# Spring, 2012 CIS 515

# Fundamentals of Linear Algebra and Optimization
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## Homework 5 & 6 + Project 3 & 4

## Note: Problems B2 and B6 are for extra credit

April 17, 2012; Due May 7, 2012

**Problem B1 (20 pts).** Let $A$ be any real or complex $n \times n$ matrix and let $\| \, \|$ be any operator norm.

Prove that for every $m \geq 1$,

$$\|I\| + \sum_{k=1}^{m} \left\| \frac{A^k}{k!} \right\| \leq e^{\|A\|}.$$

Deduce from the above that the sequence $(E_m)$ of matrices

$$E_m = I + \sum_{k=1}^{m} \frac{A^k}{k!}$$

converges to a limit denoted $e^A$, and called the *exponential* of $A$.

**Problem B2 (Extra Credit 60 pts).** Recall that the affine maps $\rho\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined such that

$$\rho \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where $\theta, w_1, w_2 \in \mathbb{R}$, are *rigid motions* (or *direct affine isometries*) and that they form the group $\mathbf{SE}(2)$.

Given any map $\rho$ as above, if we let

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and} \quad W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

then $\rho$ can be represented by the $3 \times 3$ matrix

$$A = \begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & w_1 \\ \sin\theta & \cos\theta & w_2 \\ 0 & 0 & 1 \end{pmatrix}$$

in the sense that

$$\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

iff

$$\rho(X) = RX + W.$$

(a) Consider the set of matrices of the form

$$\begin{pmatrix} 0 & -\theta & u \\ \theta & 0 & v \\ 0 & 0 & 0 \end{pmatrix}$$

where $\theta, u, v \in \mathbb{R}$. Verify that this set of matrices is a vector space isomorphic to $(\mathbb{R}^3, +)$. This vector space is denoted by $\mathfrak{se}(2)$. Show that in general, $BC \neq CB$, if $B, C \in \mathfrak{se}(2)$.

(b) Given a matrix

$$B = \begin{pmatrix} 0 & -\theta & u \\ \theta & 0 & v \\ 0 & 0 & 0 \end{pmatrix},$$

prove that if $\theta = 0$, then

$$e^B = \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix},$$

and that if $\theta \neq 0$, then

$$e^B = \begin{pmatrix} \cos\theta & -\sin\theta & \frac{u}{\theta}\sin\theta + \frac{v}{\theta}(\cos\theta - 1) \\ \sin\theta & \cos\theta & \frac{u}{\theta}(-\cos\theta + 1) + \frac{v}{\theta}\sin\theta \\ 0 & 0 & 1 \end{pmatrix}.$$

*Hint*. Prove that

$$B^3 = -\theta^2 B,$$

and that

$$e^B = I_3 + \frac{\sin\theta}{\theta} B + \frac{1 - \cos\theta}{\theta^2} B^2.$$

(c) Check that $e^B$ is a direct affine isometry in $\mathbf{SE}(2)$. Prove that the exponential map $\exp\colon \mathfrak{se}(2) \to \mathbf{SE}(2)$ is surjective. If $\theta \neq k2\pi$ ($k \in \mathbb{Z}$), how do you need to restrict $\theta$ to get an injective map?

**Problem B3 (100 + 200 pts).** Consider the affine maps $\rho\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined such that

$$\rho \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where $\theta, w_1, w_2, \alpha \in \mathbb{R}$, with $\alpha > 0$. These maps are called (direct) *affine similitudes* (for short, *similitudes*). The number $\alpha > 0$ is the *scale factor* of the similitude. These affine maps are the composition of a rotation of angle $\theta$, a rescaling by $\alpha > 0$, and a translation.

(a) Prove that these maps form a group that we denote by $\mathbf{SIM}(2)$.

