Curve Interpolation Using Quintic Bézier Splines

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1 Equations for a Generic Point x_i

We will compute the de Boor control points of a quintic spline C^4-curve so that it passes through $N+1$ prescribed data points $x_0, \ldots, x_N$. We assume that we have $N+1 \geq 8$ data points, which means that $N \geq 7$. Our method is based on the approach in which the de Boor control points of a B-spline curve are defined in terms of multiaffine maps, a method also known as "blossoming." We assume that the reader is familiar with this approach. A thorough presentation is given in Gallier [1], Part II, especially Chapter 6.

We will compute de Boor control points based on the uniform knot sequence

$$\begin{aligned}&00000,\ 00001,\ 00012,\ 00123,\ 01234,\ ,12345,\ 23456, \cdots ,\\ &N-4\,N-3\,N-2\,N-1\,N,\ N-3\,N-2\,N-1\,NN,\\ &N-2\,N-1\,NNN,\ N-1\,NNNN,\ NNNNN.\end{aligned}$$

We will denote the polar values $f(uvwxy)$ by $d_{-2} = f(00000) = x_0$, $d_{-1} = f(00001)$, $d_0 = f(00012)$, $d_1 = f(00123), d_2 = f(01234)$, and more generally

$$d_i = f(i-2\,i-1\,i\,i+1\,i+2), \quad 2 \leq i \leq N-2,$$

and ending with the polar values

$$\begin{aligned}&f(N-4\,N-3\,N-2\,N-1\,N),\ f(N-3\,N-2\,N-1\,NN),\ f(N-2\,N-1\,NNN),\\ &f(N-1\,NNNN),\ f(NNNNN),\end{aligned}$$

with $d_{N-1} = f(N-3\,N-2\,N-1\,NN)$, $d_N = f(N-2\,N-1\,NNN)$, $d_{N+1} = f(N-1\,NNNN)$, and $d_{N+2} = f(NNNNN) = x_N$.

Consequently, there are $N-1+6 = N+5$ de Boor points denoted

$$x_0 = d_{-2}, d_{-1}, d_0, d_1, \ldots, d_{N-1}, d_N, d_{N+1}, d_{N+2} = x_N.$$

There are $N-1$ equations corresponding to $x_1, \ldots, x_{N-1}$, so the four de Boor points $d_{-1}, d_0, d_N, d_{N+1}$ are not prescribed by the equations, and are thus free parameters. With our conventions, the spline curve consists of $N \geq 7$ quintic Bézier segments.

First, we compute the equation for a generic data point x_i, with $4 \leq i \leq N-4$ (which implies $N \geq 8$. To avoid notational complications with the indices, we show the computation of $f(44444) = x_4$:

$$\begin{array}{lllllllll}
f(01234) & & & & & & & & \\
 & 4/5 & f(12344) & & & & & & \\
f(12345) & & & 3/4 & f(23444) & & & & \\
 & 3/5 & f(23454) & & & 2/3 & f(34444) & & \\
f(23456) & & & 2/4 & f(34544) & & & 1/2 & f(44444). \\
 & 2/5 & f(34564) & & & 1/3 & f(45444) & & \\
f(34567) & & & 1/4 & f(45644) & & & & \\
 & 1/5 & f(45674) & & & & & & \\
f(45678) & & & & & & & &
\end{array}$$

A graphical illustration of $f(44444)$ is shown in Figure 1.

Note that in addition to

$$\begin{aligned}
f(34444) &= \frac{1}{3} f(23444) + \frac{2}{3} f(34544) \\
f(45444) &= \frac{2}{3} f(34544) + \frac{1}{3} f(45644),
\end{aligned}$$

we can also compute

$$\begin{aligned}
f(33444) &= \frac{2}{3} f(23444) + \frac{1}{3} f(34544) \\
f(55444) &= \frac{1}{3} f(34544) + \frac{2}{3} f(45644).
\end{aligned}$$

The polar values

$$f(33333),\ f(43333),\ f(44333),\ f(44433),\ f(44443),\ f(44444),$$

are the Bézier control points of the curve segment between $x_3 = f(33333)$ and $x_4 = f(44444)$, and we see that we can compute the fourth and fifth Bézier control points $f(33444)$, $f(34444)$

of the curve segment C_4 between x_3 and x_4, as well as the second and third Bézier control points $f(54444), f(55444)$ of the curve segment C_5 between x_4 and x_5.

This is a general fact, and we organize the computation as follows. For any i with $4 \leq i \leq N-4$ ($N \geq 8$), given $d_{i-2}, d_{i-1}, d_i, d_{i+1}, d_{i+2}$, we compute

$$\begin{aligned}
&d^1_{i,0}, d^1_{i,1}, d^1_{i,2}, d^1_{i,3},\\
&d^2_{i,0}, d^2_{i,1}, d^2_{i,2},\\
&d^3_{i,0}, d^3_{i,1}, d^3_{i,2}, d^3_{i,3},\\
&d^4_{i,0},
\end{aligned}$$

where $d^3_{i,0}, d^3_{i,1}$ are the fourth and fifth Bézier control points of C_i and $d^3_{i,2}, d^3_{i,3}$ are the second and third Bézier control points of C_{i+1}. For example, if $i = 4$, then $d^3_{i,0} = f(33444)$, $d^3_{i,1} = f(34444)$, $d^3_{i,2} = f(54444)$, and $d^3_{0,3} = f(55444)$.

We have

$$\begin{aligned}
d^1_{i,0} &= \frac{1}{5}d_{i-2} + \frac{4}{5}d_{i-1}\\
d^1_{i,1} &= \frac{2}{5}d_{i-1} + \frac{3}{5}d_i\\
d^1_{i,2} &= \frac{3}{5}d_i + \frac{2}{5}d_{i+1}\\
d^1_{i,3} &= \frac{4}{5}d_{i+1} + \frac{1}{5}d_{i+2},
\end{aligned}$$

$$\begin{aligned}
d^2_{i,0} &= \frac{1}{4}d^1_{i,0} + \frac{3}{4}d^1_{i,1}\\
d^2_{i,1} &= \frac{2}{4}d^1_{i,1} + \frac{2}{4}d^1_{i,2}\\
d^2_{i,2} &= \frac{3}{4}d^1_{i,2} + \frac{1}{4}d^1_{i,3},
\end{aligned}$$

$$\begin{aligned}
d^3_{i,0} &= \frac{2}{3}d^2_{i,0} + \frac{1}{3}d^2_{i,1}\\
d^3_{i,1} &= \frac{1}{3}d^2_{i,0} + \frac{2}{3}d^2_{i,1}\\
d^3_{i,2} &= \frac{2}{3}d^2_{i,1} + \frac{1}{3}d^2_{i,2}\\
d^3_{i,3} &= \frac{1}{3}d^2_{i,1} + \frac{2}{3}d^2_{i,2},
\end{aligned}$$

and finally,

$$x_i = d^4_{i,0} = \frac{1}{2}d^3_{i,1} + \frac{1}{2}d^3_{i,2}.$$

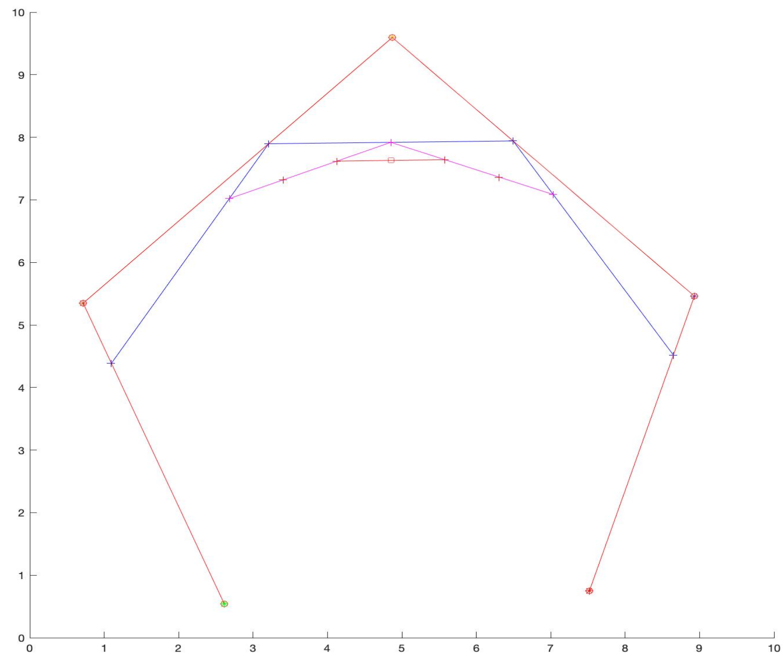

Figure 1: Construction of a point on a quintic from five de Boor control points.

The construction of the point on the spline curve, $d^4_{i,0}$, is illustrated in Figure 1. We get

$$\begin{aligned}
d^2_{i,0} &= \frac{1}{4}d^1_{i,0} + \frac{3}{4}d^1_{i,1} \\
&= \frac{1}{4}\left(\frac{1}{5}d_{i-2} + \frac{4}{5}d_{i-1}\right) + \frac{3}{4}\left(\frac{2}{5}d_{i-1} + \frac{3}{5}d_i\right) \\
&= \frac{1}{20}d_{i-2} + \frac{1}{2}d_{i-1} + \frac{9}{20}d_i \\
&= \frac{1}{20}d_{i-2} + \frac{10}{20}d_{i-1} + \frac{9}{20}d_i,
\end{aligned}$$

$$\begin{aligned}
d^2_{i,1} &= \frac{2}{4}d^1_{i,1} + \frac{2}{4}d^1_{i,2} \\
&= \frac{2}{4}\left(\frac{2}{5}d_{i-1} + \frac{3}{5}d_i\right) + \frac{2}{4}\left(\frac{3}{5}d_i + \frac{2}{5}d_{i+1}\right) \\
&= \frac{1}{5}d_{i-1} + \frac{3}{5}d_i + \frac{1}{5}d_{i+1},
\end{aligned}$$

and

$$\begin{aligned}
d_{i,2}^2 &= \frac{3}{4}d_{i,2}^1 + \frac{1}{4}d_{i,3}^1 \\
&= \frac{3}{4}\left(\frac{3}{5}d_i + \frac{2}{5}d_{i+1}\right) + \frac{1}{4}\left(\frac{4}{5}d_{i+1} + \frac{1}{5}d_{i+2},\right) \\
&= \frac{9}{20}d_i + \frac{1}{2}d_{i+1} + \frac{1}{20}d_{i+2} \\
&= \frac{9}{20}d_i + \frac{10}{20}d_{i+1} + \frac{1}{20}d_{i+2}.
\end{aligned}$$

