Fundamentals of Linear Algebra and Optimization
Jean Gallier
Project 4

November 5, 2013; Due November 21, 2013

Problem B1 (100 pts). (1) Implement the method for converting a rectangular matrix to reduced row echelon fom (rref).

(2) Use the above method to find the inverse of an invertible $n \times n$ matrix A, by applying it to the the $n \times 2n$ matrix $[A\, I]$ obtained by adding the n columns of the identity matrix to A.

(3) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 3 & 4 & 5 & \cdots & n+1 \\ 3 & 4 & 5 & 6 & \cdots & n+2 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ n & n+1 & n+2 & n+3 & \cdots & 2n-1 \end{pmatrix}.$$

Using your program, find the row reduced echelon form of A for $n = 4, \ldots, 20$.

Also run the `Matlab rref` function and compare results.

Your program probably disagrees with `rref` even for small values of n. The problem is that some pivots are very small and the normalization step (to make the pivot 1) causes roundoff errors. Use a tolerance parameter to fix this problem.

What can you conjecture about the rank of A?

(4) Prove that the matrix A has the following row reduced form:

$$R = \begin{pmatrix} 1 & 0 & -1 & -2 & \cdots & -(n-2) \\ 0 & 1 & 2 & 3 & \cdots & n-1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Deduce from the above that A has rank 2.

Hint. Some well chosen sequence of row operations.

(5) Use your program to show that if you add any number greater than or equal to $(2/25)n^2$ to every diagonal entry of A you get an invertible matrix! In fact, running the `Matlab` fuction `chol` should tell you that these matrices are SPD (symmetric, positive definite).

Problem B2 (80 pts). (1) Use your program from B1, to find a basis of the kernel of an $m \times n$ matrix A, as explained in Section 2.5 of the notes (`linalg.pdf`); see the example at the end of Section 2.5.

(2) Modify your program so that it also takes a righthand side b, and it tests whether the system $Ax = b$ is solvable or not. If the systen is solvable, find a special solution as explained in Section 2.5 of the notes.

Problem B3 (120 pts). Consider the problem of finding a basis of the subspace V_n of $n \times n$ matrices $A \in \mathrm{M}_n(\mathbb{R})$ satisfying the following properties:

1. The sum of the entries in every row has the same value (say c_1);

2. The sum of the entries in every column has the same value (say c_2).

It turns out that $c_1 = c_2$ and that the $2n-2$ equations corresponding to the above conditions are linearly independent. By the duality theorem, the dimension of the space V_n of matrices satisying the above equations is $n^2 - (2n - 2) = (n - 1)^2 + 1$.

(1) Write a program to produce the matrix A of a system of equations of the form below, asserting that the above conditions hold. For example, when $n = 4$, we have the equations

$$a_{11} + a_{12} + a_{13} + a_{14} - a_{21} - a_{22} - a_{23} - a_{24} = 0$$
$$a_{21} + a_{22} + a_{23} + a_{24} - a_{31} - a_{32} - a_{33} - a_{34} = 0$$
$$a_{31} + a_{32} + a_{33} + a_{34} - a_{41} - a_{42} - a_{43} - a_{44} = 0$$
$$a_{11} + a_{21} + a_{31} + a_{41} - a_{12} - a_{22} - a_{32} - a_{42} = 0$$
$$a_{12} + a_{22} + a_{32} + a_{42} - a_{13} - a_{23} - a_{33} - a_{43} = 0$$
$$a_{13} + a_{23} + a_{33} + a_{43} - a_{14} - a_{24} - a_{34} - a_{44} = 0.$$

Make sure that your equations are listed in the same order as the above equations.

(2) Use your program from Problem B2 to find a basis of the kernel of A for $n = 2, \ldots, 6$, and print our the corresponding matrices that form a basis of the subspace V_n.

