Problem B1 (30 pts). Let $E$ be a real vector space of finite dimension, $n \geq 1$. Say that two bases, $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, of $E$ have the same orientation iff $\det(P) > 0$, where $P$ the change of basis matrix from $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, namely, the matrix whose $j$th columns consist of the coordinates of $v_j$ over the basis $(u_1, \ldots, u_n)$.

(a) Prove that having the same orientation is an equivalence relation with two equivalence classes.

An orientation of a vector space, $E$, is the choice of any fixed basis, say $(e_1, \ldots, e_n)$, of $E$. Any other basis, $(v_1, \ldots, v_n)$, has the same orientation as $(e_1, \ldots, e_n)$ (and is said to be positive or direct) iff $\det(P) > 0$, else it is said to have the opposite orientation of $(e_1, \ldots, e_n)$ (or to be negative or indirect), where $P$ is the change of basis matrix from $(e_1, \ldots, e_n)$ to $(v_1, \ldots, v_n)$. An oriented vector space is a vector space with some chosen orientation (a positive basis).

(b) Let $B_1 = (u_1, \ldots, u_n)$ and $B_2 = (v_1, \ldots, v_n)$ be two orthonormal bases. For any sequence of vectors, $(w_1, \ldots, w_n)$, in $E$, let $\det_{B_1}(w_1, \ldots, w_n)$ be the determinant of the matrix whose columns are the coordinates of the $w_j$'s over the basis $B_1$ and similarly for $\det_{B_2}(w_1, \ldots, w_n)$.

Prove that if $B_1$ and $B_2$ have the same orientation, then

$$\det_{B_1}(w_1, \ldots, w_n) = \det_{B_2}(w_1, \ldots, w_n).$$

Given any oriented vector space, $E$, for any sequence of vectors, $(w_1, \ldots, w_n)$, in $E$, the common value, $\det_B(w_1, \ldots, w_n)$, for all positive orthonormal bases, $B$, of $E$ is denoted $\lambda_E(w_1, \ldots, w_n)$ and called a volume form of $(w_1, \ldots, w_n)$.

(c) Given any Euclidean oriented vector space, $E$, of dimension $n$ for any $n - 1$ vectors, $w_1, \ldots, w_{n-1}$, in $E$, check that the map

$$x \mapsto \lambda_E(w_1, \ldots, w_{n-1}, x)$$
is a linear form. Then, prove that there is a unique vector, denoted \( w_1 \times \cdots \times w_{n-1} \), such that
\[
\lambda_E(w_1, \ldots, w_{n-1}, x) = (w_1 \times \cdots \times w_{n-1}) \cdot x,
\]
for all \( x \in E \). The vector \( w_1 \times \cdots \times w_{n-1} \) is called the cross-product of \( (w_1, \ldots, w_{n-1}) \). It is a generalization of the cross-product in \( \mathbb{R}^3 \) (when \( n = 3 \)).

**Problem B2 (50 pts).** Given \( p \) vectors \( (u_1, \ldots, u_p) \) in a Euclidean space \( E \) of dimension \( n \geq p \), the Gram determinant (or Gramian) of the vectors \( (u_1, \ldots, u_p) \) is the determinant
\[
\text{Gram}(u_1, \ldots, u_p) = \begin{vmatrix} \|u_1\|^2 & \langle u_1, u_2 \rangle & \cdots & \langle u_1, u_p \rangle \\ \langle u_2, u_1 \rangle & \|u_2\|^2 & \cdots & \langle u_2, u_p \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_p, u_1 \rangle & \langle u_p, u_2 \rangle & \cdots & \|u_p\|^2 \end{vmatrix}.
\]

(1) Prove that
\[
\text{Gram}(u_1, \ldots, u_n) = \lambda_E(u_1, \ldots, u_n)^2.
\]
*Hint.* If \( (e_1, \ldots, e_n) \) is an orthonormal basis and \( A \) is the matrix of the vectors \( (u_1, \ldots, u_n) \) over this basis,
\[
\det(A)^2 = \det(A^\top A) = \det(A^i \cdot A^j),
\]
where \( A^i \) denotes the \( i \)th column of the matrix \( A \), and \( (A^i \cdot A^j) \) denotes the \( n \times n \) matrix with entries \( A^i \cdot A^j \).