Next, we get

$$\begin{aligned}
d_{i,0}^3 &= \frac{2}{3}d_{i,0}^2 + \frac{1}{3}d_{i,1}^2 \\
&= \frac{2}{3}\left(\frac{1}{20}d_{i-2} + \frac{1}{2}d_{i-1} + \frac{9}{20}d_i\right) + \frac{1}{3}\left(\frac{1}{5}d_{i-1} + \frac{3}{5}d_i + \frac{1}{5}d_{i+1},\right) \\
&= \frac{1}{30}d_{i-2} + \frac{2}{5}d_{i-1} + \frac{1}{2}d_i + \frac{1}{15}d_{i+1} \\
&= \frac{1}{30}d_{i-2} + \frac{12}{30}d_{i-1} + \frac{15}{30}d_i + \frac{2}{30}d_{i+1},
\end{aligned}$$

$$\begin{aligned}
d_{i,1}^3 &= \frac{1}{3}d_{i,0}^2 + \frac{2}{3}d_{i,1}^2 \\
&= \frac{1}{3}\left(\frac{1}{20}d_{i-2} + \frac{1}{2}d_{i-1} + \frac{9}{20}d_i\right) + \frac{2}{3}\left(\frac{1}{5}d_{i-1} + \frac{3}{5}d_i + \frac{1}{5}d_{i+1}\right) \\
&= \frac{1}{60}d_{i-2} + \frac{3}{10}d_{i-1} + \frac{11}{20}d_i + \frac{2}{15}d_{i+1} \\
&= \frac{1}{60}d_{i-2} + \frac{18}{60}d_{i-1} + \frac{33}{60}d_i + \frac{8}{60}d_{i+1},
\end{aligned}$$

$$\begin{aligned}
d_{i,2}^3 &= \frac{2}{3}d_{i,1}^2 + \frac{1}{3}d_{i,2}^2 \\
&= \frac{2}{3}\left(\frac{1}{5}d_{i-1} + \frac{3}{5}d_i + \frac{1}{5}d_{i+1}\right) + \frac{1}{3}\left(\frac{9}{20}d_i + \frac{1}{2}d_{i+1} + \frac{1}{20}d_{i+2}\right) \\
&= \frac{2}{15}d_{i-1} + \frac{11}{20}d_i + \frac{3}{10}d_{i+1} + \frac{1}{60}d_{i+2} \\
&= \frac{8}{60}d_{i-1} + \frac{33}{60}d_i + \frac{18}{60}d_{i+1} + \frac{1}{60}d_{i+2},
\end{aligned}$$

$$
\begin{aligned}
d^3_{i,3} &= \frac{1}{3}d^2_{i,1} + \frac{2}{3}d^2_{i,2} \\
&= \frac{1}{3}\left(\frac{1}{5}d_{i-1} + \frac{3}{5}d_i + \frac{1}{5}d_{i+1}\right) + \frac{2}{3}\left(\frac{9}{20}d_i + \frac{1}{2}d_{i+1} + \frac{1}{20}d_{i+2}\right) \\
&= \frac{1}{15}d_{i-1} + \frac{1}{2}d_i + \frac{2}{5}d_{i+1} + \frac{1}{30}d_{i+2} \\
&= \frac{2}{30}d_{i-1} + \frac{15}{30}d_i + \frac{12}{30}d_{i+1} + \frac{1}{30}d_{i+2}.
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
d^4_{i,0} &= \frac{1}{2}d^3_{i,1} + \frac{1}{2}d^3_{i,2} \\
&= \frac{1}{2}\left(\frac{1}{60}d_{i-2} + \frac{3}{10}d_{i-1} + \frac{11}{20}d_i + \frac{2}{15}d_{i+1}\right) + \frac{1}{2}\left(\frac{2}{15}d_{i-1} + \frac{11}{20}d_i + \frac{3}{10}d_{i+1} + \frac{1}{60}d_{i+2}\right) \\
&= \frac{1}{120}d_{i-2} + \frac{13}{60}d_{i-1} + \frac{11}{20}d_i + \frac{13}{60}d_{i+1} + \frac{1}{120}d_{i+2} \\
&= \frac{1}{120}d_{i-2} + \frac{26}{120}d_{i-1} + \frac{66}{120}d_i + \frac{26}{120}d_{i+1} + \frac{1}{120}d_{i+2},
\end{aligned}
$$

and since $x_i = d^4_{i,0}$, we get the equation

$$
x_i = \frac{1}{120}d_{i-2} + \frac{26}{120}d_{i-1} + \frac{66}{120}d_i + \frac{26}{120}d_{i+1} + \frac{1}{120}d_{i+2}.
$$

2 Equations for the Second Data Point x_1

We now consider the first segment C_1 of the spline curve. We compute $f(11111) = x_1$:

$$
\begin{array}{lllllllll}
f(00001) & & & & & & & & \\
 & 1/2 & f(00011) & & & & & & \\
f(00012) & & & 1/2 & f(00111) & & & & \\
 & 1/3 & f(00121) & & & 1/2 & f(01111) & & \\
f(00123) & & & 1/3 & f(01121) & & & 1/2 & f(11111). \\
 & 1/4 & f(01231) & & & 1/3 & f(11211) & & \\
f(01234) & & & 1/4 & f(12311) & & & & \\
 & 1/5 & f(12341) & & & & & & \\
f(12345) & & & & & & & &
\end{array}
$$

Given $d_{-1}, d_0, d_1, d_2, d_3$, we compute

$$
\begin{aligned}
&d^1_{1,0}, d^1_{1,1}, d^1_{1,2}, d^1_{1,3}, \\
&d^2_{1,0}, d^2_{1,1}, d^2_{1,2}, \\
&d^3_{1,0} = d^3_{1,1}, d^3_{1,2}, d^3_{1,3}, \\
&d^4_{1,0}
\end{aligned}
$$

This case is exceptional and the Bézier control points of the first segment are $x_0, d_{-1}, d^1_{1,0}$, $d^2_{1,0}, d^3_{1,0}, x_1$. The points $d^3_{1,2}, d^3_{1,3}$ are the second and third control points of C_2.

We compute

$$\begin{aligned}
d^1_{1,0} &= \frac{1}{2}d_{-1} + \frac{1}{2}d_0\\
d^1_{1,1} &= \frac{2}{3}d_0 + \frac{1}{3}d_1\\
d^1_{1,2} &= \frac{3}{4}d_1 + \frac{1}{4}d_2\\
d^1_{1,3} &= \frac{4}{5}d_2 + \frac{1}{5}d_3,
\end{aligned}$$

$$\begin{aligned}
d^2_{1,0} &= \frac{1}{2}d^1_{1,0} + \frac{1}{2}d^1_{1,1}\\
d^2_{1,1} &= \frac{2}{3}d^1_{1,1} + \frac{1}{3}d^1_{1,2}\\
d^2_{1,2} &= \frac{3}{4}d^1_{1,2} + \frac{1}{4}d^1_{1,3},
\end{aligned}$$

$$\begin{aligned}
d^3_{1,1} &= \frac{1}{2}d^2_{1,0} + \frac{1}{2}d^2_{1,1}\\
d^3_{1,2} &= \frac{2}{3}d^2_{1,1} + \frac{1}{3}d^2_{1,2}\\
d^3_{1,3} &= \frac{1}{3}d^2_{1,1} + \frac{2}{3}d^2_{1,2}
\end{aligned}$$

and finally,

$$x_1 = d^4_{1,0} = \frac{1}{2}d^3_{1,1} + \frac{1}{2}d^3_{1,2}.$$

We have

$$\begin{aligned}
d^2_{1,0} &= \frac{1}{2}\left(\frac{1}{2}d_{-1} + \frac{1}{2}d_0\right) + \frac{1}{2}\left(\frac{2}{3}d_0 + \frac{1}{3}d_1\right)\\
&= \frac{1}{4}d_{-1} + \frac{7}{12}d_0 + \frac{1}{6}d_1\\
&= \frac{3}{12}d_{-1} + \frac{7}{12}d_0 + \frac{2}{12}d_1,
\end{aligned}$$

$$\begin{aligned}
d^2_{1,1} &= \frac{2}{3}\left(\frac{2}{3}d_0 + \frac{1}{3}d_1\right) + \frac{1}{3}\left(\frac{3}{4}d_1 + \frac{1}{4}d_2\right)\\
&= \frac{4}{9}d_0 + \frac{17}{36}d_1 + \frac{1}{12}d_2\\
&= \frac{16}{36}d_0 + \frac{17}{36}d_1 + \frac{3}{36}d_2,
\end{aligned}$$

$$
\begin{aligned}
d_{1,2}^2 &= \frac{3}{4}\left(\frac{3}{4}d_1 + \frac{1}{4}d_2\right) + \frac{1}{4}\left(\frac{4}{5}d_2 + \frac{1}{5}d_3\right) \\
&= \frac{9}{16}d_1 + \frac{31}{80}d_2 + \frac{1}{20}d_3 \\
&= \frac{45}{80}d_1 + \frac{31}{80}d_2 + \frac{4}{80}d_3,
\end{aligned}
$$

then

$$
\begin{aligned}
d_{1,0}^3 = d_{1,1}^3 &= \frac{1}{2}\left(\frac{3}{12}d_{-1} + \frac{7}{12}d_0 + \frac{2}{12}d_1\right) + \frac{1}{2}\left(\frac{16}{36}d_0 + \frac{17}{36}d_1 + \frac{3}{36}d_2\right) \\
&= \frac{1}{8}d_{-1} + \frac{37}{72}d_0 + \frac{23}{72}d_1 + \frac{1}{24}d_2 \\
&= \frac{9}{72}d_{-1} + \frac{37}{72}d_0 + \frac{23}{72}d_1 + \frac{3}{72}d_2,
\end{aligned}
$$

$$
\begin{aligned}
d_{1,2}^3 &= \frac{2}{3}\left(\frac{16}{36}d_0 + \frac{17}{36}d_1 + \frac{3}{36}d_2\right) + \frac{1}{3}\left(\frac{45}{80}d_1 + \frac{31}{80}d_2 + \frac{4}{80}d_3\right) \\
&= \frac{8}{27}d_0 + \frac{217}{432}d_1 + \frac{133}{720}d_2 + \frac{1}{60}d_3 \\
&= \frac{640}{2160}d_0 + \frac{1085}{2160}d_1 + \frac{399}{2160}d_2 + \frac{36}{2160}d_3,
\end{aligned}
$$

$$
\begin{aligned}
d_{1,3}^3 &= \frac{1}{3}\left(\frac{16}{36}d_0 + \frac{17}{36}d_1 + \frac{3}{36}d_2\right) + \frac{2}{3}\left(\frac{45}{80}d_1 + \frac{31}{80}d_2 + \frac{4}{80}d_3\right) \\
&= \frac{4}{27}d_0 + \frac{115}{216}d_1 + \frac{103}{360}d_2 + \frac{1}{30}d_3 \\
&= \frac{160}{1080}d_0 + \frac{575}{1080}d_1 + \frac{309}{1080}d_2 + \frac{36}{1080}d_3,
\end{aligned}
$$

Finally,

$$
\begin{aligned}
d_{1,0}^4 &= \frac{1}{2}\left(\frac{1}{8}d_{-1} + \frac{37}{72}d_0 + \frac{23}{72}d_1 + \frac{1}{24}d_2\right) + \frac{1}{2}\left(\frac{8}{27}d_0 + \frac{217}{432}d_1 + \frac{133}{720}d_2 + \frac{1}{60}d_3\right) \\
&= \frac{1}{16}d_{-1} + \frac{175}{432}d_0 + \frac{355}{864}d_1 + \frac{163}{1440}d_2 + \frac{1}{120}d_3.
\end{aligned}
$$

Therefore, we have the equation

$$
x_1 = \frac{1}{16}d_{-1} + \frac{175}{432}d_0 + \frac{355}{864}d_1 + \frac{163}{1440}d_2 + \frac{1}{120}d_3.
$$

3 Equations for the Third Data Point x_2

We now consider the second segment C_2 of the spline curve. We compute $f(22222) = x_2$:

$$
\begin{array}{ccccccccc}
f(00012) & & & & & & & & \\
 & 2/3 & f(00122) & & & & & & \\
f(00123) & & & 2/3 & f(01222) & & & & \\
 & 2/4 & f(01232) & & & 2/3 & f(12222) & & \\
f(01234) & & & 2/4 & f(12322) & & & 1/2 & f(22222). \\
 & 2/5 & f(12342) & & & 1/3 & f(23222) & & \\
f(12345) & & & 1/4 & f(23422) & & & & \\
 & 1/5 & f(23452) & & & & & & \\
f(23456) & & & & & & & &
\end{array}
$$

Given d_0, d_1, d_2, d_3, d_4, we compute

$$
\begin{aligned}
&d^1_{2,0}, d^1_{2,1}, d^1_{2,2}, d^1_{2,3},\\
&d^2_{2,0}, d^2_{2,1}, d^2_{2,2},\\
&d^3_{2,0}, d^3_{2,1}, d^3_{2,2}, d^3_{2,3},\\
&d^4_{2,0}.
\end{aligned}
$$

The points $d^3_{2,0}, d^3_{2,1}$ are the fourth and fifth Bézier control points of C_2, and $d^3_{2,2}, d^3_{2,3}$ are the second and third Bézier control points of C_3.

