**Fall, 2015 CIS 515**

# Fundamentals of Linear Algebra and Optimization
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## Projects 4 & 5

November 24, 2015; Due December 15, 2015

**Problem B1 (100 pts).** (1) Implement in `Matlab` the method for finding a basis of the kernel of an $m \times n$ matrix $A$, as explained in Section 4.5 of the notes (`linalg.pdf`); see the example at the end of Section 4.5.

(2) Modify your program so that it also takes a righthand side $b$, and it tests whether the system $Ax = b$ is solvable or not. If the system is solvable, find a special solution as explained in Section 4.5 of the notes.

**Problem B2 (120 pts).** Consider the problem of finding a basis of the subspace $V_n$ of $n \times n$ matrices $A \in \mathrm{M}_n(\mathbb{R})$ satisfying the following properties:

1. The sum of the entries in every row has the same value (say $c_1$);
2. The sum of the entries in every column has the same value (say $c_2$).

It turns out that $c_1 = c_2$ and that the $2n-2$ equations corresponding to the above conditions are linearly independent. By the duality theorem, the dimension of the space $V_n$ of matrices satisying the above equations is $n^2 - (2n-2) = (n-1)^2 + 1$.

(1) Write a program to produce the matrix $A$ of a system of equations of the form below, asserting that the above conditions hold. For example, when $n = 4$, we have the equations

$$
\begin{aligned}
a_{11} + a_{12} + a_{13} + a_{14} - a_{21} - a_{22} - a_{23} - a_{24} &= 0 \\
a_{21} + a_{22} + a_{23} + a_{24} - a_{31} - a_{32} - a_{33} - a_{34} &= 0 \\
a_{31} + a_{32} + a_{33} + a_{34} - a_{41} - a_{42} - a_{43} - a_{44} &= 0 \\
a_{11} + a_{21} + a_{31} + a_{41} - a_{12} - a_{22} - a_{32} - a_{42} &= 0 \\
a_{12} + a_{22} + a_{32} + a_{42} - a_{13} - a_{23} - a_{33} - a_{43} &= 0 \\
a_{13} + a_{23} + a_{33} + a_{43} - a_{14} - a_{24} - a_{34} - a_{44} &= 0.
\end{aligned}
$$

Make sure that your equations are listed in the same order as the above equations.

(2) Use your program from Problem B2 to find a basis of the kernel of $A$ for $n = 2, \ldots, 6$, and print our the corresponding matrices that form a basis of the subspace $V_n$.

(3) Now consider *magic squares*. Such matrices satisfy the two conditions about the sum of the entries in each row and in each column to be the same number, and also the additional two constraints that the main descending and the main ascending diagonals add up to this common number. Furthermore, the entries are also required to be positive integers. For example, in the case $n = 4$, we have the following system of equations:

$$\begin{aligned}
a_{11} + a_{12} + a_{13} + a_{14} - a_{21} - a_{22} - a_{23} - a_{24} &= 0\\
a_{21} + a_{22} + a_{23} + a_{24} - a_{31} - a_{32} - a_{33} - a_{34} &= 0\\
a_{31} + a_{32} + a_{33} + a_{34} - a_{41} - a_{42} - a_{43} - a_{44} &= 0\\
a_{22} + a_{33} + a_{44} - a_{12} - a_{13} - a_{14} &= 0\\
a_{11} + a_{21} + a_{31} + a_{41} - a_{12} - a_{22} - a_{32} - a_{42} &= 0\\
a_{12} + a_{22} + a_{32} + a_{42} - a_{13} - a_{23} - a_{33} - a_{43} &= 0\\
a_{13} + a_{23} + a_{33} + a_{43} - a_{14} - a_{24} - a_{34} - a_{44} &= 0\\
a_{41} + a_{32} + a_{23} - a_{11} - a_{12} - a_{13} &= 0.
\end{aligned}$$

Observe that the equation asserting that the sum of the diagonal entries is equal to the sum of the entries in the first row is listed as the $n$th equation, and that the equation asserting that the sum of the ascending diagonal entries is equal to the sum of the entries in the first row is listed as the $2n$th equations. It can be shown that the above $2n$ equations are linearly independent if $n \geq 3$, so the space of (generalized) magic squares has dimension $n^2 - 2n = (n-1)^2 - 1$ (these are magic squares with no restriction on the coefficients; *i.e.*, the coefficients need not be positive integers).

Write a program to produce the matrix $M$ of a system of equations of the form above, asserting that a matrix is a generalized magic square. Make sure you use the same order for your equations as shown above.

Use your program from Problem B2 to find a basis of (generalized) magic squares for $n = 3, 4, 5$.