(2) Prove that
\[
\|u_1 \times \cdots \times u_{n-1}\|^2 = \text{Gram}(u_1, \ldots, u_{n-1}).
\]
*Hint.* Letting \( w = u_1 \times \cdots \times u_{n-1} \), observe that
\[
\lambda_E(u_1, \ldots, u_{n-1}, w) = \langle w, w \rangle = \|w\|^2,
\]
and show that
\[
\|w\|^4 = \lambda_E(u_1, \ldots, u_{n-1}, w)^2 = \text{Gram}(u_1, \ldots, u_{n-1}, w) = \text{Gram}(u_1, \ldots, u_{n-1})\|w\|^2.
\]

**Problem B3 (20 pts).** Let \( \varphi : E \times E \to \mathbb{R} \) be a bilinear form on a real vector space \( E \) of finite dimension \( n \). Given any basis \( (e_1, \ldots, e_n) \) of \( E \), let \( A = (a_{ij}) \) be the matrix defined such that
\[
a_{ij} = \varphi(e_i, e_j),
\]
\( 1 \leq i, j \leq n \). We call \( A \) the matrix of \( \varphi \) w.r.t. the basis \( (e_1, \ldots, e_n) \).
(a) For any two vectors $x$ and $y$, if $X$ and $Y$ denote the column vectors of coordinates of $x$ and $y$ w.r.t. the basis $(e_1, \ldots, e_n)$, prove that

$$\varphi(x, y) = X^TAY.$$ 

(b) Recall that $A$ is a symmetric matrix if $A = A^T$. Prove that $\varphi$ is symmetric if $A$ is a symmetric matrix.

(c) If $(f_1, \ldots, f_n)$ is another basis of $E$ and $P$ is the change of basis matrix from $(e_1, \ldots, e_n)$ to $(f_1, \ldots, f_n)$, prove that the matrix of $\varphi$ w.r.t. the basis $(f_1, \ldots, f_n)$ is

$$P^TAP.$$ 

The common rank of all matrices representing $\varphi$ is called the rank of $\varphi$.

**Problem B4 (50 pts).** Let $\varphi : E \times E \to \mathbb{R}$ be a symmetric bilinear form on a real vector space $E$ of finite dimension $n$. Two vectors $x$ and $y$ are said to be conjugate or orthogonal w.r.t. $\varphi$ if $\varphi(x, y) = 0$. The main purpose of this problem is to prove that there is a basis of vectors that are pairwise conjugate w.r.t. $\varphi$.

(a) Prove that if $\varphi(x, x) = 0$ for all $x \in E$, then $\varphi$ is identically null on $E$.

Otherwise, we can assume that there is some vector $x \in E$ such that $\varphi(x, x) \neq 0$.

Use induction to prove that there is a basis of vectors $(u_1, \ldots, u_n)$ that are pairwise conjugate w.r.t. $\varphi$.

*Hint.* For the induction step, proceed as follows. Let $(u_1, e_2, \ldots, e_n)$ be a basis of $E$, with $\varphi(u_1, u_1) \neq 0$. Prove that there are scalars $\lambda_2, \ldots, \lambda_n$ such that each of the vectors

$$v_i = e_i + \lambda_i u_1$$

is conjugate to $u_1$ w.r.t. $\varphi$, where $2 \leq i \leq n$, and that $(u_1, v_2, \ldots, v_n)$ is a basis.

(b) Let $(e_1, \ldots, e_n)$ be a basis of vectors that are pairwise conjugate w.r.t. $\varphi$, and assume that they are ordered such that

$$\varphi(e_i, e_i) = \begin{cases} \theta_i \neq 0 & \text{if } 1 \leq i \leq r, \\ 0 & \text{if } r + 1 \leq i \leq n, \end{cases}$$

where $r$ is the rank of $\varphi$. Show that the matrix of $\varphi$ w.r.t. $(e_1, \ldots, e_n)$ is a diagonal matrix, and that

$$\varphi(x, y) = \sum_{i=1}^r \theta_i x_i y_i,$$

where $x = \sum_{i=1}^n x_i e_i$ and $y = \sum_{i=1}^n y_i e_i$.

Prove that for every symmetric matrix $A$, there is an invertible matrix $P$ such that

$$P^TAP = D,$$

where $D$ is a diagonal matrix.
where $D$ is a diagonal matrix.

(c) Prove that there is an integer $p$, $0 \leq p \leq r$ (where $r$ is the rank of $\varphi$), such that $\varphi(u_i, u_i) > 0$ for exactly $p$ vectors of every basis $(u_1, \ldots, u_n)$ of vectors that are pairwise conjugate w.r.t. $\varphi$ (Sylvester’s inertia theorem).

