Fall, 2020 CIS 515

Fundamentals of Linear Algebra and Optimization
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Homework 5

November 2, 2020; Due November 16, 2020

Problem B1 (40 pts). Let H be a symmetric positive definite matrix and let K be any symmetric matrix.

(1) Prove that HK is diagonalizable, with real eigenvalues.

(2) If K is also positive definite, then prove that the eigenvalues of HK are positive.

(3) Prove that the number of positive (resp. negative) eigenvalues of HK is equal to the number of positive (resp. negative) eigenvalues of K.

Let A be any real or complex $n \times n$ matrix. It can be shown that the sequence (E_m) of matrices

$$E_m = I + \sum_{k=1}^{m} \frac{A^k}{k!}$$

converges to a limit denoted

$$e^A = I + \sum_{k=1}^{\infty} \frac{A^k}{k!}$$

and called the *exponential* of A. You may accept this fact without proof.

Problem B2 (Extra Credit 10 pts).

Let $\| \; \|$ be any operator norm. Prove that for every $m \geq 1$,

$$\|I\| + \sum_{k=1}^{m} \left\| \frac{A^k}{k!} \right\| \leq e^{\|A\|}.$$

If you know some analysis, deduce from the above that the sequence (E_m) of matrices

$$E_m = I + \sum_{k=1}^{m} \frac{A^k}{k!}$$

converges to a limit denoted e^A, and called the *exponential* of A.

Problem B3 (100 pts). (a) Let $\mathfrak{so}(3)$ be the space of 3×3 skew symmetric matrices

$$\mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

For any matrix

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \in \mathfrak{so}(3),$$

if we let $\theta = \sqrt{a^2 + b^2 + c^2}$ and

$$B = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix},$$

prove that

$$\begin{aligned} A^2 &= -\theta^2 I + B, \\ AB &= BA = 0. \end{aligned}$$

From the above, deduce that

$$A^3 = -\theta^2 A.$$

(b) Prove that the exponential map $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is given by

$$\exp A = e^A = \cos\theta \, I_3 + \frac{\sin\theta}{\theta} A + \frac{(1 - \cos\theta)}{\theta^2} B,$$

or, equivalently, by

$$e^A = I_3 + \frac{\sin\theta}{\theta} A + \frac{(1 - \cos\theta)}{\theta^2} A^2, \quad \text{if } \theta \neq 0,$$

with $\exp(0_3) = I_3$.

(c) Prove that e^A is an orthogonal matrix of determinant $+1$, i.e., a rotation matrix.

(d) Prove that the exponential map $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is surjective. For this, proceed as follows: Pick any rotation matrix $R \in \mathbf{SO}(3)$;

(1) The case $R = I$ is trivial.

(2) If $R \neq I$ and $\mathrm{tr}(R) \neq -1$, then

$$\exp^{-1}(R) = \left\{ \frac{\theta}{2\sin\theta}(R - R^T) \;\middle|\; 1 + 2\cos\theta = \mathrm{tr}(R) \right\}.$$

(Recall that $\mathrm{tr}(R) = r_{11} + r_{22} + r_{33}$, the *trace* of the matrix R).

Show that there is a unique skew-symmetric B with corresponding θ satisfying $0 < \theta < \pi$ such that $e^B = R$.

(3) If $R \neq I$ and $\mathrm{tr}(R) = -1$, then prove that the eigenvalues of R are $1, -1, -1$, that $R = R^\top$, and that $R^2 = I$. Prove that the matrix

$$S = \frac{1}{2}(R - I)$$

is a symmetric matrix whose eigenvalues are $-1, -1, 0$. Thus, S can be diagonalized with respect to an orthogonal matrix Q as

$$S = Q \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^\top.$$

Prove that there exists a skew symmetric matrix

$$U = \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}$$

so that

$$U^2 = S = \frac{1}{2}(R - I).$$

Observe that

$$U^2 = \begin{pmatrix} -(c^2 + d^2) & bc & bd \\ bc & -(b^2 + d^2) & cd \\ bd & cd & -(b^2 + c^2) \end{pmatrix},$$

and use this to conclude that if $U^2 = S$, then $b^2 + c^2 + d^2 = 1$. Then, show that

$$\exp^{-1}(R) = \left\{ (2k+1)\pi \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix},\ k \in \mathbb{Z} \right\},$$

where (b, c, d) is any unit vector such that for the corresponding skew symmetric matrix U, we have $U^2 = S$.

(e) To find a skew symmetric matrix U so that $U^2 = S = \frac{1}{2}(R - I)$ as in (d), we can solve the system

$$\begin{pmatrix} b^2 - 1 & bc & bd \\ bc & c^2 - 1 & cd \\ bd & cd & d^2 - 1 \end{pmatrix} = S.$$

We immediately get b^2, c^2, d^2, and then, since one of b, c, d is nonzero, say b, if we choose the positive square root of b^2, we can determine c and d from bc and bd.

Implement a computer program to solve the above system.