We compute

$$
\begin{aligned}
d^1_{2,0} &= \frac{1}{3}d_0 + \frac{2}{3}d_1\\
d^1_{2,1} &= \frac{2}{4}d_1 + \frac{2}{4}d_2\\
d^1_{2,2} &= \frac{3}{5}d_2 + \frac{2}{5}d_3\\
d^1_{2,3} &= \frac{4}{5}d_3 + \frac{1}{5}d_4,
\end{aligned}
$$

$$
\begin{aligned}
d^2_{2,0} &= \frac{1}{3}d^1_{2,0} + \frac{2}{3}d^1_{2,1}\\
d^2_{2,1} &= \frac{2}{4}d^1_{2,1} + \frac{2}{4}d^1_{2,2}\\
d^2_{2,2} &= \frac{3}{4}d^1_{2,2} + \frac{1}{4}d^1_{2,3},
\end{aligned}
$$

$$
\begin{aligned}
d_{2,0}^3 &= \frac{2}{3}d_{2,0}^2 + \frac{1}{3}d_{2,1}^2 \\
d_{2,1}^3 &= \frac{1}{3}d_{2,0}^2 + \frac{2}{3}d_{2,1}^2 \\
d_{2,2}^3 &= \frac{2}{3}d_{2,1}^2 + \frac{1}{3}d_{2,2}^2 \\
d_{2,3}^3 &= \frac{1}{3}d_{2,1}^2 + \frac{2}{3}d_{2,2}^2
\end{aligned}
$$

and finally,

$$x_2 = d_{2,0}^4 = \frac{1}{2}d_{2,1}^3 + \frac{1}{2}d_{2,2}^3.$$

We get

$$
\begin{aligned}
d_{2,0}^2 &= \frac{1}{3}\left(\frac{1}{3}d_0 + \frac{2}{3}d_1\right) + \frac{2}{3}\left(\frac{2}{4}d_1 + \frac{2}{4}d_2\right) \\
&= \frac{1}{9}d_0 + \frac{5}{9}d_1 + \frac{1}{3}d_2 \\
&= \frac{1}{9}d_0 + \frac{5}{9}d_1 + \frac{3}{9}d_2,
\end{aligned}
$$

$$
\begin{aligned}
d_{2,1}^2 &= \frac{2}{4}\left(\frac{2}{4}d_1 + \frac{2}{4}d_2\right) + \frac{2}{4}\left(\frac{3}{5}d_2 + \frac{2}{5}d_3\right) \\
&= \frac{1}{4}d_1 + \frac{11}{20}d_2 + \frac{1}{5}d_3 \\
&= \frac{5}{20}d_1 + \frac{11}{20}d_2 + \frac{4}{20}d_3,
\end{aligned}
$$

$$
\begin{aligned}
d_{2,2}^2 &= \frac{3}{4}\left(\frac{3}{5}d_2 + \frac{2}{5}d_3\right) + \frac{1}{4}\left(\frac{4}{5}d_3 + \frac{1}{5}d_4\right) \\
&= \frac{9}{20}d_2 + \frac{1}{2}d_3 + \frac{1}{20}d_4 \\
&= \frac{9}{20}d_2 + \frac{10}{20}d_3 + \frac{1}{20}d_4,
\end{aligned}
$$

then

$$
\begin{aligned}
d_{2,0}^3 &= \frac{2}{3}\left(\frac{1}{9}d_0 + \frac{5}{9}d_1 + \frac{1}{3}d_2\right) + \frac{1}{3}\left(\frac{1}{4}d_1 + \frac{11}{20}d_2 + \frac{1}{5}d_3\right) \\
&= \frac{2}{27}d_0 + \frac{49}{108}d_1 + \frac{73}{180}d_2 + \frac{1}{15}d_3 \\
&= \frac{40}{540}d_0 + \frac{245}{540}d_1 + \frac{219}{540}d_2 + \frac{36}{540}d_3,
\end{aligned}
$$

$$
\begin{aligned}
d_{2,1}^3 &= \frac{1}{3}\left(\frac{1}{9}d_0 + \frac{5}{9}d_1 + \frac{1}{3}d_2\right) + \frac{2}{3}\left(\frac{1}{4}d_1 + \frac{11}{20}d_2 + \frac{1}{5}d_3\right) \\
&= \frac{1}{27}d_0 + \frac{19}{54}d_1 + \frac{43}{90}d_2 + \frac{2}{15}d_3 \\
&= \frac{10}{270}d_0 + \frac{95}{270}d_1 + \frac{129}{270}d_2 + \frac{36}{270}d_3
\end{aligned}
$$

$$
\begin{aligned}
d_{2,2}^3 &= \frac{2}{3}\left(\frac{1}{4}d_1 + \frac{11}{20}d_2 + \frac{1}{5}d_3\right) + \frac{1}{3}\left(\frac{9}{20}d_2 + \frac{1}{2}d_3 + \frac{1}{20}d_4\right) \\
&= \frac{1}{6}d_1 + \frac{31}{60}d_2 + \frac{3}{10}d_3 + \frac{1}{60}d_4 \\
&= \frac{10}{60}d_1 + \frac{31}{60}d_2 + \frac{18}{60}d_3 + \frac{1}{60}d_4
\end{aligned}
$$

$$
\begin{aligned}
d_{2,3}^3 &= \frac{1}{3}\left(\frac{1}{4}d_1 + \frac{11}{20}d_2 + \frac{1}{5}d_3\right) + \frac{2}{3}\left(\frac{9}{20}d_2 + \frac{1}{2}d_3 + \frac{1}{20}d_4\right) \\
&= \frac{1}{12}d_1 + \frac{29}{60}d_2 + \frac{2}{5}d_3 + \frac{1}{30}d_4 \\
&= \frac{5}{60}d_1 + \frac{29}{60}d_2 + \frac{24}{60}d_3 + \frac{2}{60}d_4.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
d_{2,0}^4 &= \frac{1}{2}\left(\frac{1}{27}d_0 + \frac{19}{54}d_1 + \frac{43}{90}d_2 + \frac{2}{15}d_3\right) + \frac{1}{2}\left(\frac{1}{6}d_1 + \frac{31}{60}d_2 + \frac{3}{10}d_3 + \frac{1}{60}d_4\right) \\
&= \frac{1}{54}d_0 + \frac{7}{27}d_1 + \frac{179}{360}d_2 + \frac{13}{60}d_3 + \frac{1}{120}d_4 \\
&= \frac{20}{1080}d_0 + \frac{280}{1080}d_1 + \frac{537}{1080}d_2 + \frac{234}{1080}d_3 + \frac{9}{1080}d_4.
\end{aligned}
$$

Therefore, we get the equation

$$
x_2 = \frac{1}{54}d_0 + \frac{7}{27}d_1 + \frac{179}{360}d_2 + \frac{26}{120}d_3 + \frac{1}{120}d_4.
$$

4 Equations for the Fourth Data Point x_3

Next we consider the third segment C_3 of the spline curve. We compute $f(33333) = x_3$:

$$\begin{array}{ccccccccc}
f(00123) & & & & & & & & \\
 & 3/4 & f(01233) & & & & & & \\
f(01234) & & & 3/4 & f(12333) & & & & \\
 & 3/5 & f(12343) & & & 2/3 & f(23333) & & \\
f(12345) & & & 2/4 & f(23433) & & & 1/2 & f(33333). \\
 & 2/5 & f(23453) & & & 1/3 & f(34333) & & \\
f(23456) & & & 1/4 & f(34533) & & & & \\
 & 1/5 & f(34563) & & & & & & \\
f(34567) & & & & & & & &
\end{array}$$

Given $d_1, d_2, d_3,\ d_4, d_5$, we compute

$$\begin{aligned}
&d^1_{3,0}, d^1_{3,1}, d^1_{3,2}, d^1_{3,3},\\
&d^2_{3,0}, d^2_{3,1}, d^2_{3,2},\\
&d^3_{3,0}, d^3_{3,1}, d^3_{3,2}, d^3_{3,3},\\
&d^4_{3,0}.
\end{aligned}$$

The points $d^3_{3,0}, d^3_{3,1}$ are the fourth and fifth Bézier control points of C_3, and $d^3_{3,2}, d^3_{3,3}$ are the second and third Bézier control points of C_4,.

We compute

$$\begin{aligned}
d^1_{3,0} &= \frac{1}{4}d_1 + \frac{3}{4}d_2\\
d^1_{3,1} &= \frac{2}{5}d_2 + \frac{3}{5}d_3\\
d^1_{3,2} &= \frac{3}{5}d_3 + \frac{2}{5}d_4\\
d^1_{3,3} &= \frac{4}{5}d_4 + \frac{1}{5}d_5,
\end{aligned}$$

$$\begin{aligned}
d^2_{3,0} &= \frac{1}{4}d^1_{3,0} + \frac{3}{4}d^1_{3,1}\\
d^2_{3,1} &= \frac{2}{4}d^1_{3,1} + \frac{2}{4}d^1_{3,2}\\
d^2_{3,2} &= \frac{3}{4}d^1_{3,2} + \frac{1}{4}d^1_{3,3},
\end{aligned}$$

$$
\begin{aligned}
d_{3,0}^3 &= \frac{2}{3} d_{3,0}^2 + \frac{1}{3} d_{3,1}^2 \\
d_{3,1}^3 &= \frac{1}{3} d_{3,0}^2 + \frac{2}{3} d_{3,1}^2 \\
d_{3,2}^3 &= \frac{2}{3} d_{3,1}^2 + \frac{1}{3} d_{3,2}^2 \\
d_{3,3}^3 &= \frac{1}{3} d_{3,1}^2 + \frac{2}{3} d_{3,2}^2
\end{aligned}
$$

and finally,

$$
x_3 = d_{3,0}^4 = \frac{1}{2} d_{3,1}^3 + \frac{1}{2} d_{3,2}^3.
$$

We get

$$
\begin{aligned}
d_{3,0}^2 &= \frac{1}{4}\left(\frac{1}{4} d_1 + \frac{3}{4} d_2\right) + \frac{3}{4}\left(\frac{2}{5} d_2 + \frac{3}{5} d_3\right) \\
&= \frac{1}{16} d_1 + \frac{39}{80} d_2 + \frac{9}{20} d_3 \\
&= \frac{5}{80} d_1 + \frac{39}{80} d_2 + \frac{36}{80} d_3,
\end{aligned}
$$

$$
\begin{aligned}
d_{3,1}^2 &= \frac{2}{4}\left(\frac{2}{5} d_2 + \frac{3}{5} d_3\right) + \frac{2}{4}\left(\frac{3}{5} d_3 + \frac{2}{5} d_4\right) \\
&= \frac{1}{5} d_2 + \frac{3}{5} d_3 + \frac{1}{5} d_4,
\end{aligned}
$$

$$
\begin{aligned}
d_{3,2}^2 &= \frac{3}{4}\left(\frac{3}{5} d_3 + \frac{2}{5} d_4\right) + \frac{1}{4}\left(\frac{4}{5} d_4 + \frac{1}{5} d_5\right) \\
&= \frac{9}{20} d_3 + \frac{1}{2} d_4 + \frac{1}{20} d_5 \\
&= \frac{9}{20} d_3 + \frac{10}{20} d_4 + \frac{1}{20} d_5,
\end{aligned}
$$

then

$$
\begin{aligned}
d_{3,0}^3 &= \frac{2}{3}\left(\frac{1}{16} d_1 + \frac{39}{80} d_2 + \frac{9}{20} d_3\right) + \frac{1}{3}\left(\frac{1}{5} d_2 + \frac{3}{5} d_3 + \frac{1}{5} d_4\right) \\
&= \frac{1}{24} d_1 + \frac{47}{120} d_2 + \frac{1}{2} d_3 + \frac{1}{15} d_4 \\
&= \frac{5}{120} d_1 + \frac{47}{120} d_2 + \frac{60}{120} d_3 + \frac{8}{120} d_4
\end{aligned}
$$

$$
\begin{aligned}
d_{3,1}^3 &= \frac{1}{3}\left(\frac{1}{16}d_1 + \frac{39}{80}d_2 + \frac{9}{20}d_3\right) + \frac{2}{3}\left(\frac{1}{5}d_2 + \frac{3}{5}d_3 + \frac{1}{5}d_4\right) \\
&= \frac{1}{48}d_1 + \frac{71}{240}d_2 + \frac{11}{20}d_3 + \frac{2}{15}d_4 \\
&= \frac{5}{240}d_1 + \frac{71}{240}d_2 + \frac{132}{240}d_3 + \frac{32}{240}d_4
\end{aligned}
$$