Proceed as follows. Assume that in the basis $(u_1, \ldots, u_n)$, for any $x \in E$, we have

$$\varphi(x, x) = \alpha_1 x_1^2 + \cdots + \alpha_p x_p^2 - \alpha_{p+1} x_{p+1}^2 - \cdots - \alpha_r x_r^2,$$

where $x = \sum_{i=1}^n x_i u_i$, and that in the basis $(v_1, \ldots, v_n)$, for any $x \in E$, we have

$$\varphi(x, x) = \beta_1 y_1^2 + \cdots + \beta_q y_q^2 - \beta_{q+1} y_{q+1}^2 - \cdots - \beta_r y_r^2,$$

where $x = \sum_{i=1}^n y_i v_i$, with $\alpha_i > 0$, $\beta_i > 0$, $1 \leq i \leq r$.

Assume that $p > q$ and derive a contradiction. First, consider $x$ in the subspace $F$ spanned by $(u_1, \ldots, u_p, u_{r+1}, \ldots, u_n)$, and observe that $\varphi(x, x) \geq 0$ if $x \neq 0$. Next, consider $x$ in the subspace $G$ spanned by $(v_{q+1}, \ldots, v_r)$, and observe that $\varphi(x, x) < 0$ if $x \neq 0$. Prove that $F \cap G$ is nontrivial (i.e., contains some nonnull vector), and derive a contradiction. This implies that $p \leq q$. Finish the proof.

The pair $(p, r - p)$ is called the signature of $\varphi$.

(d) A symmetric bilinear form $\varphi$ is definite if for every $x \in E$, if $\varphi(x, x) = 0$, then $x = 0$.

Prove that a symmetric bilinear form is definite iff its signature is either $(n, 0)$ or $(0, n)$. In other words, a symmetric definite bilinear form has rank $n$ and is either positive or negative.

**Problem B5 (40 pts).** Let $H$ be a symmetric positive definite matrix and let $K$ be any symmetric matrix.

(1) Prove that $HK$ is diagonalizable, with real eigenvalues.

(2) If $K$ is also positive definite, then prove that the eigenvalues of $HK$ are positive.

(3) Prove that the number of positive (resp. negative) eigenvalues of $HK$ is equal to the number of positive (resp. negative) eigenvalues of $K$.

Let $A$ be any real or complex $n \times n$ matrix. It can be shown that the sequence $(E_m)$ of matrices

$$E_m = I + \sum_{k=1}^m \frac{A^k}{k!}$$

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converges to a limit denoted
\[ e^A = I + \sum_{k=1}^{\infty} \frac{A^k}{k!} \]
and called the \textit{exponential} of $A$. You may accept this fact without proof.

**Problem B6 (Extra Credit 10 pts).**

Let $\| \|$ be any operator norm. Prove that for every $m \geq 1$,
\[ \| I \| + \sum_{k=1}^{m} \left\| \frac{A^k}{k!} \right\| \leq e^{\| A \|}. \]

If you know some analysis, deduce from the above that the sequence $(E_m)$ of matrices
\[ E_m = I + \sum_{k=1}^{m} \frac{A^k}{k!} \]
converges to a limit denoted $e^A$, and called the \textit{exponential} of $A$.

**Problem B7 (100 pts).** (a) Let $\mathfrak{so}(3)$ be the space of $3 \times 3$ skew symmetric matrices
\[ \mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}. \]

For any matrix
\[ A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \in \mathfrak{so}(3), \]
if we let $\theta = \sqrt{a^2 + b^2 + c^2}$ and
\[ B = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}, \]
prove that
\[ A^2 = -\theta^2 I + B, \]
\[ AB = BA = 0. \]
From the above, deduce that
\[ A^3 = -\theta^2 A. \]
(b) Prove that the exponential map \( \exp: \mathfrak{so}(3) \rightarrow \mathbf{SO}(3) \) is given by

\[
\exp A = e^A = \cos \theta I_3 + \frac{\sin \theta}{\theta} A + \frac{(1 - \cos \theta)}{\theta^2} B,
\]

or, equivalently, by

\[
e^A = I_3 + \frac{\sin \theta}{\theta} A + \frac{(1 - \cos \theta)}{\theta^2} A^2, \quad \text{if } \theta \neq 0,
\]

with \( \exp(0) = I_3 \).

(c) Prove that \( e^A \) is an orthogonal matrix of determinant +1, i.e., a rotation matrix.

