Fall, 2018 CIS 515

Fundamentals of Linear Algebra and Optimization
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Homework 5

November 7, 2018; Due November 26, 2018

Problem B1 (60 pts). (1) Let A be any $n \times n$ matrix such that the sum of the entries of every row of A is the same (say c_1), and the sum of entries of every column of A is the same (say c_2). Prove that $c_1 = c_2$.

(2) Prove that for any $n \geq 2$, the $2n - 2$ equations asserting that the sum of the entries of every row of A is the same, and the sum of entries of every column of A is the same are lineary independent. For example, when $n = 4$, we have the following 6 equations

$$\begin{aligned}
a_{11} + a_{12} + a_{13} + a_{14} - a_{21} - a_{22} - a_{23} - a_{24} &= 0 \\
a_{21} + a_{22} + a_{23} + a_{24} - a_{31} - a_{32} - a_{33} - a_{34} &= 0 \\
a_{31} + a_{32} + a_{33} + a_{34} - a_{41} - a_{42} - a_{43} - a_{44} &= 0 \\
a_{11} + a_{21} + a_{31} + a_{41} - a_{12} - a_{22} - a_{32} - a_{42} &= 0 \\
a_{12} + a_{22} + a_{32} + a_{42} - a_{13} - a_{23} - a_{33} - a_{43} &= 0 \\
a_{13} + a_{23} + a_{33} + a_{43} - a_{14} - a_{24} - a_{34} - a_{44} &= 0.
\end{aligned}$$

Hint. Group the equations as above; that is, first list the $n - 1$ equations relating the rows, and then list the $n - 1$ equations relating the columns. Prove that the first $n - 1$ equations are linearly independent, and that the last $n - 1$ equations are also linearly independent. Then, find a relationship between the two groups of equations that will allow you to prove that they span subspace V^r and V^c such that $V^r \cap V^c = (0)$.

(3) Now consider *magic squares*. Such matrices satisfy the two conditions about the sum of the entries in each row and in each column to be the same number, and also the additional two constraints that the main descending and the main ascending diagonals add up to this common number. Traditionally, it is also required that the entries in a magic square are positive integers, but we will consider generalized magic square with arbitrary real entries. For example, in the case $n = 4$, we have the following system of 8 equations:

$$
\begin{aligned}
a_{11}+a_{12}+a_{13}+a_{14}-a_{21}-a_{22}-a_{23}-a_{24}&=0\\
a_{21}+a_{22}+a_{23}+a_{24}-a_{31}-a_{32}-a_{33}-a_{34}&=0\\
a_{31}+a_{32}+a_{33}+a_{34}-a_{41}-a_{42}-a_{43}-a_{44}&=0\\
a_{11}+a_{21}+a_{31}+a_{41}-a_{12}-a_{22}-a_{32}-a_{42}&=0\\
a_{12}+a_{22}+a_{32}+a_{42}-a_{13}-a_{23}-a_{33}-a_{43}&=0\\
a_{13}+a_{23}+a_{33}+a_{43}-a_{14}-a_{24}-a_{34}-a_{44}&=0\\
a_{22}+a_{33}+a_{44}-a_{12}-a_{13}-a_{14}&=0\\
a_{41}+a_{32}+a_{23}-a_{11}-a_{12}-a_{13}&=0.
\end{aligned}
$$

In general, the equation involving the descending diagonal is

$$a_{22}+a_{33}+\cdots+a_{nn}-a_{12}-a_{13}-\cdots-a_{1n}=0 \tag{r}$$

and the equation involving the ascending diagonal is

$$a_{n1}+a_{n-12}+\cdots+a_{2n-1}-a_{11}-a_{12}-\cdots-a_{1n-1}=0. \tag{c}$$

Prove that if $n \geq 3$, then the $2n$ equations asserting that a matrix is a generalized magic square are linearly independent.

Hint. Equations are really linear forms, so find some matrix annihilated by all equations except equation r, and some matrix annihilated by all equations except equation c.

Problem B2 (30 pts). Let A be an $m \times n$ matrix and B be an $n \times m$ matrix.

(1) Prove that

$$\det(I_m - AB) = \det(I_n - BA).$$

Hint. Consider the matrices

$$X = \begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} I_m & 0 \\ -B & I_n \end{pmatrix}.$$

(2) Prove that

$$\lambda^n \det(\lambda I_m - AB) = \lambda^m \det(\lambda I_n - BA).$$

Hint. Consider the matrices

$$X = \begin{pmatrix} \lambda I_m & A \\ B & I_n \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} I_m & 0 \\ -B & \lambda I_n \end{pmatrix}.$$

Problem B3 (30). Let Z be a $q \times p$ real matrix. Prove that if $I_p - Z^\top Z$ is positive definite, then the $(p+q) \times (p+q)$ matrix

$$S = \begin{pmatrix} I_p & Z^\top \\ Z & I_q \end{pmatrix}$$

is symmetric positive definite.

