^1 + \frac{1}{2}b_1^2 = \frac{1}{4}d_1 + \frac{7}{12}d_2 + \frac{1}{6}d_3 \\
b_1^2 &= \frac{2}{3}d_2 + \frac{1}{3}d_3 \\
b_2^2 &= \frac{1}{3}d_2 + \frac{2}{3}d_3 \\
b_3^2 &= \frac{1}{2}b_2^2 + \frac{1}{2}b_1^3 = \frac{1}{6}d_2 + \frac{4}{6}d_3 + \frac{1}{6}d_4.
\end{aligned}$$

For $i = 3, \ldots, N-4$, the curve segment C_i is specified by the "one third two third rule:"

$$
\begin{aligned}
b_0^i &= \frac{1}{2} b_2^{i-1} + \frac{1}{2} b_1^i = \frac{1}{6} d_{i-1} + \frac{4}{6} d_i + \frac{1}{6} d_{i+1} \\
b_1^i &= \frac{2}{3} d_i + \frac{1}{3} d_{i+1} \\
b_2^i &= \frac{1}{3} d_i + \frac{2}{3} d_{i+1} \\
b_3^i &= \frac{1}{2} b_2^i + \frac{1}{2} b_1^{i+1} = \frac{1}{6} d_i + \frac{4}{6} d_{i+1} + \frac{1}{6} d_{i+2}.
\end{aligned}
$$

This generic case is illustrated in Figure 1.

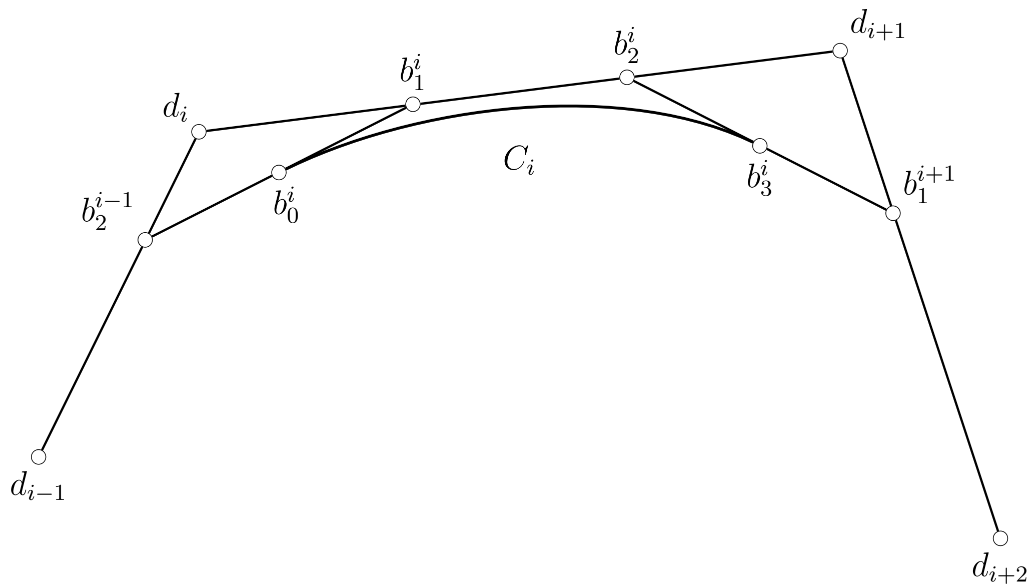

Figure 1: Computing Bézier control points from de Boor control points

The curve segment C_{N-3} is given by

$$
\begin{aligned}
b_0^{N-3} &= \frac{1}{2} b_2^{N-4} + \frac{1}{2} b_1^{N-3} = \frac{1}{6} d_{N-4} + \frac{4}{6} d_{N-3} + \frac{1}{6} d_{N-2} \\
b_1^{N-3} &= \frac{2}{3} d_{N-3} + \frac{1}{3} d_{N-2} \\
b_2^{N-3} &= \frac{1}{3} d_{N-3} + \frac{2}{3} d_{N-2} \\
b_3^{N-3} &= \frac{1}{2} b_2^{N-3} + \frac{1}{2} b_1^{N-2} = \frac{1}{6} d_{N-3} + \frac{7}{12} d_{N-2} + \frac{1}{4} d_{N-1}.
\end{aligned}
$$

Finally, C_{N-2} is specified by

$$\begin{aligned}
b_0^{N-2} &= \frac{1}{2}b_2^{N-3} + \frac{1}{2}b_1^{N-2} = \frac{1}{6}d_{N-3} + \frac{7}{12}d_{N-2} + \frac{1}{4}d_{N-1} \\
b_1^{N-2} &= \frac{1}{2}d_{N-2} + \frac{1}{2}d_{N-1} \\
b_2^{N-2} &= d_{N-1} \\
b_3^{N-2} &= d_N
\end{aligned}$$

Observe that

$$b_0^{i+1} = b_3^i, \quad 1 \leq i \leq N-3.$$

Also observe that when $N \geq 6$, the formulae for $i = 2$ and $3 \leq i \leq N-4$ can be written as a single formula taking advantage of $b_0^{i+1} = b_3^i$, $1 \leq i \leq N-3$.

If $2 \leq i \leq N-4$ and $N \geq 6$, then

$$\begin{aligned}
b_0^i &= b_3^{i-1} \\
b_1^i &= \frac{2}{3}d_i + \frac{1}{3}d_{i+1} \\
b_2^i &= \frac{1}{3}d_i + \frac{2}{3}d_{i+1} \\
b_3^i &= \frac{1}{2}b_2^i + \frac{1}{2}b_1^{i+1} = \frac{1}{6}d_i + \frac{4}{6}d_{i+1} + \frac{1}{6}d_{i+2}.
\end{aligned}$$

The case $N = 5$ is exceptional and the above formula needs to be modified if $i = 2$.