(d) Prove that the exponential map \( \exp: \mathfrak{so}(3) \rightarrow \mathbf{SO}(3) \) is surjective. For this, proceed as follows: Pick any rotation matrix \( R \in \mathbf{SO}(3) \);

1. The case \( R = I \) is trivial.
2. If \( R \neq I \) and \( \text{tr}(R) \neq -1 \), then

\[
\exp^{-1}(R) = \left\{ \frac{\theta}{2 \sin \theta} (R - R^T) \bigg| 1 + 2 \cos \theta = \text{tr}(R) \right\}.
\]

(Recall that \( \text{tr}(R) = r_{11} + r_{22} + r_{33} \), the trace of the matrix \( R \)).

Show that there is a unique skew-symmetric \( B \) with corresponding \( \theta \) satisfying \( 0 < \theta < \pi \) such that \( e^B = R \).

3. If \( R \neq I \) and \( \text{tr}(R) = -1 \), then prove that the eigenvalues of \( R \) are \( 1, -1, -1 \), that \( R = R^\top \), and that \( R^2 = I \). Prove that the matrix

\[
S = \frac{1}{2}(R - I)
\]

is a symmetric matrix whose eigenvalues are \(-1, -1, 0\). Thus, \( S \) can be diagonalized with respect to an orthogonal matrix \( Q \) as

\[
S = Q \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix} Q^\top.
\]

Prove that there exists a skew symmetric matrix

\[
U = \begin{pmatrix}
0 & -d & c \\
d & 0 & -b \\
-c & b & 0
\end{pmatrix}
\]
so that
\[ U^2 = S = \frac{1}{2}(R - I). \]

Observe that
\[
U^2 = \begin{pmatrix}
-(c^2 + d^2) & bc & bd \\
bc & -(b^2 + d^2) & cd \\
bd & cd & -(b^2 + c^2)
\end{pmatrix},
\]
and use this to conclude that if \( U^2 = S \), then \( b^2 + c^2 + d^2 = 1 \). Then, show that
\[
\exp^{-1}(R) = \begin{cases}
(2k + 1)\pi \begin{pmatrix}
0 & -d & c \\
d & 0 & -b \\
-c & b & 0
\end{pmatrix}, & k \in \mathbb{Z}
\end{cases},
\]
where \((b, c, d)\) is any unit vector such that for the corresponding skew symmetric matrix \( U \), we have \( U^2 = S \).

(e) To find a skew symmetric matrix \( U \) so that \( U^2 = S = \frac{1}{2}(R - I) \) as in (d), we can solve the system
\[
\begin{pmatrix}
b^2 - 1 & bc & bd \\
bc & c^2 - 1 & cd \\
bd & cd & d^2 - 1
\end{pmatrix} = S.
\]
We immediately get \( b^2, c^2, d^2 \), and then, since one of \( b, c, d \) is nonzero, say \( b \), if we choose the positive square root of \( b^2 \), we can determine \( c \) and \( d \) from \( bc \) and \( bd \).

Implement a computer program to solve the above system.

**Problem B8 (120 pts).** (a) Consider the set of affine maps \( \rho \) of \( \mathbb{R}^3 \) defined such that
\[
\rho(X) = \alpha RX + W,
\]
where \( R \) is a rotation matrix (an orthogonal matrix of determinant +1), \( W \) is some vector in \( \mathbb{R}^3 \), and \( \alpha \in \mathbb{R} \) with \( \alpha > 0 \). Every such a map can be represented by the \( 4 \times 4 \) matrix
\[
\begin{pmatrix}
\alpha R & W \\
0 & 1
\end{pmatrix}
\]
in the sense that
\[
\begin{pmatrix}
\rho(X) \\
1
\end{pmatrix} = \begin{pmatrix}
\alpha R & W \\
0 & 1
\end{pmatrix} \begin{pmatrix}
X \\
1
\end{pmatrix}
\]
iff
\[
\rho(X) = \alpha RX + W.
\]

Prove that these maps form a group, denoted by \( \text{SIM}(3) \) (the direct affine similitudes of \( \mathbb{R}^3 \)).
(b) Let us now consider the set of $4 \times 4$ real matrices of the form

$$B = \begin{pmatrix} \Gamma & W \\ 0 & 0 \end{pmatrix},$$

where $\Gamma$ is a matrix of the form

$$\Gamma = \lambda I_3 + \Omega,$$

with

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

so that

$$\Gamma = \begin{pmatrix} \lambda & -c & b \\ c & \lambda & -a \\ -b & a & \lambda \end{pmatrix},$$

and $W$ is a vector in $\mathbb{R}^3$.