$$
\begin{aligned}
d_{3,2}^3 &= \frac{2}{3}\left(\frac{1}{5}d_2 + \frac{3}{5}d_3 + \frac{1}{5}d_4\right) + \frac{1}{3}\left(\frac{9}{20}d_3 + \frac{1}{2}d_4 + \frac{1}{20}d_5\right) \\
&= \frac{2}{15}d_2 + \frac{11}{20}d_3 + \frac{3}{10}d_4 + \frac{1}{60}d_5 \\
&= \frac{8}{60}d_2 + \frac{33}{60}d_3 + \frac{18}{60}d_4 + \frac{1}{60}d_5
\end{aligned}
$$

$$
\begin{aligned}
d_{3,3}^3 &= \frac{1}{3}\left(\frac{1}{5}d_2 + \frac{3}{5}d_3 + \frac{1}{5}d_4\right) + \frac{2}{3}\left(\frac{9}{20}d_3 + \frac{1}{2}d_4 + \frac{1}{20}d_5\right) \\
&= \frac{1}{15}d_2 + \frac{1}{2}d_3 + \frac{2}{5}d_4 + \frac{1}{30}d_5 \\
&= \frac{4}{60}d_2 + \frac{30}{60}d_3 + \frac{24}{60}d_4 + \frac{2}{60}d_5.
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
d_{3,0}^4 &= \frac{1}{2}\left(\frac{1}{48}d_1 + \frac{71}{240}d_2 + \frac{11}{20}d_3 + \frac{2}{15}d_4\right) + \frac{1}{2}\left(\frac{2}{15}d_2 + \frac{11}{20}d_3 + \frac{3}{10}d_4 + \frac{1}{60}d_5\right) \\
&= \frac{1}{96}d_1 + \frac{103}{480}d_2 + \frac{11}{20}d_3 + \frac{13}{60}d_4 + \frac{1}{120}d_5,
\end{aligned}
$$

and so we have the equation

$$
x_3 = \frac{1}{96}d_1 + \frac{103}{480}d_2 + \frac{66}{120}d_3 + \frac{26}{120}d_4 + \frac{1}{120}d_5.
$$

5 Equations for the Data Points $x_{N-3}, x_{N-2}, x_{N-1}$

The first three equations are

$$
\begin{aligned}
\frac{355}{864}d_1 + \frac{163}{1440}d_2 + \frac{1}{120}d_3 &= x_1 - \frac{1}{16}d_{-1} - \frac{175}{432}d_0 \\
\frac{7}{27}d_1 + \frac{179}{360}d_2 + \frac{26}{120}d_3 + \frac{1}{120}d_4 &= x_2 - \frac{1}{54}d_0 \\
\frac{1}{96}d_1 + \frac{103}{480}d_2 + \frac{66}{120}d_3 + \frac{26}{120}d_4 + \frac{1}{120}d_5 &= x_3,
\end{aligned}
$$

and the generic equation is

$$\frac{1}{120}d_{i-2} + \frac{26}{120}d_{i-1} + \frac{66}{120}d_i + \frac{26}{120}d_{i+1} + \frac{1}{120}d_{i+2} = x_i.$$

Multiplying by 120, the first three equations are

$$\begin{aligned}
\frac{1775}{36}d_1 + \frac{163}{12}d_2 + d_3 &= 120x_1 - \frac{15}{2}d_{-1} - \frac{875}{18}d_0\\
\frac{280}{9}d_1 + \frac{179}{3}d_2 + 26d_3 + d_4 &= 120x_2 - \frac{20}{9}d_0\\
\frac{5}{4}d_1 + \frac{103}{4}d_2 + 66d_3 + 26d_4 + d_5 &= 120x_3,
\end{aligned}$$

and the generic equation is

$$d_{i-2} + 26d_{i-1} + 66d_i + 26d_{i+1} + d_{i+2} = 120x_i.$$

Because the spline curve begins with the polar values

$$f(00000),\ f(00001),\ f(00012),\ f(00123),\ f(01234),\ f(12345)$$

with $d_{-2} = f(00000) = x_0$, $d_{-1} = f(00001)$, $d_0 = f(00012)$, $d_1 = f(00123)$, and more generally

$$d_i = f(i-2\, i-1\, i\, i+1\, i+2), \quad 2 \leq i \leq N-2,$$

and ends with the polar values

$$\begin{aligned}
&f(N-3\, N-2\, N-1\, NN),\ f(N-2\, N-1\, NNN),\\
&f(N-1\, NNNN),\ f(NNNNN),
\end{aligned}$$

with $d_{N-1} = f(N-3\, N-2\, N-1\, NN)$, $d_N = f(N-2\, N-1\, NNN)$, $d_{N+1} = f(N-1\, NNNN)$, and $d_{N+2} = f(NNNNN) = x_N$, the last three equations for $x_{N-3}, x_{N-2}, x_{N-1}$ are just the equations for x_3, x_2, x_1 written in reverse order (with the variables substituted in a suitable fashion).

Therefore, the last three equations of the system are

$$\begin{aligned}
d_{N-5} + 26d_{N-4} + 66d_{N-3} + \frac{103}{4}d_{N-2} + \frac{5}{4}d_{N-1} &= 120x_{N-3}\\
d_{N-4} + 26d_{N-3} + \frac{179}{3}d_{N-2} + \frac{280}{9}d_{N-1} &= 120x_{N-2} - \frac{20}{9}d_N\\
d_{N-3} + \frac{163}{12}d_{N-2} + \frac{1775}{36}d_{N-1} &= 120x_{N-1} - \frac{875}{18}d_N - \frac{15}{2}d_{N+1}
\end{aligned}$$

The matrix of this linear system is

$$
\begin{pmatrix}
\frac{1775}{36} & \frac{163}{12} & 1 & & & & & & & & \\
\frac{280}{9} & \frac{179}{3} & 26 & 1 & & & & & & & \\
\frac{5}{4} & \frac{103}{4} & 66 & 26 & 1 & & & & & & \\
 & 1 & 26 & 66 & 26 & 1 & & & & & \\
 & & 1 & 26 & 66 & 26 & 1 & & & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & & 1 & 26 & 66 & 26 & 1 & & \\
 & & & & & 1 & 26 & 66 & 26 & 1 & \\
 & & & & & & 1 & 26 & 66 & \frac{103}{4} & \frac{5}{4} \\
 & & & & & & & 1 & 26 & \frac{179}{3} & \frac{280}{9} \\
 & & & & & & & & 1 & \frac{163}{12} & \frac{1775}{36}
\end{pmatrix}.
$$

This matrix is pentadiagonal. It is strictly diagonally dominant, and thus invertible (and Gaussian elimination does not require pivoting).

The right hand side is

$$
\begin{pmatrix}
120x_1 - \frac{15}{2}d_{-1} - \frac{875}{18}d_0 \\
120x_2 - \frac{20}{9}d_0 \\
120x_3 \\
120x_4 \\
\vdots \\
120x_{N-4} \\
120x_{N-3} \\
120x_{N-2} - \frac{20}{9}d_N \\
120x_{N-1} - \frac{15}{2}d_N - \frac{875}{18}d_{N+1}
\end{pmatrix}.
$$

We have not investigated methods for determining $d_{-1}, d_0, d_N, d_{N+1}$ (end conditions), except for the simple-minded method of setting $d_{-1} = d_0 = x_0 = d_{-2}$ and $d_N = d_{N+1} = x_N = d_{N+2}$. We have implemented this crude method, and it appears to give good results, but further investigation of end conditions remains to be conducted.

Here are three examples using the above crude method. Figure 2 shows an interpolating curve for 20 data points (so $N = 19$). The de Boor control points shown in blue are d_1 and d_{N-1}.

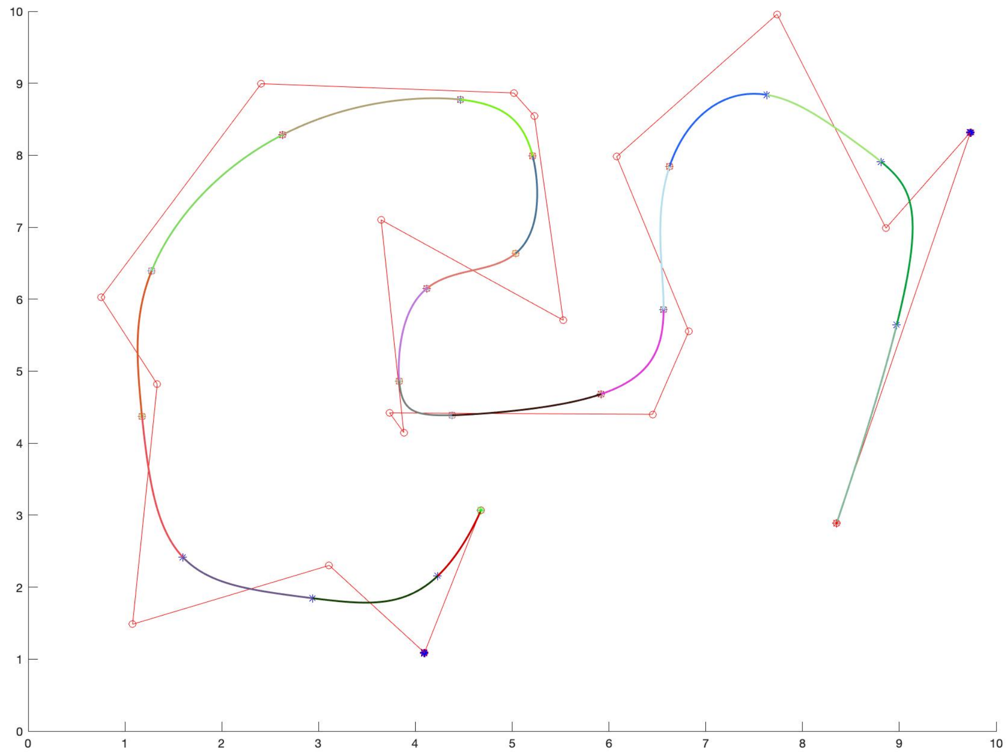

Figure 2: A quintic interpolating B-spline for 20 data points.

Figure 3 shows an interpolating curve for 22 data points (so $N = 21$). The construction of the Bézier control points is also shown. The de Boor control points shown in blue are d_1 and d_{N-1}.

Figure 4 shows an interpolating curve for 44 data points (so $N = 43$). The de Boor control points shown in blue are d_1 and d_{N-1}.

6 Control Points for the Bézier Curve Segments

Recall that we are assuming that we have $N + 1 \geq 8$ data points, which means that $N \geq 7$. There are $N + 5$ de Boor control points

$$d_{-2}, d_{-1}, d_0, d_1, \ldots, d_{N-1}, d_N, d_{N+1}, d_{N+2},$$

with $d_{-2} = x_0$ and $d_{N+2} = x_N$, and there are N quintic Bézier segments. The first three and the last three are exceptional.

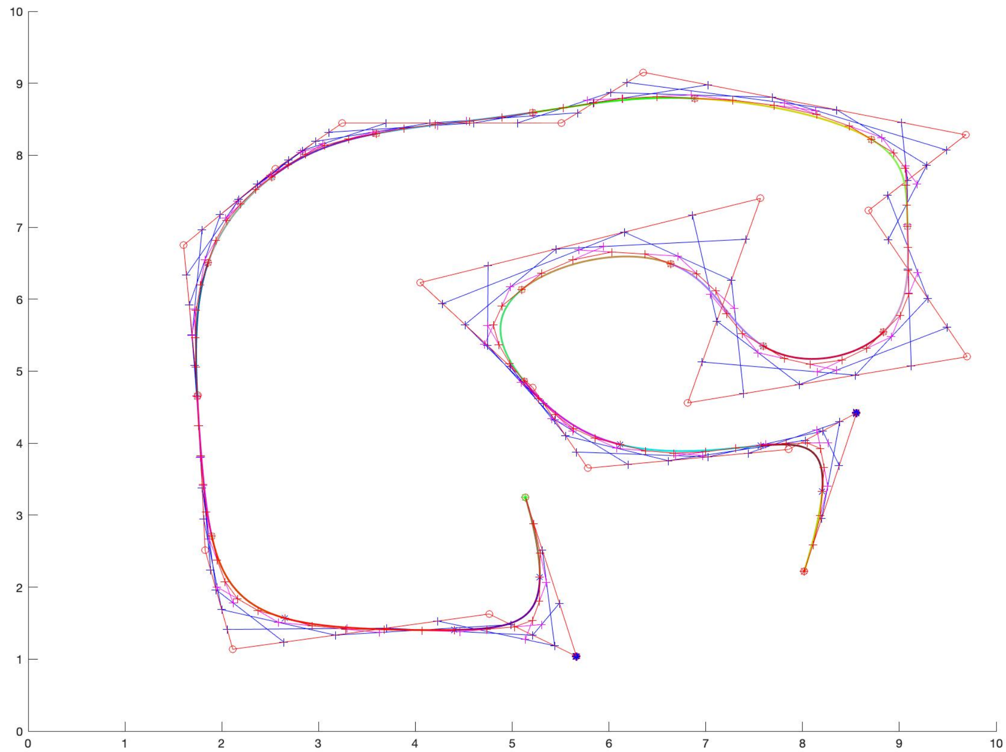

Figure 3: A quintic interpolating B-spline for 22 data points.