Problem B4 (120). (1) Prove that the columns of the following $n \times n$ matrix are linearly independent when $n \geq 3$:

$$B = \begin{pmatrix} 1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ 1 & -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & -1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & -1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & -1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & -1 \end{pmatrix}$$

In fact, prove that

$$\det(B) = (-1)^n (n-2)2^{n-1}.$$

(2) Consider the $n \times n$ matrices $R^{i,j}$ defined for all i, j with $1 \leq i < j \leq n$ and $n \geq 3$, such that the only nonzero entries are

$$\begin{aligned} R^{i,j}(i,j) &= -1 \\ R^{i,j}(i,i) &= 0 \\ R^{i,j}(j,i) &= 1 \\ R^{i,j}(j,j) &= 0 \\ R^{i,j}(k,k) &= 1, \quad 1 \leq k \leq n, k \neq i, j. \end{aligned}$$

For example,

$$R^{i,j} = \begin{pmatrix} 1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 0 & 0 & \cdots & 0 & -1 & & & \\ & & & 0 & 1 & \cdots & 0 & 0 & & & \\ & & & \vdots & \vdots & \ddots & \vdots & \vdots & & & \\ & & & 0 & 0 & \cdots & 1 & 0 & & & \\ & & & 1 & 0 & \cdots & 0 & 0 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{pmatrix}$$

Prove that the $R^{i,j}$ are rotation matrices. Use the matrices R^{ij} to form a basis of the $n \times n$ skew-symmetric matrices.

(3) Consider the $n \times n$ symmetric matrices $S^{i,j}$ defined for all i, j with $1 \leq i < j \leq n$ and $n \geq 3$, such that the only nonzero entries are

$$\begin{aligned}
S^{i,j}(i,j) &= 1 \\
S^{i,j}(i,i) &= 0 \\
S^{i,j}(j,i) &= 1 \\
S^{i,j}(j,j) &= 0 \\
S^{i,j}(k,k) &= 1, \quad 1 \leq k \leq n, k \neq i, j,
\end{aligned}$$

and if $i + 2 \leq j$ then $S^{i,j}(i+1, i+1) = -1$, else if $i > 1$ and $j = i+1$ then $S^{i,j}(1,1) = -1$, and if $i = 1$ and $j = 2$, then $S^{i,j}(3,3) = -1$.

For example,

$$S^{i,j} = \begin{pmatrix}
1 & & & & & & & & & & \\
& \ddots & & & & & & & & & \\
& & 1 & & & & & & & & \\
& & & 0 & 0 & \cdots & 0 & 1 & & & \\
& & & 0 & -1 & \cdots & 0 & 0 & & & \\
& & & \vdots & \vdots & \ddots & \vdots & \vdots & & & \\
& & & 0 & 0 & \cdots & 1 & 0 & & & \\
& & & 1 & 0 & \cdots & 0 & 0 & & & \\
& & & & & & & & 1 & & \\
& & & & & & & & & \ddots & \\
& & & & & & & & & & 1
\end{pmatrix}$$

Note that $S^{i,j}$ has a single diagonal entry equal to -1. Prove that the $S^{i,j}$ are rotations matrices.

Use (1) together with the $S^{i,j}$ to form a basis of the $n \times n$ symmetric matrices.

(4) Prove that if $n \geq 3$, the set of all linear combinations of matrices in $\mathbf{SO}(n)$ is the space $\mathrm{M}_n(\mathbb{R})$ of all $n \times n$ matrices.

Prove that if $n \geq 3$ and if a matrix $A \in \mathrm{M}_n(\mathbb{R})$ commutes with all rotations matrices, then A commutes with all matrices in $\mathrm{M}_n(\mathbb{R})$.

What happens for $n = 2$?

Prove that if $n \geq 2$, the set of all linear combinations of matrices in $\mathbf{SU}(n)$ is the space $\mathrm{M}_n(\mathbb{C})$ of all $n \times n$ complex matrices.

Extra Credit Problem B5 (100 pts). Give an example of a norm on $\mathbb{C}^n$ and of a *real* matrix A such that

$$\|A\|_{\mathbb{R}} < \|A\|,$$

where $\|-\|_{\mathbb{R}}$ and $\|-\|$ are the operator norms associated with the vector norm $\|-\|$, as defined in the notes.

Hint. This can already be done for $n = 2$.

Problem B6 (30 pts). Let $\| \ \|$ be any operator norm. Given an invertible $n \times n$ matrix A, if $c = 1/(2\,\|A^{-1}\|)$, then for every $n \times n$ matrix H, if $\|H\| \leq c$, then $A + H$ is invertible. Furthermore, show that if $\|H\| \leq c$, then $\|(A + H)^{-1}\| \leq 1/c$.

Problem B7 (20 pts). Let A be any $m \times n$ matrix and let $\lambda \in \mathbb{R}$ be any positive real number $\lambda > 0$.

(1) Prove that $A^\top A + \lambda I_n$ and $AA^\top + \lambda I_m$ are invertible.

(2) Prove that

$$A^\top (AA^\top + \lambda I_m)^{-1} = (A^\top A + \lambda I_n)^{-1} A^\top.$$

Remark: The expressions above correspond to the matrix for which the function

$$\Phi(x) = (Ax - b)^\top (Ax - b) + \lambda x^\top x$$

achieves a minimum. It shows up in machine learning (kernel methods).