Verify that this set of matrices is a vector space isomorphic to $(\mathbb{R}^7, +)$. This vector space is denoted by $\text{sim}(3)$.

(c) Given a matrix

$$B = \begin{pmatrix} \Gamma & W \\ 0 & 0 \end{pmatrix}$$
as in (b), prove that

$$B^n = \begin{pmatrix} \Gamma^n & \Gamma^{n-1}W \\ 0 & 0 \end{pmatrix},$$

where $\Gamma^0 = I_3$. Prove that

$$e^B = \begin{pmatrix} e^\Gamma & VW \\ 0 & 1 \end{pmatrix},$$

where

$$V = I_3 + \sum_{k \geq 1} \frac{\Gamma^k}{(k+1)!}.$$  

(d) Prove that if $\Gamma = \lambda I_3 + \Omega$ as in (b), then

$$V = I_3 + \sum_{k \geq 1} \frac{\Gamma^k}{(k+1)!} = \int_0^1 e^{\Gamma t} dt.$$  

(e) For any matrix $\Gamma = \lambda I_3 + \Omega$, with

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$
if we let $\theta = \sqrt{a^2 + b^2 + c^2}$, then prove that

$$e^\Gamma = e^\lambda e^\Omega = e^\lambda \left( I_3 + \frac{\sin \theta}{\theta} \Omega + \frac{(1 - \cos \theta)}{\theta^2} \Omega^2 \right), \quad \text{if } \theta \neq 0,$$

and $e^\Gamma = e^\lambda I_3$ if $\theta = 0$.

**Hint.** You may use the fact that if $AB = BA$, then $e^{A+B} = e^A e^B$. In general, $e^{A+B} \neq e^A e^B$!

(f) Prove that

1. If $\theta = 0$ and $\lambda = 0$, then
   $$V = I_3.$$  
2. If $\theta = 0$ and $\lambda \neq 0$, then
   $$V = \frac{(e^\lambda - 1)}{\lambda} I_3;$$
3. If $\theta \neq 0$ and $\lambda = 0$, then
   $$V = I_3 + \frac{(1 - \cos \theta)}{\theta^2} \Omega + \frac{(\theta - \sin \theta)}{\theta^3} \Omega^2.$$  
4. If $\theta \neq 0$ and $\lambda \neq 0$, then
   $$V = \frac{(e^\lambda - 1)}{\lambda} I_3 + \frac{(\theta(1 - e^\lambda \cos \theta) + e^\lambda \lambda \sin \theta)}{\theta(\lambda^2 + \theta^2)} \Omega + \left( \frac{(e^\lambda - 1)}{\lambda \theta^2} - \frac{e^\lambda \sin \theta}{\theta(\lambda^2 + \theta^2)} - \frac{\lambda(e^\lambda \cos \theta - 1)}{\theta^2(\lambda^2 + \theta^2)} \right) \Omega^2.$$  

**Hint.** You will need to compute $\int_0^1 e^{\lambda t} \sin \theta t \, dt$ and $\int_0^1 e^{\lambda t} \cos \theta t \, dt$.

(g) Prove that $V$ is invertible iff $\lambda \neq 0$ or $\theta \neq k2\pi$, with $k \in \mathbb{Z} - \{0\}$.

**Hint.** Express the eigenvalues of $V$ in terms of the eigenvalues of $\Gamma$.

In the special case where $\lambda = 0$, show that

$$V^{-1} = I - \frac{1}{2} \Omega + \frac{1}{\theta^2} \left( 1 - \frac{\theta \sin \theta}{2(1 - \cos \theta)} \right) \Omega^2, \quad \text{if } \theta \neq 0.$$  

**Hint.** Assume that the inverse of $V$ is of the form

$$Z = I_3 + a\Omega + b\Omega^2,$$

and show that $a, b$, are given by a system of linear equations that always has a unique solution.
(h) Prove that the exponential map \( \exp : \mathfrak{sim}(3) \to \text{SIM}(3) \), given by \( \exp(B) = e^B \), is surjective. You may use the fact that \( \exp : \mathfrak{so}(3) \to \text{SO}(3) \) is surjective, proved in another Problem.

Remark: Curves in \( \text{SIM}(3) \) can be used to describe certain deformations of bodies in \( \mathbb{R}^3 \).

TOTAL: 410 points 10 points Extra credit