Given any map $\rho$ as above, if we let

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and} \quad W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

then $\rho$ can be represented by the $3 \times 3$ matrix

$$A = \begin{pmatrix} \alpha R & W \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha\cos\theta & -\alpha\sin\theta & w_1 \\ \alpha\sin\theta & \alpha\cos\theta & w_2 \\ 0 & 0 & 1 \end{pmatrix}$$

in the sense that

$$\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

iff

$$\rho(X) = \alpha R X + W.$$

(b) Consider the set of matrices of the form

$$\begin{pmatrix} \lambda & -\theta & u \\ \theta & \lambda & v \\ 0 & 0 & 0 \end{pmatrix}$$

where $\theta, \lambda, u, v \in \mathbb{R}$. Verify that this set of matrices is a vector space isomorphic to $(\mathbb{R}^4, +)$. This vector space is denoted by $\mathfrak{sim}(2)$.

(c) Given a matrix

$$\Omega = \begin{pmatrix} \lambda & -\theta \\ \theta & \lambda \end{pmatrix},$$

prove that

$$e^{\Omega} = e^{\lambda} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

*Hint*. Write

$$\Omega = \lambda I + \theta J,$$

with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Observe that $J^2 = -I$, and prove by induction on $k$ that

$$\Omega^k = \frac{1}{2}\left((\lambda + i\theta)^k + (\lambda - i\theta)^k\right) I + \frac{1}{2i}\left((\lambda + i\theta)^k - (\lambda - i\theta)^k\right) J.$$

(d) As in (c), write

$$\Omega = \begin{pmatrix} \lambda & -\theta \\ \theta & \lambda \end{pmatrix},$$

let

$$U = \begin{pmatrix} u \\ v \end{pmatrix},$$

and let

$$B = \begin{pmatrix} \Omega & U \\ 0 & 0 \end{pmatrix}.$$

Prove that

$$B^n = \begin{pmatrix} \Omega^n & \Omega^{n-1}U \\ 0 & 0 \end{pmatrix}$$

where $\Omega^0 = I_2$.

Prove that

$$e^B = \begin{pmatrix} e^\Omega & VU \\ 0 & 1 \end{pmatrix},$$

where

$$V = I_2 + \sum_{k \geq 1} \frac{\Omega^k}{(k+1)!}.$$

(e) Prove that

$$V = I_2 + \sum_{k \geq 1} \frac{\Omega^k}{(k+1)!} = \int_0^1 e^{\Omega t} dt.$$

Use this formula to prove that if $\lambda = \theta = 0$, then

$$V = I_2,$$

else

$$V = \frac{1}{\lambda^2 + \theta^2} \begin{pmatrix} \lambda(e^\lambda \cos\theta - 1) + e^\lambda \theta \sin\theta & -\theta(1 - e^\lambda \cos\theta) - e^\lambda \lambda \sin\theta \\ \theta(1 - e^\lambda \cos\theta) + e^\lambda \lambda \sin\theta & \lambda(e^\lambda \cos\theta - 1) + e^\lambda \theta \sin\theta \end{pmatrix}.$$

Observe that for $\lambda = 0$, the above gives back the expression in B2(b) for $\theta \neq 0$.

Conclude that if $\lambda = \theta = 0$, then

$$e^B = \begin{pmatrix} I & U \\ 0 & 1 \end{pmatrix},$$

else

$$e^B = \begin{pmatrix} e^\Omega & VU \\ 0 & 1 \end{pmatrix},$$

with

$$e^\Omega = e^\lambda \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and

$$V = \frac{1}{\lambda^2+\theta^2}\begin{pmatrix} \lambda(e^\lambda\cos\theta - 1) + e^\lambda\theta\sin\theta & -\theta(1-e^\lambda\cos\theta) - e^\lambda\lambda\sin\theta \\ \theta(1-e^\lambda\cos\theta) + e^\lambda\lambda\sin\theta & \lambda(e^\lambda\cos\theta - 1) + e^\lambda\theta\sin\theta \end{pmatrix},$$

and that $e^B \in \mathbf{SIM}(2)$, with scale factor $e^\lambda$.

(f) Prove that the exponential map $\exp\colon \mathfrak{sim}(2) \to \mathbf{SIM}(2)$ is surjective.