If $N \geq 5$, the last curve segment C_{N-2} is always given by the equations

$$\begin{aligned}
b_0^{N-2} &= \frac{1}{2}b_2^{N-3} + \frac{1}{2}b_1^{N-2} = \frac{1}{6}d_{N-3} + \frac{7}{12}d_{N-2} + \frac{1}{4}d_{N-1} \\
b_1^{N-2} &= \frac{1}{2}d_{N-2} + \frac{1}{2}d_{N-1} \\
b_2^{N-2} &= d_{N-1} \\
b_3^{N-2} &= d_N
\end{aligned}$$

Generally, if $N \geq 5$, having C_{N-4} and C_{N-2}, the curve C_{N-3} is given by

$$\begin{aligned}
b_0^{N-3} &= b_3^{N-4} \\
b_1^{N-3} &= \frac{2}{3}d_{N-3} + \frac{1}{3}d_{N-2} \\
b_2^{N-3} &= \frac{1}{3}d_{N-3} + \frac{2}{3}d_{N-2} \\
b_3^{N-3} &= b_0^{N-2}.
\end{aligned}$$

Thus it is preferable to compute the control points for $C_1, C_2, \ldots, C_{N-4}, C_{N-2}$, and C_{N-3} last.

Using the above equations, the cases $N = 5, 6$ are easily adapted from the general case.

When $N = 5$, there are three Bézier curves defined as follows. C_1 and C_3 are given by the above formulae, namely C_1 is given by

$$\begin{aligned}
b_0^1 &= d_0 \\
b_1^1 &= d_1 \\
b_2^1 &= \frac{1}{2}d_1 + \frac{1}{2}d_2 \\
b_3^1 &= \frac{1}{4}d_1 + \frac{7}{12}d_2 + \frac{1}{6}d_3,
\end{aligned}$$

C_3 is given by

$$\begin{aligned}
b_0^3 &= \frac{1}{6}d_2 + \frac{7}{12}d_3 + \frac{1}{4}d_4 \\
b_1^3 &= \frac{1}{2}d_3 + \frac{1}{2}d_4 \\
b_2^3 &= d_4 \\
b_3^3 &= d_5,
\end{aligned}$$

and C_2 is given by

$$\begin{aligned}
b_0^2 &= b_3^1 \\
b_1^2 &= \frac{2}{3}d_2 + \frac{1}{3}d_3 \\
b_2^2 &= \frac{1}{3}d_2 + \frac{2}{3}d_3 \\
b_3^2 &= b_0^3.
\end{aligned}$$

Note that C_3 should be computed before C_2.

When $N = 6$, there are four Bézier curves defined as follows. C_1, C_2 and C_4 are given by the above formulae, namely C_1 is given by

$$\begin{aligned}
b_0^1 &= d_0 \\
b_1^1 &= d_1 \\
b_2^1 &= \frac{1}{2}d_1 + \frac{1}{2}d_2 \\
b_3^1 &= \frac{1}{4}d_1 + \frac{7}{12}d_2 + \frac{1}{6}d_3,
\end{aligned}$$

C_2 is given by

$$
\begin{aligned}
b_0^2 &= b_3^1 \\
b_1^2 &= \frac{2}{3}d_2 + \frac{1}{3}d_3 \\
b_2^2 &= \frac{1}{3}d_2 + \frac{2}{3}d_3 \\
b_3^2 &= \frac{1}{6}d_2 + \frac{4}{6}d_3 + \frac{1}{6}d_4,
\end{aligned}
$$

and C_4 is given by

$$
\begin{aligned}
b_0^4 &= \frac{1}{6}d_3 + \frac{7}{12}d_4 + \frac{1}{4}d_5 \\
b_1^4 &= \frac{1}{2}d_4 + \frac{1}{2}d_5 \\
b_2^4 &= d_5 \\
b_3^4 &= d_6,
\end{aligned}
$$

Then C_3 is given by

$$
\begin{aligned}
b_0^3 &= b_3^2 \\
b_1^3 &= \frac{2}{3}d_3 + \frac{1}{3}d_4 \\
b_2^3 &= \frac{1}{3}d_3 + \frac{2}{3}d_4 \\
b_3^3 &= b_0^4.
\end{aligned}
$$

Note that C_4 should be computed before C_3.

When $N = 4$, use the formulae for C_1 and $C_{N-2} = C_2$ with

$$b_3^1 = b_0^2 = \frac{1}{4}d_1 + \frac{1}{2}d_2 + \frac{1}{4}d_3.$$

Thus C_1 is given by

$$
\begin{aligned}
b_0^1 &= d_0 \\
b_1^1 &= d_1 \\
b_2^1 &= \frac{1}{2}d_1 + \frac{1}{2}d_2 \\
b_3^1 &= \frac{1}{4}d_1 + \frac{1}{2}d_2 + \frac{1}{4}d_3,
\end{aligned}
$$

and C_2 is given by

$$
\begin{aligned}
b_0^2 &= b_3^1 = \frac{1}{4}d_1 + \frac{1}{2}d_2 + \frac{1}{4}d_3 \\
b_1^2 &= \frac{1}{2}d_2 + \frac{1}{2}d_3 \\
b_2^2 &= d_3 \\
b_3^2 &= d_4.
\end{aligned}
$$