$\boxed{\textbf{Fall, 2013} \quad \textbf{CIS 515}}$

Fundamentals of Linear Algebra and Optimization

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Homework 5

November 7, 2013; Due November 26, 2013
Beginning of classes

Problem B1 (60 pts). (1) Let A be any $n \times n$ matrix such that the sum of the entries of every row of A is the same (say c_1), and the sum of entries of every column of A is the same (say c_2). Prove that $c_1 = c_2$.

(2) Prove that for any $n \geq 2$, the $2n - 2$ equations asserting that the sum of the entries of every row of A is the same, and the sum of entries of every column of A is the same are lineary independent. For example, when $n = 4$, we have the following 6 equations

$$
\begin{aligned}
a_{11} + a_{12} + a_{13} + a_{14} - a_{21} - a_{22} - a_{23} - a_{24} &= 0 \\
a_{21} + a_{22} + a_{23} + a_{24} - a_{31} - a_{32} - a_{33} - a_{34} &= 0 \\
a_{31} + a_{32} + a_{33} + a_{34} - a_{41} - a_{42} - a_{43} - a_{44} &= 0 \\
a_{11} + a_{21} + a_{31} + a_{41} - a_{12} - a_{22} - a_{32} - a_{42} &= 0 \\
a_{12} + a_{22} + a_{32} + a_{42} - a_{13} - a_{23} - a_{33} - a_{43} &= 0 \\
a_{13} + a_{23} + a_{33} + a_{43} - a_{14} - a_{24} - a_{34} - a_{44} &= 0.
\end{aligned}
$$

Hint. Group the equations as above; that is, first list the $n - 1$ equations relating the rows, and then list the $n - 1$ equations relating the columns. Prove that the first $n - 1$ equations are linearly independent, and that the last $n - 1$ equations are also linearly independent. Then, find a relationship between the two groups of equations that will allow you to prove that they span subspace V^r and V^c such that $V^r \cap V^c = (0)$.

(3) Now consider *magic squares*. Such matrices satisfy the two conditions about the sum of the entries in each row and in each column to be the same number, and also the additional two constraints that the main descending and the main ascending diagonals add up to this common number. Traditionally, it is also required that the entries in a magic square are positive integers, but we will consider generalized magic square with arbitrary real entries. For example, in the case $n = 4$, we have the following system of 8 equations:

$$\begin{aligned}
a_{11} + a_{12} + a_{13} + a_{14} - a_{21} - a_{22} - a_{23} - a_{24} &= 0 \\
a_{21} + a_{22} + a_{23} + a_{24} - a_{31} - a_{32} - a_{33} - a_{34} &= 0 \\
a_{31} + a_{32} + a_{33} + a_{34} - a_{41} - a_{42} - a_{43} - a_{44} &= 0 \\
a_{11} + a_{21} + a_{31} + a_{41} - a_{12} - a_{22} - a_{32} - a_{42} &= 0 \\
a_{12} + a_{22} + a_{32} + a_{42} - a_{13} - a_{23} - a_{33} - a_{43} &= 0 \\
a_{13} + a_{23} + a_{33} + a_{43} - a_{14} - a_{24} - a_{34} - a_{44} &= 0 \\
a_{22} + a_{33} + a_{44} - a_{12} - a_{13} - a_{14} &= 0 \\
a_{41} + a_{32} + a_{23} - a_{11} - a_{12} - a_{13} &= 0.
\end{aligned}$$

In general, the equation involving the descending diagonal is

$$a_{22} + a_{33} + \cdots + a_{nn} - a_{12} - a_{13} - \cdots - a_{1n} = 0 \tag{r}$$

and the equation involving the ascending diagonal is

$$a_{n1} + a_{n-12} + \cdots + a_{2n-1} - a_{11} - a_{12} - \cdots - a_{1n-1} = 0. \tag{c}$$

Prove that if $n \geq 3$, then the $2n$ equations asserting that a matrix is a generalized magic square are linearly independent.

Hint. Equations are really linear forms, so find some matrix annihilated by all equations except equation r, and some matrix annihilated by all equations except equation c.