(g) Similitudes can be used to describe certain deformations (or flows) of a deformable body $\mathcal{B}_t$ in the plane. Given some initial shape $\mathcal{B}$ in the plane (for example, a circle), a deformation of $\mathcal{B}$ is given by a piecewise differentiable curve

$$\mathcal{D}\colon [0,T] \to \mathbf{SIM}(2),$$

where each $\mathcal{D}(t)$ is a similitude (for some $T > 0$). The deformed body $\mathcal{B}_t$ at time $t$ is given by

$$\mathcal{B}_t = \mathcal{D}(t)(\mathcal{B}).$$

The surjectivity of the exponential map $\exp\colon \mathfrak{sim}(2) \to \mathbf{SIM}(2)$ implies that there is a map $\log\colon \mathbf{SIM}(2) \to \mathfrak{sim}(2)$, although it is multivalued. The exponential map and the log "function" allows us to work in the simpler (noncurved) Euclidean space $\mathfrak{sim}(2)$.

For instance, given two similitudes $A_1, A_2 \in \mathbf{SIM}(2)$ specifying the shape of $\mathcal{B}$ at two different times, we can compute $\log(A_1)$ and $\log(A_2)$, which are just elements of the Euclidean space $\mathfrak{sim}(2)$, form the linear interpolant $(1-t)\log(A_1) + t\log(A_2)$, and then apply the exponential map to get an interpolating deformation

$$t \mapsto e^{(1-t)\log(A_1) + t\log(A_2)}, \quad t \in [0,1].$$

Also, given a sequence of "snapshots" of the deformable body $\mathcal{B}$, say $A_0, A_1, \ldots, A_m$, where each is $A_i$ is a similitude, we can try to find an interpolating deformation (a curve in $\mathbf{SIM}(2)$) by finding a simpler curve $t \mapsto C(t)$ in $\mathfrak{sim}(2)$ (say, a $B$-spline) interpolating $\log A_1, \log A_1, \ldots, \log A_m$. Then, the curve $t \mapsto e^{C(t)}$ yields a deformation in $\mathbf{SIM}(2)$ interpolating $A_0, A_1, \ldots, A_m$.

(1) **(75 pts).** Write a program interpolating between two deformations.

(2) **(125 pts).** Write a program using your cubic spline interpolation program from the first project, to interpolate a sequence of deformations given by similitudes $A_0, A_1, \ldots, A_m$.

**Problem B4 (40 pts).** Recall that the coordinates of the cross product $u \times v$ of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ are

$$(u_2v_3 - u_3v_2,\; u_3v_1 - u_1v_3,\; u_1v_2 - u_2v_1).$$

(a) If we let $U$ be the matrix

$$U = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix},$$

check that the coordinates of the cross product $u \times v$ are given by

$$\begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

(b) Show that the set of matrices of the form

$$U = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

is a vector space isomorphic to $(\mathbb{R}^3 +)$. This vector space is denoted by $\mathfrak{so}(3)$. Show that such matrices are never invertible. Find the kernel of the linear map associated with a matrix $U$. Describe geometrically the action of the linear map defined by a matrix $U$. Show that when restricted to the plane orthogonal to $u = (u_1, u_2, u_3)$, if $u$ is a unit vector, then $U$ behaves like a rotation by $\pi/2$.

**Problem B5 (100 pts).** (a) For any matrix

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

if we let $\theta = \sqrt{a^2 + b^2 + c^2}$ and

$$B = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix},$$

prove that

$$\begin{aligned} A^2 &= -\theta^2 I + B, \\ AB &= BA = 0. \end{aligned}$$

From the above, deduce that

$$A^3 = -\theta^2 A.$$

(b) Prove that the exponential map $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is given by

$$\exp A = e^A = \cos\theta\, I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} B,$$

or, equivalently, by

$$e^A = I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} A^2,$$

if $\theta \neq 0$, with $\exp(0_3) = I_3$.

(c) Prove that $e^A$ is an orthogonal matrix of determinant $+1$, i.e., a rotation matrix.

(d) Prove that the exponential map $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is surjective. For this, proceed as follows: Pick any rotation matrix $R \in \mathbf{SO}(3)$;

(1) The case $R = I$ is trivial.