We first treat the case where $N \geq 8$, and then the special case where $N = 7$. The Bézier control points of the generic Bézier curve C_{i+1} between x_i and x_{i+1}, with $4 \leq i \leq N-4$ ($N \geq 8$), are

$$x_i, d^3_{i,2}, d^3_{i,3}, d^3_{i+1,0}, d^3_{i+1,1}, x_{i+1},$$

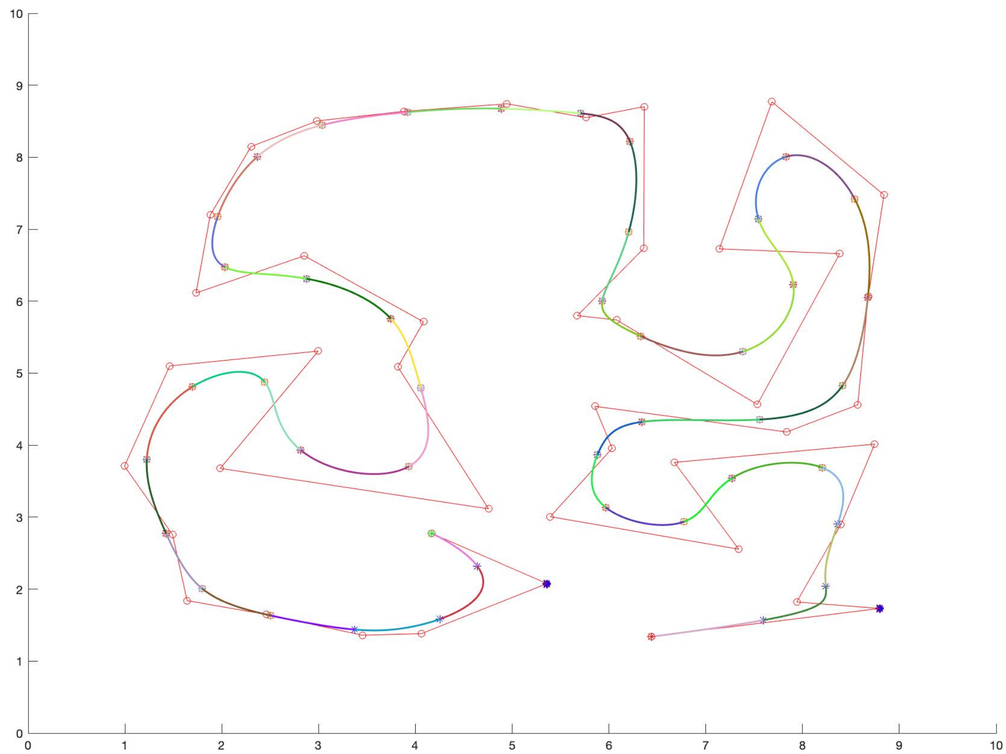

Figure 4: A quintic interpolating B-spline for 44 data points.

namely

$$
\begin{aligned}
b_{i+1}^0 &= x_i \\
b_{i+1}^1 &= \frac{2}{15}d_{i-1} + \frac{11}{20}d_i + \frac{3}{10}d_{i+1} + \frac{1}{60}d_{i+2} \\
b_{i+1}^2 &= \frac{1}{15}d_{i-1} + \frac{1}{2}d_i + \frac{2}{5}d_{i+1} + \frac{1}{30}d_{i+2} \\
b_{i+1}^3 &= \frac{1}{30}d_{i-1} + \frac{2}{5}d_i + \frac{1}{2}d_{i+1} + \frac{1}{15}d_{i+2} \\
b_{i+1}^4 &= \frac{1}{60}d_{i-1} + \frac{3}{10}d_i + \frac{11}{20}d_{i+1} + \frac{2}{15}d_{i+2} \\
b_{i+1}^5 &= x_{i+1}.
\end{aligned}
$$

When computing from the de Boor points, we have $b_{i+1}^0 = d_{i,0}^4$ and $b_{i+1}^5 = d_{i+1,0}^4$. The construction of a generic quintic Bézier curve determined by the control points

$$d_{i,0}^4, d_{i,2}^3, d_{i,3}^3, d_{i+1,0}^3, d_{i+1,1}^3, d_{i+1,0}^4$$

is illustrated in Figure 5.

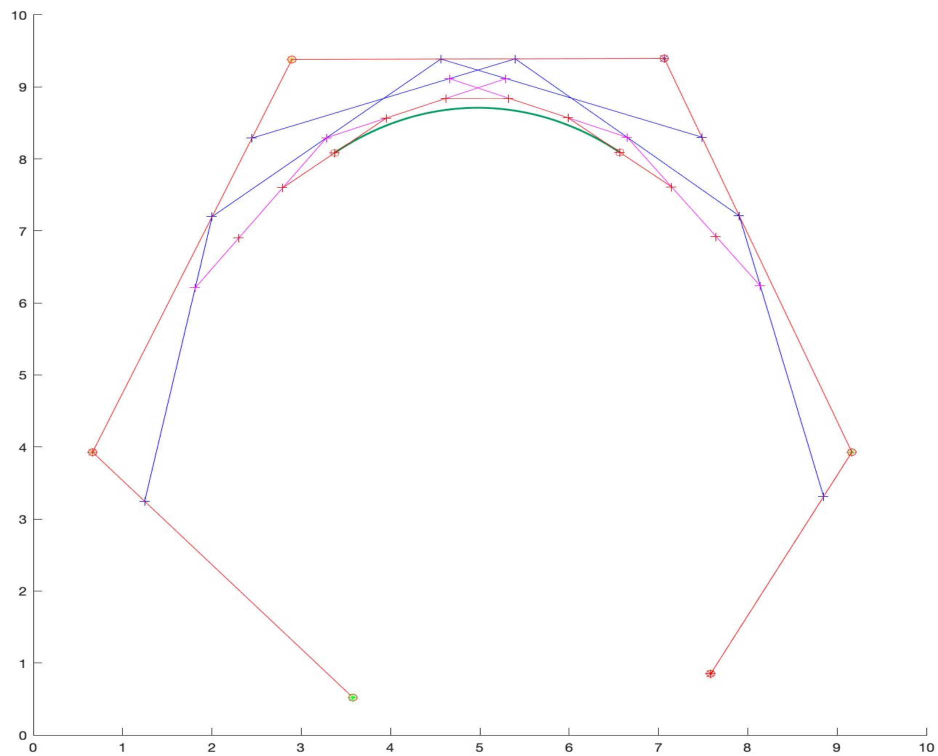

Figure 5: Construction of a generic quintic Bézier curve.

The Bézier control points of the first segment are $x_0, d_{-1}, d^1_{1,0}, d^2_{1,0}, d^3_{1,0}, x_1$, namely

$$\begin{aligned}
b_1^0 &= x_0 \\
b_1^1 &= d_{-1} \\
b_1^2 &= \frac{1}{2} d_{-1} + \frac{1}{2} d_0 \\
b_1^3 &= \frac{1}{4} d_{-1} + \frac{7}{12} d_0 + \frac{1}{6} d_1 \\
b_1^4 &= \frac{1}{8} d_{-1} + \frac{37}{72} d_0 + \frac{23}{72} d_1 + \frac{1}{24} d_2 \\
b_1^5 &= x_1.
\end{aligned}$$

When computing from the de Boor points, we have $b_1^0 = d_{-2}$, $b_1^5 = d^4_{1,0}$.

The Bézier control points of the second segment are $x_1, d^3_{1,2}, d^3_{1,3}, d^3_{2,0}, d^3_{2,1}, x_2$, namely

$$
\begin{aligned}
b^0_2 &= x_1\\
b^1_2 &= \frac{8}{27}d_0 + \frac{217}{432}d_1 + \frac{133}{720}d_2 + \frac{1}{60}d_3\\
b^2_2 &= \frac{4}{27}d_0 + \frac{115}{216}d_1 + \frac{103}{360}d_2 + \frac{1}{30}d_3\\
b^3_2 &= \frac{2}{27}d_0 + \frac{49}{108}d_1 + \frac{73}{180}d_2 + \frac{1}{15}d_3\\
b^4_2 &= \frac{1}{27}d_0 + \frac{19}{54}d_1 + \frac{43}{90}d_2 + \frac{2}{15}d_3\\
b^5_2 &= x_2.
\end{aligned}
$$

When computing from the de Boor points, we have $b^0_2 = d^4_{1,0}$ and $b^5_2 = d^4_{2,0}$.

The Bézier control points of the third segment are $x_2, d^3_{2,2}, d^3_{2,3}, d^3_{3,0}, d^3_{3,1}, x_3$, namely

$$
\begin{aligned}
b^0_3 &= x_2\\
b^1_3 &= \frac{1}{6}d_1 + \frac{31}{60}d_2 + \frac{3}{10}d_3 + \frac{1}{60}d_4\\
b^2_3 &= \frac{1}{12}d_1 + \frac{29}{60}d_2 + \frac{2}{5}d_3 + \frac{1}{30}d_4\\
b^3_3 &= \frac{1}{24}d_1 + \frac{47}{120}d_2 + \frac{1}{2}d_3 + \frac{1}{15}d_4\\
b^4_3 &= \frac{1}{48}d_1 + \frac{71}{240}d_2 + \frac{11}{20}d_3 + \frac{2}{15}d_4\\
b^5_3 &= x_3.
\end{aligned}
$$

When computing from the de Boor points, we have $b^0_3 = d^4_{2,0}$ and $b^5_3 = d^4_{3,0}$.

When $N \geq 8$, the point x_4 is a generic point and the Bézier control points of the fourth segment are $x_3, d^3_{3,2}, d^3_{3,3}, d^3_{4,0}, d^3_{4,1}, x_4$, namely

$$
\begin{aligned}
b^0_4 &= x_3\\
b^1_4 &= \frac{2}{15}d_2 + \frac{11}{20}d_3 + \frac{3}{10}d_4 + \frac{1}{60}d_5\\
b^2_4 &= \frac{1}{15}d_2 + \frac{1}{2}d_3 + \frac{2}{5}d_4 + \frac{1}{30}d_5\\
b^3_4 &= \frac{1}{30}d_2 + \frac{2}{5}d_3 + \frac{1}{2}d_4 + \frac{1}{15}d_5\\
b^4_4 &= \frac{1}{60}d_2 + \frac{3}{10}d_3 + \frac{11}{20}d_4 + \frac{2}{15}d_5\\
b^5_4 &= x_4,
\end{aligned}
$$

When computing from the de Boor points, we have $b_4^0 = d_{3,0}^4$ and $b_4^5 = d_{4,0}^4$.

When $N = 7$, the points x_4 is analogous to the point x_3, in the sense that it is the fourth point from the last data point, x_7. The Bézier control points of the fourth segment are still $x_3, d_{3,2}^3, d_{3,3}^3, d_{4,0}^3, d_{4,1}^3, x_4$, but $d_{4,0}^3, d_{4,1}^3$ need to be computed differently. We use the reversal method used in Section 5.

Since x_{N+1-k} is the kth point from right to left, the equations associated with x_{N+1-k} are obtained from the equations associated with x_k by replacing $d_{-2}, d_{-1}, d_0, d_1, \ldots, x_\ell, \ldots$ by $d_{N+2}, d_{N+1}, d_N, d_{N-1}, \ldots, x_{N-\ell}, \ldots$, and replacing $d_{N+1-k,i}^1$ by $d_{k,3-i}^1$, and similarly $d_{N+1-k,i}^2$ by $d_{k,2-i}^2$, $d_{N+1-k,i}^3$ by $d_{k,3-i}^3$, and $d_{N+1-k,0}^4$ by $d_{k,0}^4$.