Problem B8 (80 pts). Let A be a real 2×2 matrix

$$A = \begin{pmatrix} a_{1\,1} & a_{1\,2} \\ a_{2\,1} & a_{2\,2} \end{pmatrix}.$$

(1) Prove that the squares of the singular values $\sigma_1 \geq \sigma_2$ of A are the roots of the quadratic equation

$$X^2 - \mathrm{tr}(A^\top A)X + |\det(A)|^2 = 0.$$

(2) If we let

$$\mu(A) = \frac{a_{1\,1}^2 + a_{1\,2}^2 + a_{2\,1}^2 + a_{2\,2}^2}{2|a_{1\,1}a_{2\,2} - a_{1\,2}a_{2\,1}|},$$

prove that

$$\mathrm{cond}_2(A) = \frac{\sigma_1}{\sigma_2} = \mu(A) + (\mu(A)^2 - 1)^{1/2}.$$

(3) Consider the subset $\mathcal{S}$ of 2×2 invertible matrices whose entries $a_{i\,j}$ are integers such that $0 \leq a_{ij} \leq 100$.

Prove that the functions $\mathrm{cond}_2(A)$ and $\mu(A)$ reach a maximum on the set $\mathcal{S}$ for the same values of A.

Check that for the matrix

$$A_m = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix}$$

we have

$$\mu(A_m) = 19,603 \quad \det(A_m) = -1$$

and

$$\mathrm{cond}_2(A_m) \approx 39,206.$$

(4) Prove that for all $A \in \mathcal{S}$, if $|\det(A)| \geq 2$ then $\mu(A) \leq 10,000$. Conclude that the maximum of $\mu(A)$ on $\mathcal{S}$ is achieved for matrices such that $\det(A) = \pm 1$. Prove that finding matrices that maximize μ on $\mathcal{S}$ is equivalent to finding some integers n_1, n_2, n_3, n_4 such that

$$\begin{aligned}
&0 \leq n_4 \leq n_3 \leq n_2 \leq n_1 \leq 100 \\
&n_1^2 + n_2^2 + n_3^2 + n_4^2 \geq 100^2 + 99^2 + 99^2 + 98^2 = 39,206 \\
&|n_1 n_4 - n_2 n_3| = 1.
\end{aligned}$$

You may use without proof that the fact that the only solution to the above constraints is the multiset

$$\{100, 99, 99, 98\}.$$

(5) Deduce from part (4) that the matrices in $\mathcal{S}$ for which μ has a maximum value are

$$A_m = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix} \quad \begin{pmatrix} 98 & 99 \\ 99 & 100 \end{pmatrix} \quad \begin{pmatrix} 99 & 100 \\ 98 & 99 \end{pmatrix} \quad \begin{pmatrix} 99 & 98 \\ 100 & 99 \end{pmatrix}$$

and check that μ has the same value for these matrices. Conclude that

$$\max_{A \in \mathcal{S}} \mathrm{cond}_2(A) = \mathrm{cond}_2(A_m).$$

(6) Solve the system

$$\begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 199 \\ 197 \end{pmatrix}.$$

Perturb the right-hand side b by

$$\delta b = \begin{pmatrix} -0.0097 \\ 0.0106 \end{pmatrix}$$

and solve the new system

$$A_m y = b + \delta b$$

where $y = (y_1, y_2)$. Check that

$$\delta x = y - x = \begin{pmatrix} 2 \\ -2.0203 \end{pmatrix}.$$

Compute $\|x\|_2$, $\|\delta x\|_2$, $\|b\|_2$, $\|\delta b\|_2$, and estimate

$$c = \frac{\|\delta x\|_2}{\|x\|_2} \left(\frac{\|\delta b\|_2}{\|b\|_2} \right)^{-1}.$$

Ckeck that

$$c \approx \mathrm{cond}_2(A_m) = 39,206.$$

Problem B9 (50 pts). Given any two subspaces V_1, V_2 of a finite-dimensional vector space E, prove that

$$\begin{aligned} (V_1 + V_2)^0 &= V_1^0 \cap V_2^0 \\ (V_1 \cap V_2)^0 &= V_1^0 + V_2^0. \end{aligned}$$

Beware that in the second equation, V_1 and V_2 are subspaces of E, not E^*.

Hint. To prove the second equation, prove the inclusions $V_1^0 + V_2^0 \subseteq (V_1 \cap V_2)^0$ and $(V_1 \cap V_2)^0 \subseteq V_1^0 + V_2^0$. Proving the second inclusion is a little tricky. First, prove that we can pick a subspace W_1 of V_1 and a subspace W_2 of V_2 such that

1. V_1 is the direct sum $V_1 = (V_1 \cap V_2) \oplus W_1$.
2. V_2 is the direct sum $V_2 = (V_1 \cap V_2) \oplus W_2$.
3. $V_1 + V_2$ is the direct sum $V_1 + V_2 = (V_1 \cap V_2) \oplus W_1 \oplus W_2$.

TOTAL: 420 points + 100 points Extra Credit