(2) If $R \neq I$ and $\mathrm{tr}(R) \neq -1$, then

$$\exp^{-1}(R) = \left\{ \frac{\theta}{2\sin\theta}(R - R^T) \;\middle|\; 1 + 2\cos\theta = \mathrm{tr}(R) \right\}.$$

(Recall that $\mathrm{tr}(R) = r_{1\,1} + r_{2\,2} + r_{3\,3}$, the *trace* of the matrix $R$). Note that both $\theta$ and $2\pi - \theta$ yield the same matrix $\exp(R)$.

(3) If $R \neq I$ and $\mathrm{tr}(R) = -1$, then prove that the eigenvalues of $R$ are $1, -1, -1$, that $R = R^\top$, and that $R^2 = I$. Prove that the matrix

$$S = \frac{1}{2}(R - I)$$

is a symmetric matrix whose eigenvalues are $-1, -1, 0$. Thus, $S$ can be diagonalized with respect to an orthogonal matrix $Q$ as

$$S = Q \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^\top.$$

Prove that there exists a skew symmetric matrix

$$U = \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}$$

so that

$$U^2 = S = \frac{1}{2}(R - I).$$

Observe that

$$U^2 = \begin{pmatrix} -(c^2+d^2) & bc & bd \\ bc & -(b^2+d^2) & cd \\ bd & cd & -(b^2+c^2) \end{pmatrix}.$$

and use this to conclude that if $U^2 = S$, then $b^2 + c^2 + d^2 = 1$. Then, show that

$$\exp^{-1}(R) = \left\{ (2k+1)\pi \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}, \; k \in \mathbb{Z} \right\},$$

where $(b, c, d)$ is any unit vector such that for the corresponding skew symmetric matrix $U$, we have $U^2 = S$.

(e) To find a skew symmetric matrix $U$ so that $U^2 = S = \frac{1}{2}(R - I)$ as in (d), we can solve the system

$$\begin{pmatrix} b^2-1 & bc & bd \\ bc & c^2-1 & cd \\ bd & cd & d^2-1 \end{pmatrix} = S.$$

We immediately get $b^2, c^2, d^2$, and then, since one of $b, c, d$ is nonzero, say $b$, if we choose the positive square root of $b^2$, we can determine $c$ and $d$ from $bc$ and $bd$.

Implement a computer program to solve the above system.

**Problem B6 (Extra Credit 100 pts).** (a) Consider the set of affine maps $\rho$ of $\mathbb{R}^3$ defined such that

$$\rho(X) = RX + W,$$

where $R$ is a rotation matrix (an orthogonal matrix of determinant $+1$) and $W$ is some vector in $\mathbb{R}^3$. Every such a map can be represented by the $4 \times 4$ matrix

$$\begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix}$$

in the sense that

$$\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

iff

$$\rho(X) = RX + W.$$

Prove that these maps are affine bijections and that they form a group, denoted by $\mathbf{SE}(3)$ (the *direct affine isometries, or rigid motions*, of $\mathbb{R}^3$).

(b) Let us now consider the set of $4 \times 4$ matrices of the form

$$B = \begin{pmatrix} \Omega & W \\ 0 & 0 \end{pmatrix},$$

where $\Omega$ is a skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

and $W$ is a vector in $\mathbb{R}^3$.

Verify that this set of matrices is a vector space isomorphic to $(\mathbb{R}^6, +)$. This vector space is denoted by $\mathfrak{se}(3)$. Show that in general, $BC \neq CB$.