Problem B2 (120 pts). (Affine frames and affine maps) For any vector $u \in \mathbb{R}^n$, let $\widehat{u} \in \mathbb{R}^{n+1}$ be the vector defined by

$$\widehat{u}_i = \begin{cases} u_i & \text{if } 1 \leq i \leq n, \\ 1 & \text{if } i = n+1. \end{cases}$$

(1) For any $m+1$ vectors $(u_0, u_1, \ldots, u_m)$ with $u_i \in \mathbb{R}^n$ and $m \leq n$, prove that if the m vectors $(u_1 - u_0, \ldots, u_m - u_0)$ are linearly independent, then the $m+1$ vectors $(\widehat{u}_0, \ldots, \widehat{u}_m)$ are linearly independent.

(2) Prove that if the $m+1$ vectors $(\widehat{u}_0, \ldots, \widehat{u}_m)$ are linearly independent, then for any choice of i, with $0 \leq i \leq m$, the m vectors $u_j - u_i$ for $j \in \{0, \ldots, m\}$ with $j - i \neq 0$ are linearly independent.

Any $m+1$ vectors $(u_0, u_1, \ldots, u_m)$ such that the $m+1$ vectors $(\widehat{u}_0, \ldots, \widehat{u}_m)$ are linearly independent are said to be *affinely independent*.

From (1) and (2), the vector $(u_0, u_1, \ldots, u_m)$ are affinely independent iff for any any choice of i, with $0 \leq i \leq m$, the m vectors $u_j - u_i$ for $j \in \{0, \ldots, m\}$ with $j - i \neq 0$ are linearly independent. If $m = n$, we say that $n+1$ affinely independent vectors $(u_0, u_1, \ldots, u_n)$ form an *affine frame* of $\mathbb{R}^n$.

(3) if $(u_0, u_1, \ldots, u_n)$ is an affine frame of $\mathbb{R}^n$, then prove that for every vector $v \in \mathbb{R}^n$, there is a unique $(n+1)$-tuple $(\lambda_0, \lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{n+1}$, with $\lambda_0 + \lambda_1 + \cdots + \lambda_n = 1$, such that

$$v = \lambda_0 u_0 + \lambda_1 u_1 + \cdots + \lambda_n u_n.$$

The scalars $(\lambda_0, \lambda_1, \ldots, \lambda_n)$ are called the *barycentric* (or *affine*) *coordinates* of v w.r.t. the affine frame $(u_0, u_1, \ldots, u_n)$.

If we write $e_i = u_i - u_0$, for $i = 1, \ldots, n$, then prove that we have

$$v = u_0 + \lambda_1 e_1 + \cdots + \lambda_n e_n,$$

and since $(e_1, \ldots, e_n)$ is a basis of $\mathbb{R}^n$ (by (1) & (2)), the n-tuple $(\lambda_1, \ldots, \lambda_n)$ consists of the standard coordinates of $v - u_0$ over the basis $(e_1, \ldots, e_n)$.

Conversely, for any vector $u_0 \in \mathbb{R}^n$ and for any basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$, let $u_i = u_0 + e_i$ for $i = 1, \ldots, n$. Prove that $(u_0, u_1, \ldots, u_n)$ is an affine frame of $\mathbb{R}^n$, and for any $v \in \mathbb{R}^n$, if

$$v = u_0 + x_1 e_1 + \cdots + x_n e_n,$$

with $(x_1, \ldots, x_n) \in \mathbb{R}^n$ (unique), then

$$v = (1 - (x_1 + \cdots + x_x))u_0 + x_1 u_1 + \cdots + x_n u_n,$$

so that $(1 - (x_1 + \cdots + x_x)), x_1, \cdots, x_n)$, are the barycentric coordinates of v w.r.t. the affine frame $(u_0, u_1, \ldots, u_n)$.

The above shows that there is a one-to-one correspondence between affine frames $(u_0, \ldots, u_n)$ and pairs $(u_0, (e_1, \ldots, e_n))$, with $(e_1, \ldots, e_n)$ a basis. Given an affine frame $(u_0, \ldots, u_n)$, we obtain the basis $(e_1, \ldots, e_n)$ with $e_i = u_i - u_0$, for $i = 1, \ldots, n$; given the pair $(u_0, (e_1, \ldots, e_n))$ where $(e_1, \ldots, e_n)$ is a basis, we obtain the affine frame $(u_0, \ldots, u_n)$, with $u_i = u_0 + e_i$, for $i = 1, \ldots, n$. There is also a one-to-one correspondence between barycentric coordinates w.r.t. the affine frame $(u_0, \ldots, u_n)$ and standard coordinates w.r.t. the basis $(e_1, \ldots, e_n)$. The barycentric cordinates $(\lambda_0, \lambda_1, \ldots, \lambda_n)$ (with $\lambda_0 + \lambda_1 + \cdots + \lambda_n = 1$) yield the standard coordinates $(\lambda_1, \ldots, \lambda_n)$; the standard coordinates $(x_1, \ldots, x_n)$ yield the barycentric coordinates $(1 - (x_1 + \cdots + x_n), x_1, \ldots, x_n)$.