Problem B4 (120 pts). (a) Consider the set of affine maps ρ of $\mathbb{R}^3$ defined such that

$$\rho(X) = \alpha RX + W,$$

where R is a rotation matrix (an orthogonal matrix of determinant $+1$), W is some vector in $\mathbb{R}^3$, and $\alpha \in \mathbb{R}$ with $\alpha > 0$. Every such a map can be represented by the 4×4 matrix

$$\begin{pmatrix} \alpha R & W \\ 0 & 1 \end{pmatrix}$$

in the sense that

$$\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

iff

$$\rho(X) = \alpha RX + W.$$

Prove that these maps form a group, denoted by $\mathbf{SIM}(3)$ (the *direct affine similitudes* of $\mathbb{R}^3$).

(b) Let us now consider the set of 4×4 real matrices of the form

$$B = \begin{pmatrix} \Gamma & W \\ 0 & 0 \end{pmatrix},$$

where Γ is a matrix of the form

$$\Gamma = \lambda I_3 + \Omega,$$

with

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

so that

$$\Gamma = \begin{pmatrix} \lambda & -c & b \\ c & \lambda & -a \\ -b & a & \lambda \end{pmatrix},$$

and W is a vector in $\mathbb{R}^3$.

Verify that this set of matrices is a vector space isomorphic to $(\mathbb{R}^7, +)$. This vector space is denoted by $\mathfrak{sim}(3)$.

(c) Given a matrix

$$B = \begin{pmatrix} \Gamma & W \\ 0 & 0 \end{pmatrix}$$

as in (b), prove that

$$B^n = \begin{pmatrix} \Gamma^n & \Gamma^{n-1} W \\ 0 & 0 \end{pmatrix}$$

where $\Gamma^0 = I_3$. Prove that

$$e^B = \begin{pmatrix} e^{\Gamma} & VW \\ 0 & 1 \end{pmatrix},$$

where

$$V = I_3 + \sum_{k \geq 1} \frac{\Gamma^k}{(k+1)!}.$$

(d) Prove that if $\Gamma = \lambda I_3 + \Omega$ as in (b), then

$$V = I_3 + \sum_{k \geq 1} \frac{\Gamma^k}{(k+1)!} = \int_0^1 e^{\Gamma t} dt.$$

(e) For any matrix $\Gamma = \lambda I_3 + \Omega$, with

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

if we let $\theta = \sqrt{a^2 + b^2 + c^2}$, then prove that

$$e^{\Gamma} = e^{\lambda} e^{\Omega} = e^{\lambda}\left(I_3 + \frac{\sin\theta}{\theta}\Omega + \frac{(1-\cos\theta)}{\theta^2}\Omega^2\right), \quad \text{if } \theta \neq 0,$$

and $e^{\Gamma} = e^{\lambda} I_3$ if $\theta = 0$.

Hint. You may use the fact that if $AB = BA$, then $e^{A+B} = e^A e^B$. In general, $e^{A+B} \neq e^A e^B$!

(f) Prove that

1. If $\theta = 0$ and $\lambda = 0$, then
$$V = I_3.$$
2. If $\theta = 0$ and $\lambda \neq 0$, then
$$V = \frac{(e^{\lambda} - 1)}{\lambda} I_3;$$
3. If $\theta \neq 0$ and $\lambda = 0$, then
$$V = I_3 + \frac{(1-\cos\theta)}{\theta^2}\Omega + \frac{(\theta - \sin\theta)}{\theta^3}\Omega^2.$$
4. If $\theta \neq 0$ and $\lambda \neq 0$, then
$$\begin{aligned} V = {} & \frac{(e^{\lambda}-1)}{\lambda} I_3 + \frac{(\theta(1 - e^{\lambda}\cos\theta) + e^{\lambda}\lambda\sin\theta)}{\theta(\lambda^2+\theta^2)}\Omega \\ & + \left(\frac{(e^{\lambda}-1)}{\lambda\theta^2} - \frac{e^{\lambda}\sin\theta}{\theta(\lambda^2+\theta^2)} - \frac{\lambda(e^{\lambda}\cos\theta - 1)}{\theta^2(\lambda^2+\theta^2)}\right)\Omega^2. \end{aligned}$$

Hint. You will need to compute $\int_0^1 e^{\lambda t} \sin \theta t \, dt$ and $\int_0^1 e^{\lambda t} \cos \theta t \, dt$.

(g) Prove that V is invertible iff $\lambda \neq 0$ or $\theta \neq k2\pi$, with $k \in \mathbb{Z} - \{0\}$.

Hint. Express the eigenvalues of V in terms of the eigenvalues of Γ.

In the special case where $\lambda = 0$, show that

$$V^{-1} = I - \frac{1}{2}\Omega + \frac{1}{\theta^2}\left(1 - \frac{\theta \sin \theta}{2(1 - \cos \theta)}\right)\Omega^2, \quad \text{if } \theta \neq 0.$$

Hint. Assume that the inverse of V is of the form

$$Z = I_3 + a\Omega + b\Omega^2,$$

and show that a, b, are given by a system of linear equations that always has a unique solution.

(h) Prove that the exponential map $\exp\colon \mathfrak{sim}(3) \to \mathbf{SIM}(3)$, given by $\exp(B) = e^B$, is surjective. You may use the fact that $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is surjective, proved in another Problem.

Remark: Curves in $\mathbf{SIM}(3)$ can be used to describe certain deformations of bodies in $\mathbb{R}^3$.

TOTAL: 260 points+ 10 points Extra credit