(c) Given a matrix

$$B = \begin{pmatrix} \Omega & W \\ 0 & 0 \end{pmatrix}$$

as in (b), prove that

$$B^n = \begin{pmatrix} \Omega^n & \Omega^{n-1}W \\ 0 & 0 \end{pmatrix}$$

where $\Omega^0 = I_3$. Given

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

let $\theta = \sqrt{a^2 + b^2 + c^2}$. Prove that if $\theta = 0$, then

$$e^B = \begin{pmatrix} I_3 & W \\ 0 & 1 \end{pmatrix},$$

and that if $\theta \neq 0$, then

$$e^B = \begin{pmatrix} e^\Omega & VW \\ 0 & 1 \end{pmatrix},$$

where

$$V = I_3 + \sum_{k \geq 1} \frac{\Omega^k}{(k+1)!}.$$

(d) Prove that

$$e^\Omega = I_3 + \frac{\sin\theta}{\theta}\Omega + \frac{(1 - \cos\theta)}{\theta^2}\Omega^2$$

and

$$V = I_3 + \frac{(1 - \cos\theta)}{\theta^2}\Omega + \frac{(\theta - \sin\theta)}{\theta^3}\Omega^2.$$

*Hint*. Use the fact that $\Omega^3 = -\theta^2\Omega$.

(e) Prove that $e^B$ is a direct affine isometry in $\mathbf{SE}(3)$. Prove that $V$ is invertible and that

$$Z = I - \frac{1}{2}\Omega + \frac{1}{\theta^2}\left(1 - \frac{\theta\sin\theta}{2(1-\cos\theta)}\right)\Omega^2,$$

for $\theta \neq 0$.

*Hint*. Assume that the inverse of $V$ is of the form

$$Z = I_3 + a\Omega + b\Omega^2,$$

and show that $a, b$, are given by a system of linear equations that always has a unique solution.

Prove that the exponential map $\exp\colon \mathfrak{se}(3) \to \mathbf{SE}(3)$ is surjective.

**Remark:** As in the case of the plane, rigid motions in $\mathbf{SE}(3)$ can be used to describe certain deformations of bodies in $\mathbb{R}^3$.

**Problem B7 (80 pts).** Let $A$ be a real $2 \times 2$ matrix

$$A = \begin{pmatrix} a_{1\,1} & a_{1\,2} \\ a_{2\,1} & a_{2\,2} \end{pmatrix}.$$

(1) Prove that the squares of the singular values $\sigma_1 \geq \sigma_2$ of $A$ are the roots of the quadratic equation

$$X^2 - \mathrm{tr}(A^\top A)X + |\det(A)|^2 = 0.$$

(2) If we let

$$\mu(A) = \frac{a_{1\,1}^2 + a_{1\,2}^2 + a_{2\,1}^2 + a_{2\,2}^2}{2|a_{1\,1}a_{2\,2} - a_{1\,2}a_{2\,1}|},$$

prove that

$$\mathrm{cond}_2(A) = \frac{\sigma_1}{\sigma_2} = \mu(A) + (\mu(A)^2 - 1)^{1/2}.$$

(3) Consider the subset $\mathcal{S}$ of $2 \times 2$ invertible matrices whose entries $a_{i\,j}$ are integers such that $0 \leq a_{ij} \leq 100$.

Prove that the functions $\mathrm{cond}_2(A)$ and $\mu(A)$ reach a maximum on the set $\mathcal{S}$ for the same values of $A$.

Check that for the matrix

$$A_m = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix}$$

we have

$$\mu(A_m) = 19,603 \quad \det(A_m) = -1$$

and

$$\mathrm{cond}_2(A_m) \approx 39,206.$$

(4) Prove that for all $A \in \mathcal{S}$, if $|\det(A)| \geq 2$ then $\mu(A) \leq 10,000$. Conclude that the maximum of $\mu(A)$ on $\mathcal{S}$ is achieved for matrices such that $\det(A) = \pm 1$. Prove that finding matrices that maximize $\mu$ on $\mathcal{S}$ is equivalent to finding some integers $n_1, n_2, n_3, n_4$ such that

$$\begin{aligned}
&0 \leq n_4 \leq n_3 \leq n_2 \leq n_1 \leq 100\\
&n_1^2 + n_2^2 + n_3^2 + n_4^2 \geq 100^2 + 99^2 + 99^2 + 98^2 = 39,206\\
&|n_1 n_4 - n_2 n_3| = 1.
\end{aligned}$$