When $N = 7$, the equations for the $d_{4,h}^i$ are obtained from the equations for the $d_{3,j}^k$ and we get:

$$
\begin{aligned}
d_{4,3}^1 &= \frac{1}{4} d_6 + \frac{3}{4} d_5 \\
d_{4,2}^1 &= \frac{2}{5} d_5 + \frac{3}{5} d_4 \\
d_{4,1}^1 &= \frac{3}{5} d_4 + \frac{2}{5} d_3 \\
d_{4,0}^1 &= \frac{4}{5} d_3 + \frac{1}{5} d_2,
\end{aligned}
$$

$$
\begin{aligned}
d_{4,2}^2 &= \frac{1}{4} d_{4,3}^1 + \frac{3}{4} d_{4,2}^1 \\
d_{4,1}^2 &= \frac{2}{4} d_{4,2}^1 + \frac{2}{4} d_{4,1}^1 \\
d_{4,0}^2 &= \frac{3}{4} d_{4,1}^1 + \frac{1}{4} d_{4,0}^1,
\end{aligned}
$$

$$
\begin{aligned}
d_{4,3}^3 &= \frac{2}{3} d_{4,2}^2 + \frac{1}{3} d_{4,1}^2 \\
d_{4,2}^3 &= \frac{1}{3} d_{4,2}^2 + \frac{2}{3} d_{4,1}^2 \\
d_{4,1}^3 &= \frac{2}{3} d_{4,1}^2 + \frac{1}{3} d_{4,0}^2 \\
d_{4,0}^3 &= \frac{1}{3} d_{4,1}^2 + \frac{2}{3} d_{4,0}^2,
\end{aligned}
$$

$$
d_{4,0}^4 = \frac{1}{2} d_{4,2}^3 + \frac{1}{2} d_{4,1}^3.
$$

The equations involved in computing $d_{4,0}^3$ and $d_{4,1}^3$ are

$$\begin{aligned}
d_{4,0}^1 &= \frac{1}{5}d_2 + \frac{4}{5}d_3 \\
d_{4,1}^1 &= \frac{2}{5}d_3 + \frac{3}{5}d_4 \\
d_{4,2}^1 &= \frac{3}{5}d_4 + \frac{2}{5}d_5 \\
d_{4,0}^2 &= \frac{1}{4}d_{4,0}^1 + \frac{3}{4}d_{4,1}^1 \\
d_{4,1}^2 &= \frac{2}{4}d_{4,1}^1 + \frac{2}{4}d_{4,2}^1 \\
d_{4,0}^3 &= \frac{2}{3}d_{4,0}^2 + \frac{1}{3}d_{4,1}^2 \\
d_{4,1}^3 &= \frac{1}{3}d_{4,0}^2 + \frac{2}{3}d_{4,1}^2,
\end{aligned}$$

These are identical to the equations for $d_{i,0}^1, d_{i,1}^1, d_{i,2}^1, d_{i,0}^2, d_{i,1}^2, d_{i,0}^3, d_{i,1}^3$ in the generic case $i \geq 5$, and so we get

$$\begin{aligned}
d_{4,0}^3 &= \frac{1}{30}d_2 + \frac{2}{5}d_3 + \frac{1}{2}d_4 + \frac{1}{15}d_5 \\
d_{4,1}^3 &= \frac{1}{60}d_2 + \frac{3}{10}d_3 + \frac{11}{20}d_4 + \frac{2}{15}d_5,
\end{aligned}$$

which are identical to the equations obtained when $N \geq 8$. Therefore, the equations for the control points for the fourth curve segment C_4 are the same in the special case $N = 7$ as the equations in the general case $N \geq 8$, and they agree with the equations for the generic curve segment C_i for $i \geq 5$.

Using the reversal method described above, the Bézier control points of the Nth segment are $x_{N-1}, d_{N-1,3}^3, d_{N-1,2}^2, d_{N-1,3}^1, d_{N+1}, x_N$, namely

$$\begin{aligned}
b_N^0 &= x_{N-1} \\
b_N^1 &= \frac{1}{24}d_{N-2} + \frac{23}{72}d_{N-1} + \frac{37}{72}d_N + \frac{1}{8}d_{N+1} \\
b_N^2 &= \frac{1}{6}d_{N-1} + \frac{7}{12}d_N + \frac{1}{4}d_{N+1} \\
b_N^3 &= \frac{1}{2}d_N + \frac{1}{2}d_{N+1} \\
b_N^4 &= d_{N+1} \\
b_N^5 &= x_N.
\end{aligned}$$

When computing from the de Boor points, we have $b_N^0 = d_{N-1,0}^4$, $b_N^5 = d_{N+2}$.

The Bézier control points of the $(N-1)$th segment are $x_{N-2}, d^3_{N-2,2}, d^3_{N-2,3}, d^3_{N-1,0}, d^3_{N-1,1}$, x_{N-1}, which yields

$$
\begin{aligned}
b^0_{N-1} &= x_{N-2}\\
b^1_{N-1} &= \frac{2}{15}d_{N-3} + \frac{43}{90}d_{N-2} + \frac{19}{54}d_{N-1} + \frac{1}{27}d_N\\
b^2_{N-1} &= \frac{1}{15}d_{N-3} + \frac{73}{180}d_{N-2} + \frac{49}{108}d_{N-1} + \frac{2}{27}d_N\\
b^3_{N-1} &= \frac{1}{30}d_{N-3} + \frac{103}{360}d_{N-2} + \frac{115}{216}d_{N-1} + \frac{4}{27}d_N\\
b^4_{N-1} &= \frac{1}{60}d_{N-3} + \frac{133}{720}d_{N-2} + \frac{217}{432}d_{N-1} + \frac{8}{27}d_N\\
b^5_{N-1} &= x_{N-1}.
\end{aligned}
$$

When computing from the de Boor points, we have $b^0_{N-1} = d^4_{N-2,0}$ and $b^5_{N-1} = d^4_{N-1,0}$.

The Bézier control points of the $(N-2)$th segment are $x_{N-3}, d^3_{N-3,2}, d^3_{N-3,3}, d^3_{N-2,0}, d^3_{N-2,1}$, x_{N-2}, namely

$$
\begin{aligned}
b^0_{N-2} &= x_{N-3}\\
b^4_{N-2} &= \frac{2}{15}d_{N-4} + \frac{11}{20}d_{N-3} + \frac{71}{240}d_{N-2} + \frac{1}{48}d_{N-1}\\
b^3_{N-2} &= \frac{1}{15}d_{N-4} + \frac{1}{2}d_{N-3} + \frac{47}{120}d_{N-2} + \frac{1}{24}d_{N-1}\\
b^2_{N-2} &= \frac{1}{30}d_{N-4} + \frac{2}{5}d_{N-3} + \frac{29}{60}d_{N-2} + \frac{1}{12}d_{N-1}\\
b^1_{N-2} &= \frac{1}{60}d_{N-4} + \frac{3}{10}d_{N-3} + \frac{31}{60}d_{N-2} + \frac{1}{6}d_{N-1}\\
b^5_{N-2} &= x_{N-2}.
\end{aligned}
$$

When computing from the de Boor points, we have $b^0_{N-2} = d^4_{N-3,0}$ and $b^5_{N-2} = d^4_{N-2,0}$.

Finally, the Bézier control points of the $(N-3)$th segment are $x_{N-4}, d^3_{N-4,2}, d^3_{N-4,3}, d^3_{N-3,0}$, $d^3_{N-3,1}$, x_{N-3}, namely

$$
\begin{aligned}
b^0_{N-3} &= x_{N-4}\\
b^1_{N-3} &= \frac{2}{15}d_{N-5} + \frac{11}{20}d_{N-4} + \frac{3}{10}d_{N-3} + \frac{1}{60}d_{N-2}\\
b^2_{N-3} &= \frac{1}{15}d_{N-5} + \frac{1}{2}d_{N-4} + \frac{2}{5}d_{N-3} + \frac{1}{30}d_{N-2}\\
b^3_{N-3} &= \frac{1}{30}d_{N-5} + \frac{2}{5}d_{N-4} + \frac{1}{2}d_{N-3} + \frac{1}{15}d_{N-2}\\
b^4_{N-3} &= \frac{1}{60}d_{N-5} + \frac{3}{10}d_{N-4} + \frac{11}{20}d_{N-3} + \frac{2}{15}d_{N-2}\\
b^5_{N-3} &= x_{N-3},
\end{aligned}
$$

When computing from the de Boor points, we have $b^0_{N-3} = d^4_{N-4,0}$ and $b^5_{N-3} = d^4_{N-3,0}$.

Examples quintic B-splines are shown in Figures 6–8.

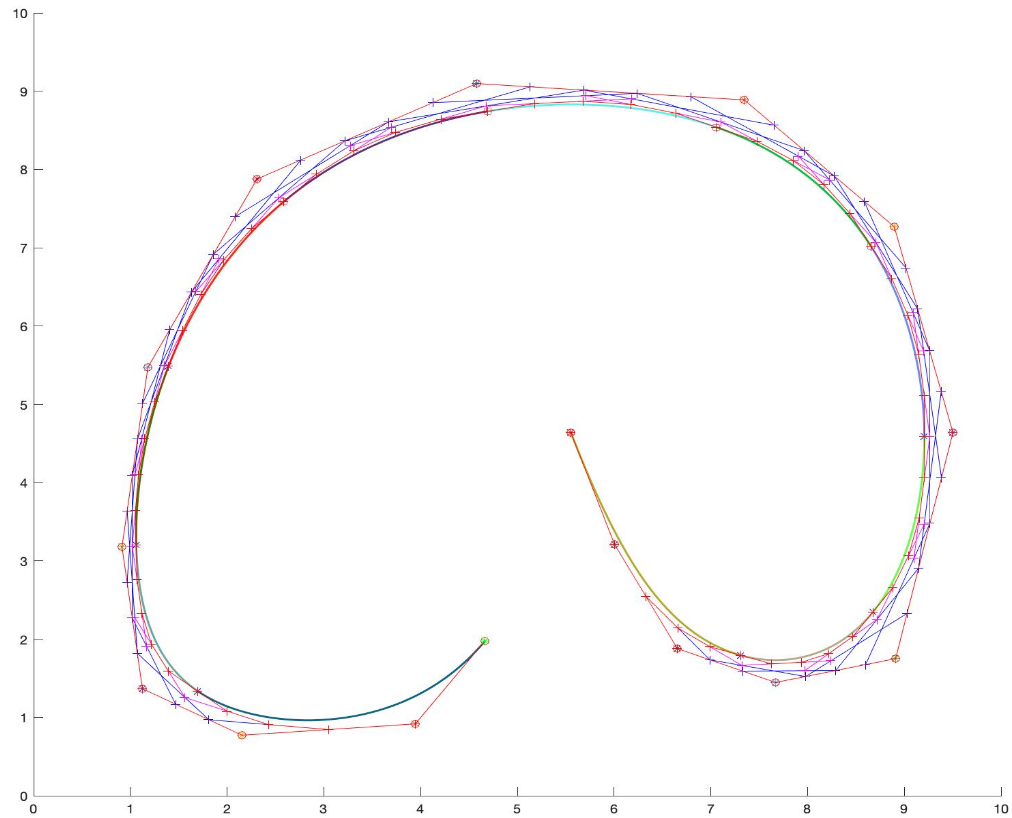

Figure 6: A quintic B-spline with 16 de Boor control points.

For an implementation of a program computing the Bézier control points from the de Boor points, it is also convenient to have the following equations.