(4) Let $(u_0, \ldots, u_n)$ be any affine frame in $\mathbb{R}^n$ and let $(v_0, \ldots, v_n)$ be any vectors in $\mathbb{R}^m$. Prove that there is a *unique* affine map $f \colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$f(u_i) = v_i, \quad i = 0, \ldots, n.$$

(5) Let $(a_0, \ldots, a_n)$ be any affine frame in $\mathbb{R}^n$ and let $(b_0, \ldots, b_n)$ be any $n + 1$ points in $\mathbb{R}^n$. Prove that the $(n + 1) \times (n + 1)$ matrix A corresponding to the unique affine map f such that

$$f(a_i) = b_i, \quad i = 0, \ldots, n,$$

is given by

$$A = \left(\widehat{b}_0 \quad \widehat{b}_1 \quad \cdots \quad \widehat{b}_n\right) \left(\widehat{a}_0 \quad \widehat{a}_1 \quad \cdots \quad \widehat{a}_n\right)^{-1}.$$

In the special case where $(a_0, \ldots, a_n)$ is the canonical affine frame with $a_i = e_{i+1}$ for $i = 0, \ldots, n-1$ and $a_n = (0, \ldots, 0)$ (where e_i is the ith canonical basis vector), show that

$$\left(\widehat{a}_0 \quad \widehat{a}_1 \quad \cdots \quad \widehat{a}_n\right) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

and

$$\left(\widehat{a}_0 \quad \widehat{a}_1 \quad \cdots \quad \widehat{a}_n\right)^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ -1 & -1 & \cdots & -1 & 1 \end{pmatrix}.$$

For example, when $n = 2$, if we write $a_i = (x_i, y_i)$, then we have

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} x_1 - x_3 & x_2 - x_3 & x_3 \\ y_1 - y_3 & y_2 - y_3 & y_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

(6) Recall that a nonempty affine subspace $\mathcal{A}$ of $\mathbb{R}^n$ is any nonempty subset of $\mathbb{R}^n$ closed under affine combinations. For any affine map $f\colon \mathbb{R}^n \to \mathbb{R}^m$, for any affine subspace U of $\mathbb{R}^n$, and any affine subspace V of $\mathbb{R}^m$, prove that $f(U)$ is an affine subspace of $\mathbb{R}^m$, and that $f^{-1}(V)$ is an affine subspace of $\mathbb{R}^n$.

Problem B3 (40 pts). Give an example of a norm on $\mathbb{C}^n$ and of a *real* matrix A such that

$$\|A\|_{\mathbb{R}} < \|A\|,$$

where $\|-\|_{\mathbb{R}}$ and $\|-\|$ are the operator norms associated with the vector norm $\|-\|$, as defined in the notes.

Hint. This can already be done for $n = 2$.

Problem B4 (40 pts). Let A be an $n \times n$ matrix which is strictly row diagonally dominant, which means that

$$|a_{i\,i}| > \sum_{j \neq i} |a_{i\,j}|,$$

for $i = 1, \ldots, n$, and let

$$\delta = \min_i \left\{ |a_{i\,i}| - \sum_{j \neq i} |a_{i\,j}| \right\}.$$

The fact that A is strictly row diagonally dominant is equivalent to the condition $\delta > 0$.

(1) For any nonzero vector v, prove that

$$\|Av\|_\infty \geq \|v\|_\infty \delta.$$

Use the above to prove that A is invertible.