You may use without proof that the fact that the only solution to the above constraints is the multiset

$$\{100, 99, 99, 98\}.$$

(5) Deduce from part (4) that the matrices in $\mathcal{S}$ for which $\mu$ has a maximum value are

$$A_m = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix} \quad \begin{pmatrix} 98 & 99 \\ 99 & 100 \end{pmatrix} \quad \begin{pmatrix} 99 & 100 \\ 98 & 99 \end{pmatrix} \quad \begin{pmatrix} 99 & 98 \\ 100 & 99 \end{pmatrix}$$

and check that $\mu$ has the same value for these matrices. Conclude that

$$\max_{A \in \mathcal{S}} \mathrm{cond}_2(A) = \mathrm{cond}_2(A_m).$$

(6) Solve the system

$$\begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 199 \\ 197 \end{pmatrix}.$$

Perturb the right-hand side $b$ by

$$\delta b = \begin{pmatrix} -0.0097 \\ 0.0106 \end{pmatrix}$$

and solve the new system

$$A_m y = b + \delta b$$

where $y = (y_1, y_2)$. Check that

$$\delta x = y - x = \begin{pmatrix} 2 \\ -2.0203 \end{pmatrix}.$$

Compute $\|x\|_2$, $\|\delta x\|_2$, $\|b\|_2$, $\|\delta b\|_2$, and estimate

$$c = \frac{\|\delta x\|_2}{\|x\|_2} \left( \frac{\|\delta b\|_2}{\|b\|_2} \right)^{-1}.$$

Ckeck that

$$c \approx \mathrm{cond}_2(A_m) = 39,206.$$

**Problem B8 (20 pts)**. Let $E$ be a real vector space of finite dimension, $n \geq 1$. Say that two bases, $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, of $E$ have the *same orientation* iff $\det(P) > 0$, where $P$ the change of basis matrix from $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, namely, the matrix whose $j$th columns consist of the coordinates of $v_j$ over the basis $(u_1, \ldots, u_n)$.

(a) Prove that having the same orientation is an equivalence relation with two equivalence classes.

An *orientation* of a vector space, $E$, is the choice of any fixed basis, say $(e_1, \ldots, e_n)$, of $E$. Any other basis, $(v_1, \ldots, v_n)$, has the *same orientation* as $(e_1, \ldots, e_n)$ (and is said to be *positive* or *direct*) iff $\det(P) > 0$, else it is said to have the *opposite orientation* of $(e_1, \ldots, e_n)$ (or to be *negative* or *indirect*), where $P$ is the change of basis matrix from $(e_1, \ldots, e_n)$ to $(v_1, \ldots, v_n)$. An *oriented* vector space is a vector space with some chosen orientation (a positive basis).

(b) Let $B_1 = (u_1, \ldots, u_n)$ and $B_2 = (v_1, \ldots, v_n)$ be two orthonormal bases. For any sequence of vectors, $(w_1, \ldots, w_n)$, in $E$, let $\det_{B_1}(w_1, \ldots, w_n)$ be the determinant of the matrix whose columns are the coordinates of the $w_j$'s over the basis $B_1$ and similarly for $\det_{B_2}(w_1, \ldots, w_n)$.

Prove that if $B_1$ and $B_2$ have the same orientation, then

$$\det_{B_1}(w_1, \ldots, w_n) = \det_{B_2}(w_1, \ldots, w_n).$$

Given any oriented vector space, $E$, for any sequence of vectors, $(w_1, \ldots, w_n)$, in $E$, the common value, $\det_B(w_1, \ldots, w_n)$, for all positive orthonormal bases, $B$, of $E$ is denoted

$$\lambda_E(w_1, \ldots, w_n)$$

and called a *volume form* of $(w_1, \ldots, w_n)$.

(c) Given any Euclidean oriented vector space, $E$, of dimension $n$ for any $n-1$ vectors, $w_1, \ldots, w_{n-1}$, in $E$, check that the map

$$x \mapsto \lambda_E(w_1, \ldots, w_{n-1}, x)$$

is a linear form. Then, prove that there is a unique vector, denoted $w_1 \times \cdots \times w_{n-1}$, such that

$$\lambda_E(w_1, \ldots, w_{n-1}, x) = (w_1 \times \cdots \times w_{n-1}) \cdot x,$$

for all $x \in E$. The vector $w_1 \times \cdots \times w_{n-1}$ is called the *cross-product* of $(w_1, \ldots, w_{n-1})$. It is a generalization of the cross-product in $\mathbb{R}^3$ (when $n = 3$).