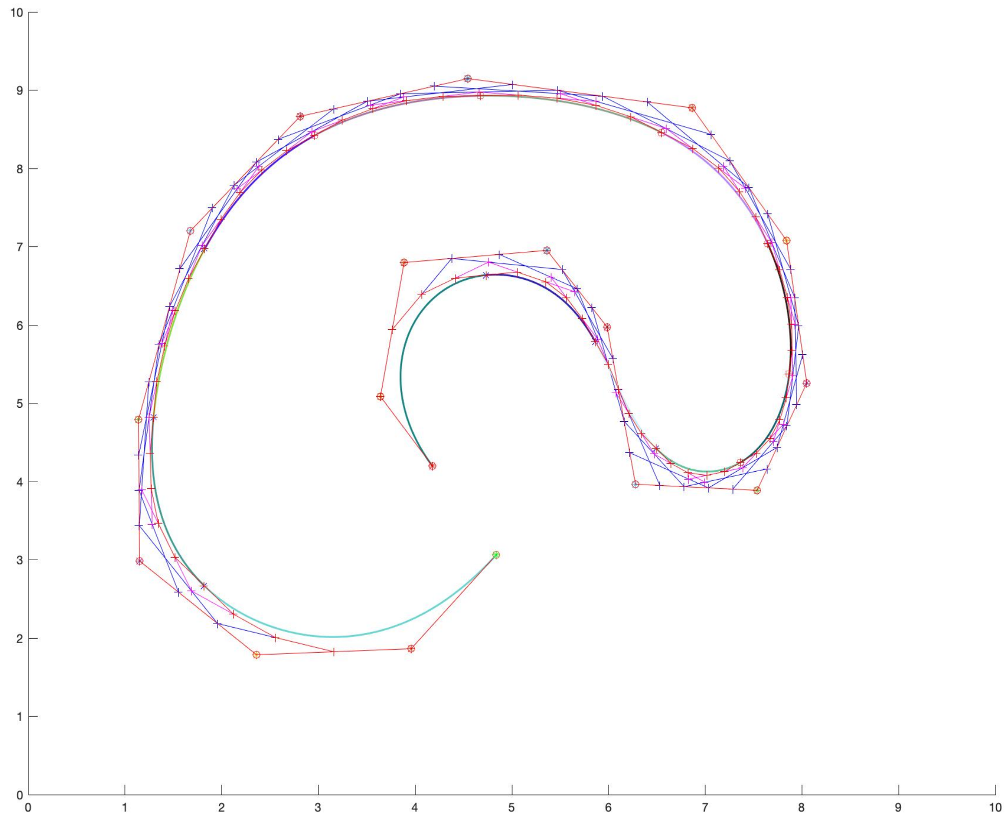

Figure 7: A quintic B-spline with 18 de Boor control points.

For x_{N-3}:

$$\begin{aligned}
d^1_{N-3,3} &= \frac{1}{4}d_{N-1} + \frac{3}{4}d_{N-2}\\
d^1_{N-3,2} &= \frac{2}{5}d_{N-2} + \frac{3}{5}d_{N-3}\\
d^1_{N-3,1} &= \frac{3}{5}d_{N-3} + \frac{2}{5}d_{N-4}\\
d^1_{N-3,0} &= \frac{4}{5}d_{N-4} + \frac{1}{5}d_{N-5},
\end{aligned}$$

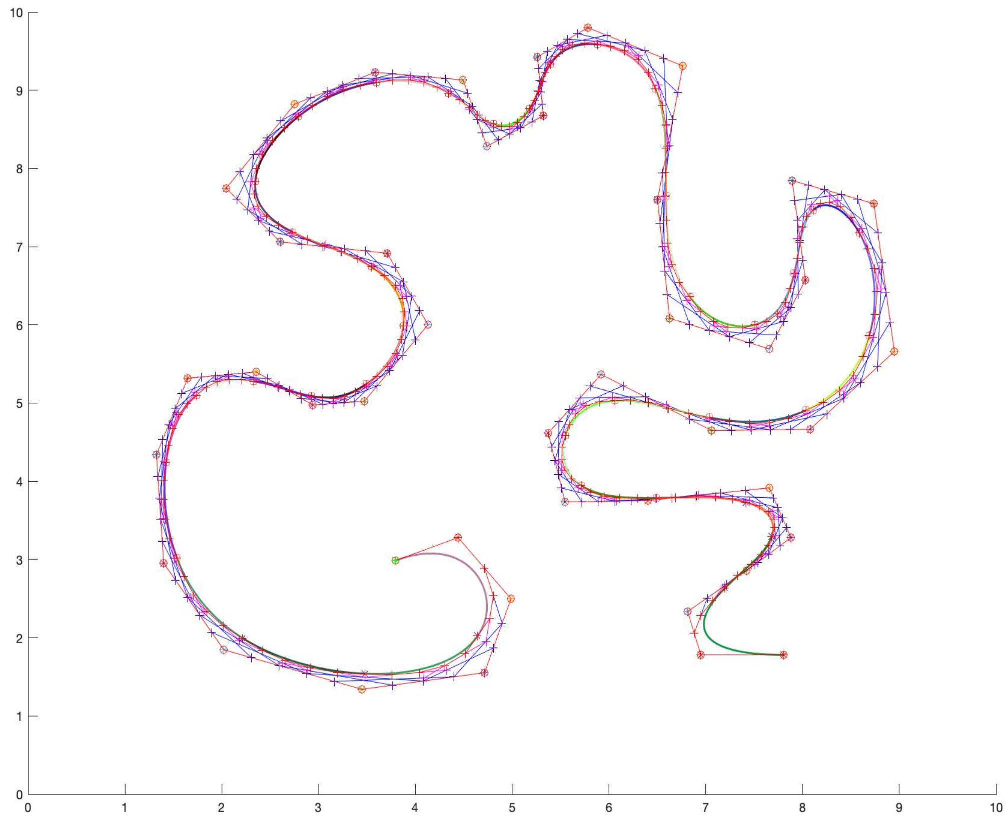

Figure 8: A quintic B-spline with 43 de Boor control points.

$$
\begin{aligned}
d^2_{N-3,2} &= \frac{1}{4} d^1_{N-3,3} + \frac{3}{4} d^1_{N-3,2} \\
d^2_{N-3,1} &= \frac{2}{4} d^1_{N-3,2} + \frac{2}{4} d^1_{N-3,1} \\
d^2_{N-3,0} &= \frac{3}{4} d^1_{N-3,1} + \frac{1}{4} d^1_{N-3,0},
\end{aligned}
$$

$$
\begin{aligned}
d^3_{N-3,3} &= \frac{2}{3}d^2_{N-3,2} + \frac{1}{3}d^2_{N-3,1} \\
d^3_{N-3,2} &= \frac{1}{3}d^2_{N-3,2} + \frac{2}{3}d^2_{N-3,1} \\
d^3_{N-3,1} &= \frac{2}{3}d^2_{N-3,1} + \frac{1}{3}d^2_{N-3,0} \\
d^3_{N-3,0} &= \frac{1}{3}d^2_{N-3,1} + \frac{2}{3}d^2_{N-3,0},
\end{aligned}
$$

$$
d^4_{N-3,0} = \frac{1}{2}d^3_{N-3,2} + \frac{1}{2}d^3_{N-3,1}.
$$

Then, the Bézier control points for the segment from x_{N-4} to x_{N-3} are $d^4_{N-4,0}, d^3_{N-4,2}, d^3_{N-4,3}, d^3_{N-3,0}, d^3_{N-3,1}, d^4_{N-3,0}$, and the Bézier control points for the segment from x_{N-3} to x_{N-2} are $d^4_{N-3,0}, d^3_{N-3,2}, d^3_{N-3,3}, d^3_{N-2,0}, d^3_{N-2,1}, d^4_{N-2,0}$. Note that d_{N-4} is computed using the generic formula.

For x_{N-2}:

$$
\begin{aligned}
d^1_{N-2,3} &= \frac{1}{3}d_N + \frac{2}{3}d_{N-1} \\
d^1_{N-2,2} &= \frac{2}{4}d_{N-1} + \frac{2}{4}d_{N-2} \\
d^1_{N-2,1} &= \frac{3}{5}d_{N-2} + \frac{2}{5}d_{N-3} \\
d^1_{N-2,0} &= \frac{4}{5}d_{N-3} + \frac{1}{5}d_{N-4},
\end{aligned}
$$

$$
\begin{aligned}
d^2_{N-2,2} &= \frac{1}{3}d^1_{N-2,3} + \frac{2}{3}d^1_{N-2,2} \\
d^2_{N-2,1} &= \frac{2}{4}d^1_{N-2,2} + \frac{2}{4}d^1_{N-2,1} \\
d^2_{N-2,0} &= \frac{3}{4}d^1_{N-2,1} + \frac{1}{4}d^1_{N-2,0},
\end{aligned}
$$

$$
\begin{aligned}
d^3_{N-2,3} &= \frac{2}{3}d^2_{N-2,2} + \frac{1}{3}d^2_{N-2,1} \\
d^3_{N-2,2} &= \frac{1}{3}d^2_{N-2,2} + \frac{2}{3}d^2_{N-2,1} \\
d^3_{N-2,1} &= \frac{2}{3}d^2_{N-2,1} + \frac{1}{3}d^2_{N-2,0} \\
d^3_{N-2,0} &= \frac{1}{3}d^2_{N-2,1} + \frac{2}{3}d^2_{N-2,0},
\end{aligned}
$$

$$
d^4_{N-2,0} = \frac{1}{2}d^3_{N-2,2} + \frac{1}{2}d^3_{N-2,1}
$$

with Bézier control points for the segment from x_{N-2} to x_{N-1} given by $d^4_{N-2,0}, d^3_{N-2,2}, d^3_{N-2,3}$, $d^3_{N-1,0}, d^3_{N-1,1}, d^4_{N-1,0}$.

For x_{N-1}:

$$
\begin{aligned}
d^1_{N-1,3} &= \frac{1}{2}d_{N+1} + \frac{1}{2}d_N \\
d^1_{N-1,2} &= \frac{2}{3}d_N + \frac{1}{3}d_{N-1} \\
d^1_{N-1,1} &= \frac{3}{4}d_{N-1} + \frac{1}{4}d_{N-2} \\
d^1_{N-1,0} &= \frac{4}{5}d_{N-2} + \frac{1}{5}d_{N-3},
\end{aligned}
$$

$$
\begin{aligned}
d^2_{N-1,2} &= \frac{1}{2}d^1_{N-1,3} + \frac{1}{2}d^1_{N-1,2} \\
d^2_{N-1,1} &= \frac{2}{3}d^1_{N-1,2} + \frac{1}{3}d^1_{N-1,1} \\
d^2_{N-1,0} &= \frac{3}{4}d^1_{N-1,1} + \frac{1}{4}d^1_{N-1,0},
\end{aligned}
$$

$$
\begin{aligned}
d^3_{N-1,3} &= \frac{1}{2}d^2_{N-1,2} + \frac{1}{2}d^2_{N-1,1} \\
d^3_{N-1,2} &= \frac{1}{2}d^2_{N-1,2} + \frac{1}{2}d^2_{N-1,1} \\
d^3_{N-1,1} &= \frac{2}{3}d^2_{N-1,1} + \frac{1}{3}d^2_{N-1,0} \\
d^3_{N-1,0} &= \frac{1}{3}d^2_{N-1,1} + \frac{2}{3}d^2_{N-1,0},
\end{aligned}
$$

$$
d^4_{N-1,0} = \frac{1}{2}d^3_{N-1,2} + \frac{1}{2}d^3_{N-1,1},
$$

with Bézier control points for the segment from x_{N-1} to x_N given by $d^4_{N-1,0}, d^3_{N-1,3}, d^2_{N-1,2}$, $d^1_{N-1,3}, d_{N+1}, d_{N+2}$.