For $n = 3$, show that a generic magic square is of the form

$$\begin{pmatrix} (2x_1 + 2x_2 - x_3)/3 & (2x_1 - x_2 + 2x_3)/3 & (-x_1 + 2x_2 + 2x_3)/3 \\ (-2x_1 + x_2 + 4x_3)/3 & (x_1 + x_2 + x_3)/3 & (4x_1 + x_2 - 2x_3)/3 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

For $n = 4$, show that a generic magic square is of the form

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & a_1 \\ m_{31} & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix},$$

with

$$
\begin{aligned}
m_{11} &= a_1 + a_4 - a_5 \\
m_{12} &= a_1 - a_2 + a_3 + a_4 - a_5 - a_6 + a_8 \\
m_{13} &= -a_1 + a_2 - a_3 - a_4 + 2a_5 + a_6 \\
m_{14} &= -a_1 - a_4 + a_5 + a_6 + a_7 \\
m_{21} &= -a_1 + a_2 + a_3 \\
m_{22} &= -a_1 - a_3 - a_4 + 2a_5 + a_6 + a_7 \\
m_{23} &= a_1 - a_2 + a_4 - a_5 + a_8 \\
m_{31} &= -a_2 - a_3 - a_4 + a_5 + a_6 + a_7 + a_8.
\end{aligned}
$$

(4) A *normal magic square* is a magic square whose entries are the integers $1, 2, \ldots, n^2$. Show that there are no normal magic squares for $n = 2$. For $n = 3$, show that we must have

$$x_1 + x_2 + x_3 = 15.$$

Eliminating $x_3$, a generic normal magic square is of the form

$$\begin{pmatrix} x_1 + x_2 - 5 & 10 - x_2 & 10 - x_1 \\ 20 - 2x_1 - x_2 & 5 & 2x_1 + x_2 - 10 \\ x_1 & x_2 & 15 - x_1 - x_2 \end{pmatrix}.$$

**Extra credit (40 pts))** Show that there is a unique normal magic square (up to rotations and reflections) given by:

$$\begin{pmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{pmatrix}.$$

For $n = 4$, a normal magic square must have

$$a_5 + a_6 + a_7 + a_8 = 34.$$

We can eliminate $a_8$, but we still have 7 variables ranging from 1 to 16, so it is a nontrivial task to find all 880 distinct normal magic squares!

**Problem B3(40 pts).** Refer to Problem B4 of HW6. Write a `Matlab` program to compute a logarithm $B$ of a rotation matrix $R \in \mathbf{SO}(3)$, namely a skew-symmetric matrix $B$ such that $e^B = R$, using the method described in Problem B4. **Write your own program, do not use the built-in Matlab function**. In particular, compute a log $B$ of $R$ such that the angle $\theta$ associated with $B$ satisfies the condition $0 \leq \theta \leq \pi$. If $\mathrm{tr}(R) \neq -1$, then $\theta < \pi$ and the matrix $B$ associated with $\theta$ is uniquely determined. Otherwise $\theta = \pi$ and $B$ is determined up to sign.

**Problem B4(210 pts).** Refer to Problem B5 of HW6. Similitudes can be used to describe certain deformations (or flows) of a deformable body $\mathcal{B}_t$ in 3D. Given some initial shape $\mathcal{B}$ in $\mathbb{R}^3$ (for example, a sphere, a cube, etc.), a deformation of $\mathcal{B}$ is given by a piecewise differentiable curve

$$\mathcal{D}\colon [0,T] \to \mathbf{SIM}(3),$$

where each $\mathcal{D}(t)$ is a similitude (for some $T > 0$). The deformed body $\mathcal{B}_t$ at time $t$ is given by

$$\mathcal{B}_t = \mathcal{D}(t)(\mathcal{B}),$$

where $\mathcal{D}(t) \in \mathbf{SIM}(3)$ is a similitude.

The surjectivity of the exponential map $\exp\colon \mathfrak{sim}(3) \to \mathbf{SIM}(3)$ implies that there is a map $\log\colon \mathbf{SIM}(3) \to \mathfrak{sim}(3)$, although it is multivalued. The exponential map and the log "function" allows us to work in the simpler (noncurved) Euclidean space $\mathfrak{sim}(3)$ (which has dimension 7).

For instance, given two similitudes $A_1, A_2 \in \mathbf{SIM}(3)$ specifying the shape of $\mathcal{B}$ at two different times, we can compute $\log(A_1)$ and $\log(A_2)$, which are just elements of the Euclidean space $\mathfrak{sim}(3)$, form the linear interpolant $(1-t)\log(A_1) + t\log(A_2)$, and then apply the exponential map to get an interpolating deformation

$$t \mapsto e^{(1-t)\log(A_1)+t\log(A_2)}, \quad t \in [0,1].$$

Also, given a sequence of "snapshots" of the deformable body $\mathcal{B}$, say $A_0, A_1, \ldots, A_m$, where each is $A_i$ is a similitude, we can try to find an interpolating deformation (a curve in $\mathbf{SIM}(3)$) by finding a simpler curve $t \mapsto C(t)$ in $\mathfrak{sim}(3)$ (say, a $B$-spline) interpolating $\log A_1, \log A_1, \ldots, \log A_m$. Then, the curve $t \mapsto e^{C(t)}$ yields a deformation in $\mathbf{SIM}(3)$ interpolating $A_0, A_1, \ldots, A_m$.

(1) **(60 pts).** Write a program interpolating between two deformations, using the formulae found in Problems B4 and B5 of HW6 (not the built-in `Matlab` functions!).

(2) **(150 pts).** Write a program using your cubic spline interpolation program from the first project, to interpolate a sequence of deformations given by similitudes $A_0, A_1, \ldots, A_m$ in $\mathbf{SIM}(3)$. Use the formulae found in Problems B4 and B5 of HW6 (not the built-in `Matlab` functions!).

**TOTAL: 470 + 40 points.**