(2) Prove that

$$\left\|A^{-1}\right\|_\infty \leq \delta^{-1}.$$

Hint. Prove that

$$\sup_{v \neq 0} \frac{\|A^{-1}v\|_\infty}{\|v\|_\infty} = \sup_{w \neq 0} \frac{\|w\|_\infty}{\|Aw\|_\infty}.$$

Problem B5 (80 pts). Let A be a real 2×2 matrix

$$A = \begin{pmatrix} a_{1\,1} & a_{1\,2} \\ a_{2\,1} & a_{2\,2} \end{pmatrix}.$$

(1) Prove that the squares of the singular values $\sigma_1 \geq \sigma_2$ of A are the roots of the quadratic equation

$$X^2 - \mathrm{tr}(A^\top A)X + |\det(A)|^2 = 0.$$

(2) If we let

$$\mu(A) = \frac{a_{1\,1}^2 + a_{1\,2}^2 + a_{2\,1}^2 + a_{2\,2}^2}{2|a_{1\,1}a_{2\,2} - a_{1\,2}a_{2\,1}|},$$

prove that

$$\mathrm{cond}_2(A) = \frac{\sigma_1}{\sigma_2} = \mu(A) + (\mu(A)^2 - 1)^{1/2}.$$

(3) Consider the subset $\mathcal{S}$ of 2×2 invertible matrices whose entries $a_{i\,j}$ are integers such that $0 \leq a_{ij} \leq 100$.

Prove that the functions $\mathrm{cond}_2(A)$ and $\mu(A)$ reach a maximum on the set $\mathcal{S}$ for the same values of A.

Check that for the matrix

$$A_m = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix}$$

we have

$$\mu(A_m) = 19{,}603 \quad \det(A_m) = -1$$

and

$$\mathrm{cond}_2(A_m) \approx 39{,}206.$$

(4) Prove that for all $A \in \mathcal{S}$, if $|\det(A)| \geq 2$ then $\mu(A) \leq 10,000$. Conclude that the maximum of $\mu(A)$ on $\mathcal{S}$ is achieved for matrices such that $\det(A) = \pm 1$. Prove that finding matrices that maximize μ on $\mathcal{S}$ is equivalent to finding some integers n_1, n_2, n_3, n_4 such that

$$\begin{aligned}
&0 \leq n_4 \leq n_3 \leq n_2 \leq n_1 \leq 100 \\
&n_1^2 + n_2^2 + n_3^2 + n_4^2 \geq 100^2 + 99^2 + 99^2 + 98^2 = 39,206 \\
&|n_1 n_4 - n_2 n_3| = 1.
\end{aligned}$$

You may use without proof that the fact that the only solution to the above constraints is the multiset

$$\{100, 99, 99, 98\}.$$

(5) Deduce from part (4) that the matrices in $\mathcal{S}$ for which μ has a maximum value are

$$A_m = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix} \quad \begin{pmatrix} 98 & 99 \\ 99 & 100 \end{pmatrix} \quad \begin{pmatrix} 99 & 100 \\ 98 & 99 \end{pmatrix} \quad \begin{pmatrix} 99 & 98 \\ 100 & 99 \end{pmatrix}$$

and check that μ has the same value for these matrices. Conclude that

$$\max_{A \in \mathcal{S}} \mathrm{cond}_2(A) = \mathrm{cond}_2(A_m).$$

(6) Solve the system

$$\begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 199 \\ 197 \end{pmatrix}.$$

Perturb the right-hand side b by

$$\delta b = \begin{pmatrix} -0.0097 \\ 0.0106 \end{pmatrix}$$

and solve the new system

$$A_m y = b + \delta b$$

where $y = (y_1, y_2)$. Check that

$$\delta x = y - x = \begin{pmatrix} 2 \\ -2.0203 \end{pmatrix}.$$

Compute $\|x\|_2$, $\|\delta x\|_2$, $\|b\|_2$, $\|\delta b\|_2$, and estimate

$$c = \frac{\|\delta x\|_2}{\|x\|_2} \left(\frac{\|\delta b\|_2}{\|b\|_2} \right)^{-1}.$$

Ckeck that

$$c \approx \mathrm{cond}_2(A_m) = 39,206.$$

Problem B6 (20 pts). Let E be a real vector space of finite dimension, $n \geq 1$. Say that two bases, $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, of E have the *same orientation* iff $\det(P) > 0$, where P the change of basis matrix from $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, namely, the matrix whose jth columns consist of the coordinates of v_j over the basis $(u_1, \ldots, u_n)$.

(a) Prove that having the same orientation is an equivalence relation with two equivalence classes.