7 A Simple Variation of the Interpolation Problem

A simple way to deal with the the beginning and the end of the interpolating spline is to use the uniform knot sequence

$$
01234\,, 12345,\ 23456,\ 34567,\ 45678, \cdots, N+3\ N+4\ N+5\ N+6\ N+7,
$$

with $N + 3 \geq 5$, that is $N \geq 2$. In this case, the polar values

$$
f(44444), \ldots, f(N+3\ N+3\ N+3\ N+3\ N+3)
$$

correspond to $N \geq 2$ data points that we denote $x_1, \ldots, x_N$. We denote the de Boor points by

$$d_{-1}, d_0, d_1, \ldots, d_N, d_{N+1}, d_{N+2}.$$

Then, we simply have N generic equations,

$$d_{i-2} + 26d_{i-1} + 66d_i + 26d_{i+1} + d_{i+2} = 120x_i, \quad i = 1, \ldots, N,$$

one for each x_i, with $d_{-1}, d_0, d_{N+1}, d_{N+2}$ as free parameters, and we don't have to deal with the first three and the last three curve segments in a special way, as in the previous solution. We can use the following system to solve for $d_1, \ldots, d_N$ in terms of $d_{-1}, d_0, d_{N+1}, d_{N+2}$:

$$\begin{pmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ & 1 & 26 & 66 & 26 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & 26 & 66 & 26 & 1 & \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} d_{-1} \\ d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-1} \\ d_N \\ d_{N+1} \\ d_{N+2} \end{pmatrix} = \begin{pmatrix} d_{-1} \\ d_0 \\ 120x_1 \\ 120x_2 \\ \vdots \\ 120x_{N-1} \\ 120x_N \\ d_{N+1} \\ d_{N+2} \end{pmatrix}.$$

We actually have a better behaved matrix if we move the terms involving d_{-1}, d_0, d_{N+1}, d_{N+2} on the right hand side of the system. We obtain the system

$$\begin{pmatrix} 66 & 26 & 1 & & & & \\ 26 & 66 & 26 & 1 & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & 1 & 26 & 66 & 26 & 1 \\ & & & 1 & 26 & 66 & 26 \\ & & & & 1 & 26 & 66 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{N-2} \\ d_{N-1} \\ d_N \end{pmatrix} = \begin{pmatrix} 120x_1 - 26d_0 - d_{-1} \\ 120x_2 - d_0 \\ 120x_3 \\ \vdots \\ 120x_{N-2} \\ 120x_{N-1} - d_{N+1} \\ 120x_N - 26d_{N+1} - d_{N+2} \end{pmatrix}.$$

The above hods for $N \geq 4$. For $N = 3$, we get the system

$$\begin{pmatrix} 66 & 26 & 1 \\ 26 & 66 & 26 \\ 1 & 26 & 66 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 120x_1 - 26d_0 - d_{-1} \\ 120x_2 - d_0 - d_4 \\ 120x_3 - 26d_4 - d_5 \end{pmatrix},$$

and for $N = 2$, we get the system

$$\begin{pmatrix} 66 & 26 \\ 26 & 66 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 120x_1 - 26d_0 - d_{-1}d_4 \\ 120x_2 - d_0 - 26d_3 - d_4 \end{pmatrix}.$$

In all cases, the matrix is symmetric, and in fact, positive definite.

Because our numbering is designed such that x_i is computed from $d_{i-2}, d_{i-1}, d_i, d_{i+1}, d_{i+2}$, as in the previous sections, the control points of the Bézier curve C_i between x_i and x_{i+1} are given by

$$\begin{aligned} b_i^0 &= x_i \\ b_i^1 &= \frac{2}{15}d_{i-1} + \frac{11}{20}d_i + \frac{3}{10}d_{i+1} + \frac{1}{60}d_{i+2} \\ b_i^2 &= \frac{1}{15}d_{i-1} + \frac{1}{2}d_i + \frac{2}{5}d_{i+1} + \frac{1}{30}d_{i+2} \\ b_i^4 &= \frac{1}{30}d_{i-1} + \frac{2}{5}d_i + \frac{1}{2}d_{i+1} + \frac{1}{15}d_{i+2} \\ b_i^4 &= \frac{1}{60}d_{i-1} + \frac{3}{10}d_i + \frac{11}{20}d_{i+1} + \frac{2}{15}d_{i+2} \\ b_i^5 &= x_{i+1}. \end{aligned}$$

for $i = 1, \ldots, N - 1$.

A simple-minded way to pick $d_{-1}, d_0, d_{N+1}, d_{N+2}$ is to set

$$d_{-1} = d_0 = x_1, \quad d_{N+1} = d_{N+2} = x_N.$$

An implementation in `Matlab` shows that this works well! Here are three examples using the above crude method. Figure 9 shows an interpolating curve for 18 data points (so $N = 18$). The de Boor control points shown in blue are d_1 and d_N.

Figure 10 shows an interpolating curve for 10 data points (so $N = 10$). The construction of the Bézier control points is also shown. The de Boor control points shown in blue are d_1 and d_N.

Figure 11 shows an interpolating curve for 43 data points (so $N = 43$). The de Boor control points shown in blue are d_1 and d_N.

Observe that in all cases the de Boor control points d_1 and d_N are "outside" of the interpolating spline curve, which is not suprising since $x_1 = d_{-1}$ and $x_N = d_{N+2}$ are generic de Boor control points. This could cause some unexpected behavior of the interpolating curve. We have not witnessed such a behavior but this issue, and more generally the determination of "good" end conditions, should be explored further.

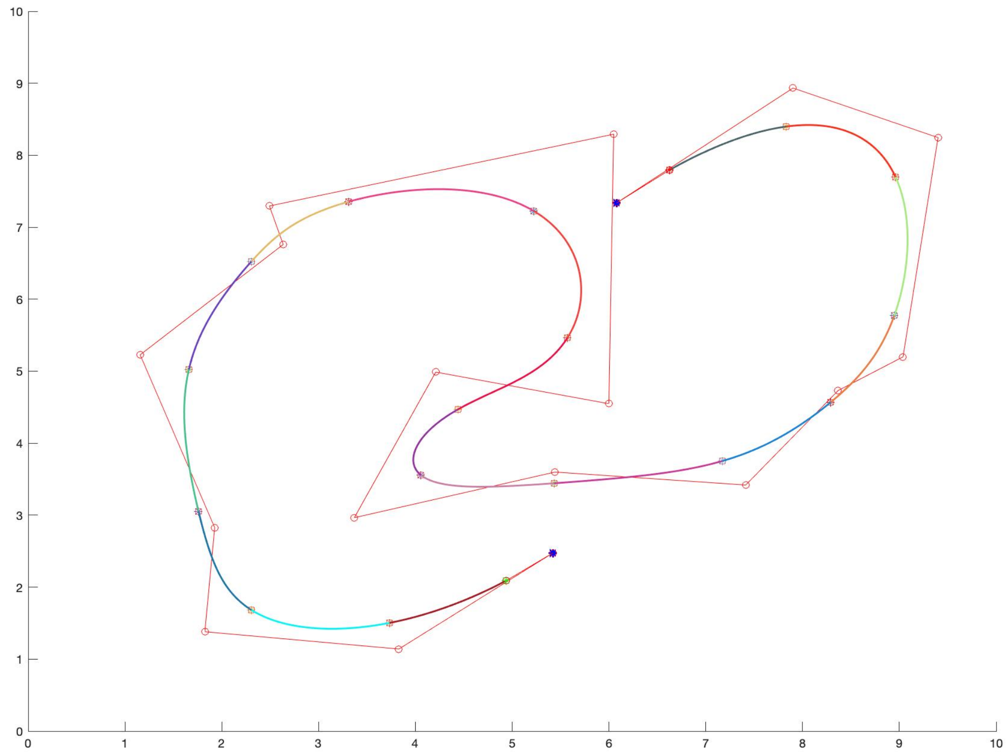

Figure 9: A quintic interpolating B-spline for 18 data points.

References

[1] Jean H. Gallier. *Curves and Surfaces In Geometric Modeling: Theory And Algorithms*. Morgan Kaufmann, 1999.

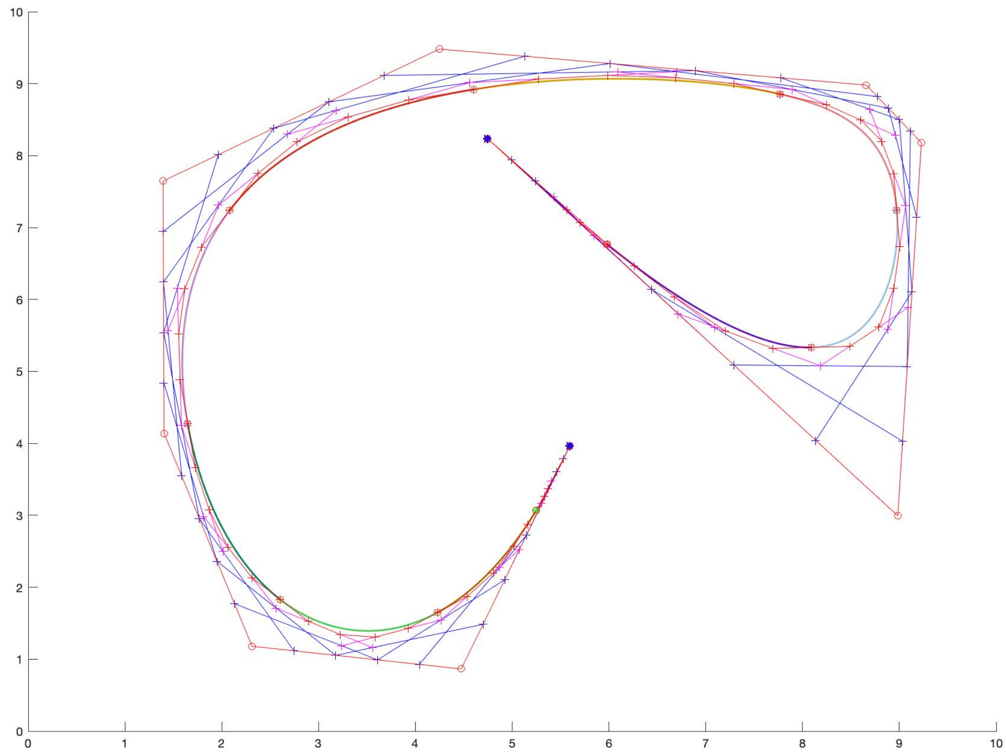

Figure 10: A quintic interpolating B-spline for 10 data points.

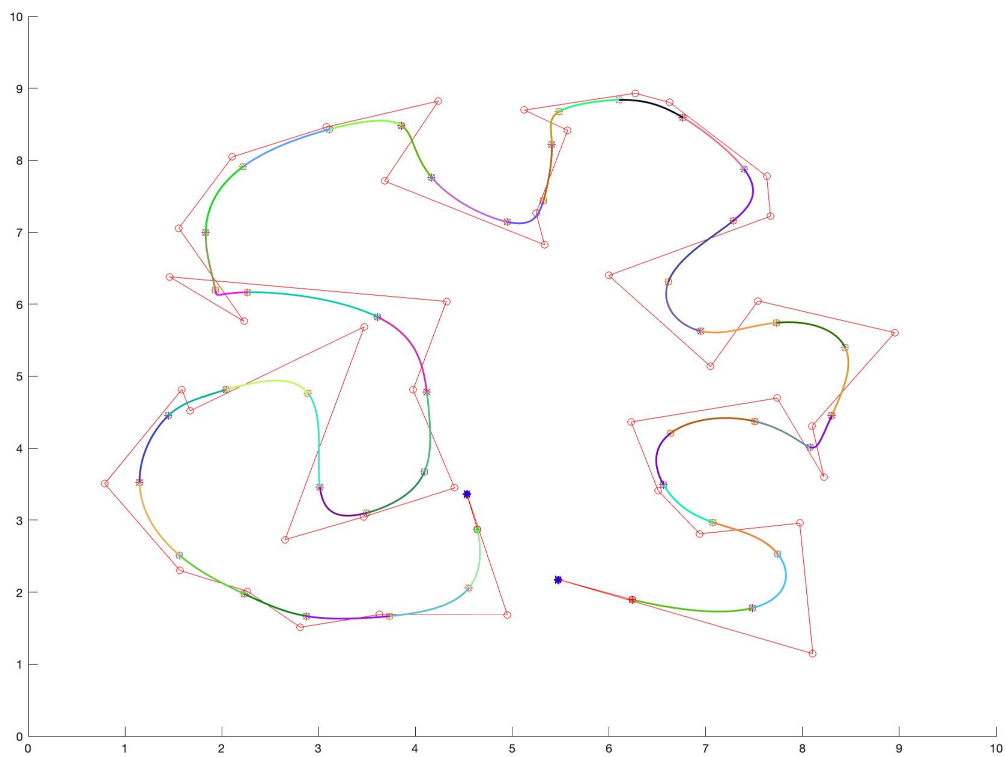

Figure 11: A quintic interpolating B-spline for 43 data points.