

# Fundamentals of Linear Algebra and Optimization

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## Project 2

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The purpose of this project is to investigate properties and applications of Haar wavelets. In particular, methods for compressing audio signals and digital images are investigated.

(1) In the notes, I explain how a vector  $u = (u_1, \dots, u_m)$  corresponds to a piecewise linear function over the interval  $[0, 1)$ . The function  $\text{plf}(u)$  is defined such that

$$\text{plf}(u)(x) = u_i, \quad \frac{i-1}{m} \leq x < \frac{i}{m}, \quad 1 \leq i \leq m.$$

In words, the function  $\text{plf}(u)$  has the value  $u_1$  on the interval  $[0, 1/m)$ , the value  $u_2$  on  $[1/m, 2/m)$ , etc., and the value  $u_m$  on the interval  $[(m-1)/m, 1)$ .

Write a `Matlab` program that takes as input a vector  $u$  and plots the corresponding function  $\text{plf}(u)$ . Test your program on several inputs including

$$u = [0 \ 2 \ 4 \ 6 \ 6 \ 4 \ 2 \ 1 \ -1 \ -2 \ -4 \ -6 \ -6 \ -4 \ -2 \ 0],$$

and the vectors  $w$  obtained by concatenating  $u$  with itself 8 and 9 times. Note that  $u$  contains 16 integers, and the last one is 0, not  $-20$  (spaces are important).

(2) Write two `Matlab` functions `haar` and `haar_inv` implementing the method for computing the Haar transform of a vector and the reconstruction of a vector from its Haar coefficients, as described in the notes.

Test your programs on many input, including

$$u = [0 \ 2 \ 4 \ 6 \ 6 \ 4 \ 2 \ 1 \ -1 \ -2 \ -4 \ -6 \ -6 \ -4 \ -2 \ 0]$$

and the string strings  $w$  from (1). What do you observe?

(3) Write a `Matlab` function `haar_step` that performs only  $k$  rounds of averaging and differencing on some input vector  $u$ ; the function `haar_step` takes as input  $u$  and  $k$ .

Test your program on the vector  $w$  obtained by concatenating  $u$  with itself 8 eight times. Something remarkable happens starting with  $k = 4$ . Explain the behavior that you observe for  $k = 4, 5, 6, 7$ .

Load the audio file `handel` using `load handel`. This file is saved in the variable `y`. Keep the first 65536 elements of this vector by doing `handel = y(1:65536);`. To play and hear the music, do `sound(handel)`. Run `haar_step` on the vector `handel` for  $k = 1$ . Then play the result. What happens. Can you explain it? Do this again for  $k = 2, 3$ . What do you observe?

Write a function `haar_inv_step` inverting the function `haar_step`. The function `haar_inv_step` takes as input a vector  $v$  and the number of rounds  $k$ .

To check that this function is correct, first apply `haar_step` and then `haar_inv_step` for the same number of steps  $k$ . You should get back the original vector.

Run `haar` on the vector `handel` to get the Haar transform  $c$ . Set the detail coefficients to zero by doing `c1 = c; c1(32768:end) = 0;`. Then apply `haar_inv` to `c1` to get `handel1`. Play `handel1`. What difference do you observe compared to playing `handel`? Experiment with other compressions of  $c$ .

(4) Write two Matlab functions `haar2D` and `haar_inv2D` implementing the method for computing the Haar transform of a matrix and the reconstruction of an image from its matrix of Haar coefficients, as described in the notes.

Apply the function `haar_inv2D` to the matrix

$$T = \begin{pmatrix} 1212 & -306 & -146 & -54 & -24 & -68 & -40 & 4 \\ 30 & 36 & -90 & -2 & 8 & -20 & 8 & -4 \\ -50 & -10 & -20 & -24 & 0 & 72 & -16 & -16 \\ 82 & 38 & -24 & 68 & 48 & -64 & 32 & 8 \\ 8 & 8 & -32 & 16 & -48 & -48 & -16 & 16 \\ 20 & 20 & -56 & -16 & -16 & 32 & -16 & -16 \\ -8 & 8 & -48 & 0 & -16 & -16 & -16 & -16 \\ 44 & 36 & 0 & 8 & 80 & -16 & -16 & 0 \end{pmatrix}.$$

Compare your result with the matrix  $P$  of Example 4.1 of the paper by Greg Ames (see the web page for CIS515). The matrix in Ames's paper seems to have a typo! What is it?

You can load and display various images in Matlab using the following lines of code:

```
clear X map
load('durer', 'X')
Xdurer = X(1:512, :);
Xdurer(:, 510:512) = 50;
figure
colormap(gray)
imagesc(Xdurer)
```

The above loads the file `durer`. There are a few other images such as `detail`, `flujet`, `earth`, `mandrill`, `spine`, and `clown`. You may have to resize these images to have dimensions that are powers of 2. To display an image, use `imagesc`.

Convert `Xdur` to its Haar transform and decode it. Compare the original and the reconstructed image.

(5) Write two `Matlab` functions `haar2D_n` and `haar_inv2D_n` implementing the method for computing the normalized Haar transform of a matrix and the reconstruction of an image from its matrix of normalized Haar coefficients, as described in the notes.

Consider the image given by the following matrix:

$$A = \begin{pmatrix} 100 & 103 & 99 & 97 & 93 & 94 & 78 & 73 \\ 102 & 97 & 100 & 111 & 113 & 104 & 96 & 82 \\ 99 & 109 & 104 & 95 & 93 & 92 & 88 & 76 \\ 114 & 104 & 99 & 102 & 93 & 82 & 74 & 74 \\ 96 & 91 & 91 & 87 & 79 & 78 & 77 & 76 \\ 90 & 88 & 83 & 78 & 77 & 74 & 76 & 76 \\ 92 & 81 & 73 & 72 & 69 & 65 & 66 & 62 \\ 75 & 70 & 69 & 65 & 60 & 55 & 61 & 65 \end{pmatrix}$$

Use `haar2D_n` to compute the normalized matrix  $C$  of Haar coefficients of  $A$ .

It is claimed in Ames's paper (Section 7) that the reconstructed matrix

$$A_2 = \begin{pmatrix} 100 & 100 & 95 & 95 & 92 & 92 & 76 & 76 \\ 103 & 103 & 98 & 98 & 106 & 106 & 90 & 90 \\ 99 & 109 & 99 & 99 & 96 & 96 & 81 & 81 \\ 114 & 104 & 104 & 104 & 91 & 91 & 76 & 76 \\ 91 & 91 & 86 & 86 & 76 & 76 & 76 & 76 \\ 91 & 91 & 86 & 86 & 76 & 76 & 76 & 76 \\ 82 & 82 & 76 & 76 & 66 & 66 & 66 & 66 \\ 74 & 74 & 69 & 69 & 58 & 58 & 59 & 59 \end{pmatrix}$$

is obtained from the normalized matrix

$$C_1 = \begin{pmatrix} 255 & 52 & 15 & 21 & 0 & 0 & 0 & 0 \\ 78 & 0 & 0 & 22 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 38 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

but this not quite correct. First, the coefficient 255 should be 682, and other nonzero entries are missing. Find the matrix  $C_2$ , a compressed version of  $C$ , that gives back  $A_2$ .

*Hint.* The command `round` is helpful.

(6) **Extra Credit.** Write versions of `haar2D` and `haar_inv2D` that perform only  $k$  rounds of averaging and differencing. Test your programs on `Xdurer` (and possibly other images).

**TOTAL: 300 + 50 points.**