(3) Now consider *magic squares*. Such matrices satisfy the two conditions about the sum of the entries in each row and in each column to be the same number, and also the additional two constraints that the main descending and the main ascending diagonals add up to this common number. Furthermore, the entries are also required to be positive integers. For example, in the case $n = 4$, we have the following system of equations:

$$a_{11} + a_{12} + a_{13} + a_{14} - a_{21} - a_{22} - a_{23} - a_{24} = 0$$
$$a_{21} + a_{22} + a_{23} + a_{24} - a_{31} - a_{32} - a_{33} - a_{34} = 0$$
$$a_{31} + a_{32} + a_{33} + a_{34} - a_{41} - a_{42} - a_{43} - a_{44} = 0$$
$$a_{22} + a_{33} + a_{44} - a_{12} - a_{13} - a_{14} = 0$$
$$a_{11} + a_{21} + a_{31} + a_{41} - a_{12} - a_{22} - a_{32} - a_{42} = 0$$
$$a_{12} + a_{22} + a_{32} + a_{42} - a_{13} - a_{23} - a_{33} - a_{43} = 0$$
$$a_{13} + a_{23} + a_{33} + a_{43} - a_{14} - a_{24} - a_{34} - a_{44} = 0$$
$$a_{41} + a_{32} + a_{23} - a_{11} - a_{12} - a_{13} = 0.$$

Observe that the equation asserting that the sum of the diagonal entries is equal to the sum of the entries in the first row is listed as the nth equation, and that the equation asserting that the sum of the ascending diagonal entries is equal to the sum of the entries in the first row is listed as the $2n$th equations. It can be shown that the above $2n$ equations are linearly independent if $n \geq 3$, so the space of (generalized) magic squares has dimension $n^2 - 2n = (n-1)^2 - 1$ (these are magic squares with no restriction on the coefficients; *i.e.*, the coefficients need not be positive integers).

Write a program to produce the matrix M of a system of equations of the form above, asserting that a matrix is a generalized magic square. Make sure you use the same order for your equations as shown above.

Use your program from Problem B2 to find a basis of (generalized) magic squares for $n = 3, 4, 5$.

For $n = 3$, show that a generic magic square is of the form

$$\begin{pmatrix} (2x_1 + 2x_2 - x_3)/3 & (2x_1 - x_2 + 2x_3)/3 & (-x_1 + 2x_2 + 2x_3)/3 \\ (-2x_1 + x_2 + 4x_3)/3 & (x_1 + x_2 + x_3)/3 & (4x_1 + x_2 - 2x_3)/3 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

For $n = 4$, show that a generic magic square is of the form

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & a_1 \\ m_{31} & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix},$$

with

$$m_{11} = a_1 + a_4 - a_5$$
$$m_{12} = a_1 - a_2 + a_3 + a_4 - a_5 - a_6 + a_8$$
$$m_{13} = -a_1 + a_2 - a_3 - a_4 + 2a_5 + a_6$$
$$m_{14} = -a_1 - a_4 + a_5 + a_6 + a_7$$
$$m_{21} = -a_1 + a_2 + a_3$$
$$m_{22} = -a_1 - a_3 - a_4 + 2a_5 + a_6 + a_7$$
$$m_{23} = a_1 - a_2 + a_4 - a_5 + a_8$$
$$m_{31} = -a_2 - a_3 - a_4 + a_5 + a_6 + a_7 + a_8.$$

(4) A *normal magic square* is a magic square whose entries are the integers $1, 2, \ldots, n^2$. Show that there are no normal magic squares for $n = 2$. For $n = 3$, show that we must have

$$x_1 + x_2 + x_3 = 15.$$

Eliminating x_3, a generic normal magic square is of the form

$$\begin{pmatrix} x_1 + x_2 - 5 & 10 - x_2 & 10 - x_1 \\ 20 - 2x_1 - x_2 & 5 & 2x_1 + x_2 - 10 \\ x_1 & x_2 & 15 - x_1 - x_2 \end{pmatrix}.$$

Extra credit (40 pts)) Show that there is a unique normal magic square (up to rotations and reflections) given by:

$$\begin{pmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{pmatrix}.$$

For $n = 4$, a normal magic square must have

$$a_5 + a_6 + a_7 + a_8 = 34.$$

We can eliminate a_8, but we still have 7 variables ranging from 1 to 16, so it is a formidable task to find all 880 distinct normal magic squares!

TOTAL: 300 + 40 points.