An *orientation* of a vector space, E, is the choice of any fixed basis, say $(e_1, \ldots, e_n)$, of E. Any other basis, $(v_1, \ldots, v_n)$, has the *same orientation* as $(e_1, \ldots, e_n)$ (and is said to be *positive* or *direct*) iff $\det(P) > 0$, else it is said to have the *opposite orientation* of $(e_1, \ldots, e_n)$ (or to be *negative* or *indirect*), where P is the change of basis matrix from $(e_1, \ldots, e_n)$ to $(v_1, \ldots, v_n)$. An *oriented* vector space is a vector space with some chosen orientation (a positive basis).

(b) Let $B_1 = (u_1, \ldots, u_n)$ and $B_2 = (v_1, \ldots, v_n)$ be two orthonormal bases. For any sequence of vectors, $(w_1, \ldots, w_n)$, in E, let $\det_{B_1}(w_1, \ldots, w_n)$ be the determinant of the matrix whose columns are the coordinates of the w_j's over the basis B_1 and similarly for $\det_{B_2}(w_1, \ldots, w_n)$.

Prove that if B_1 and B_2 have the same orientation, then

$$\det{}_{B_1}(w_1, \ldots, w_n) = \det{}_{B_2}(w_1, \ldots, w_n).$$

Given any oriented vector space, E, for any sequence of vectors, $(w_1, \ldots, w_n)$, in E, the common value, $\det_B(w_1, \ldots, w_n)$, for all positive orthonormal bases, B, of E is denoted

$$\lambda_E(w_1, \ldots, w_n)$$

and called a *volume form* of $(w_1, \ldots, w_n)$.

(c) Given any Euclidean oriented vector space, E, of dimension n for any $n-1$ vectors, $w_1, \ldots, w_{n-1}$, in E, check that the map

$$x \mapsto \lambda_E(w_1, \ldots, w_{n-1}, x)$$

is a linear form. Then, prove that there is a unique vector, denoted $w_1 \times \cdots \times w_{n-1}$, such that

$$\lambda_E(w_1, \ldots, w_{n-1}, x) = (w_1 \times \cdots \times w_{n-1}) \cdot x,$$

for all $x \in E$. The vector $w_1 \times \cdots \times w_{n-1}$ is called the *cross-product* of $(w_1, \ldots, w_{n-1})$. It is a generalization of the cross-product in $\mathbb{R}^3$ (when $n = 3$).

Problem B7 (40 pts). Given p vectors $(u_1, \ldots, u_p)$ in a Euclidean space E of dimension $n \geq p$, the *Gram determinant (or Gramian)* of the vectors $(u_1, \ldots, u_p)$ is the determinant

$$\mathrm{Gram}(u_1, \ldots, u_p) = \begin{vmatrix} \|u_1\|^2 & \langle u_1, u_2\rangle & \ldots & \langle u_1, u_p\rangle \\ \langle u_2, u_1\rangle & \|u_2\|^2 & \ldots & \langle u_2, u_p\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_p, u_1\rangle & \langle u_p, u_2\rangle & \ldots & \|u_p\|^2 \end{vmatrix}.$$

(1) Prove that

$$\mathrm{Gram}(u_1, \ldots, u_n) = \lambda_E(u_1, \ldots, u_n)^2.$$

Hint. If $(e_1, \ldots, e_n)$ is an orthonormal basis and A is the matrix of the vectors $(u_1, \ldots, u_n)$ over this basis,

$$\det(A)^2 = \det(A^\top A) = \det(A^i \cdot A^j),$$

where A^i denotes the ith column of the matrix A, and $(A^i \cdot A^j)$ denotes the $n \times n$ matrix with entries $A^i \cdot A^j$.

(2) Prove that

$$\|u_1 \times \cdots \times u_{n-1}\|^2 = \mathrm{Gram}(u_1, \ldots, u_{n-1}).$$

Hint. Letting $w = u_1 \times \cdots \times u_{n-1}$, observe that

$$\lambda_E(u_1, \ldots, u_{n-1}, w) = \langle w, w \rangle = \|w\|^2,$$

and show that

$$\begin{aligned} \|w\|^4 &= \lambda_E(u_1, \ldots, u_{n-1}, w)^2 = \mathrm{Gram}(u_1, \ldots, u_{n-1}, w) \\ &= \mathrm{Gram}(u_1, \ldots, u_{n-1})\|w\|^2. \end{aligned}$$

TOTAL: 400 points.