**Problem B9 (40 pts).** Given $p$ vectors $(u_1, \ldots, u_p)$ in a Euclidean space $E$ of dimension $n \geq p$, the *Gram determinant (or Gramian)* of the vectors $(u_1, \ldots, u_p)$ is the determinant

$$\mathrm{Gram}(u_1, \ldots, u_p) = \begin{vmatrix} \|u_1\|^2 & \langle u_1, u_2\rangle & \ldots & \langle u_1, u_p\rangle \\ \langle u_2, u_1\rangle & \|u_2\|^2 & \ldots & \langle u_2, u_p\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_p, u_1\rangle & \langle u_p, u_2\rangle & \ldots & \|u_p\|^2 \end{vmatrix}.$$

(1) Prove that

$$\mathrm{Gram}(u_1, \ldots, u_n) = \lambda_E(u_1, \ldots, u_n)^2.$$

*Hint*. If $(e_1, \ldots, e_n)$ is an orthonormal basis and $A$ is the matrix of the vectors $(u_1, \ldots, u_n)$ over this basis,

$$\det(A)^2 = \det(A^\top A) = \det(A^i \cdot A^j),$$

where $A^i$ denotes the $i$th column of the matrix $A$, and $(A^i \cdot A^j)$ denotes the $n \times n$ matrix with entries $A^i \cdot A^j$.

(2) Prove that

$$\|u_1 \times \cdots \times u_{n-1}\|^2 = \mathrm{Gram}(u_1, \ldots, u_{n-1}).$$

*Hint*. Letting $w = u_1 \times \cdots \times u_{n-1}$, observe that

$$\lambda_E(u_1, \ldots, u_{n-1}, w) = \langle w, w\rangle = \|w\|^2,$$

and show that

$$\begin{aligned} \|w\|^4 &= \lambda_E(u_1, \ldots, u_{n-1}, w)^2 = \mathrm{Gram}(u_1, \ldots, u_{n-1}, w) \\ &= \mathrm{Gram}(u_1, \ldots, u_{n-1})\|w\|^2. \end{aligned}$$

**Problem B10 (60 pts).** (1) Implement the Gram-Schmidt orthonormalization procedure and the modified Gram-Schmidt procedure. You may use the pseudo-code showed in (2).

(2) Implement the method to compute the QR decomposition of an invertible matrix. You may use the following pseudo-code:

**function** $qr(A$: matrix): $[Q, R]$ pair of matrices
  **begin**
    $n = \dim(A)$;
    $R = 0$; (the zero matrix)
    $Q1(:, 1) = A(:, 1)$;
    $R(1, 1) = \mathrm{sqrt}(Q1(:, 1)^\top \cdot Q1(:, 1))$;
    $Q(:, 1) = Q1(:, 1)/R(1, 1)$;
    **for** $k := 1$ **to** $n - 1$ **do**

$w = A(:, k+1)$;
**for** $i := 1$ **to** $k$ **do**
$R(i, k+1) = A(:, k+1)^\top \cdot Q(:, i)$;
$w = w - R(i, k+1)Q(:, i)$;
**endfor**;
$Q1(:, k+1) = w$;
$R(k+1, k+1) = \mathrm{sqrt}(Q1(:, k+1)^\top \cdot Q1(:, k+1))$;
$Q(:, k+1) = Q1(:, k+1)/R(k+1, k+1)$;
**endfor**;
**end**

Test it on various matrices, including those involved in Project 1.

(3) Given any invertible matrix $A$, define the sequences $A_k$, $Q_k, R_k$ as follows:

$$\begin{aligned} A_1 &= A \\ Q_k R_k &= A_k \\ A_{k+1} &= R_k Q_k \end{aligned}$$

for all $k \geq 1$, where in the second equation, $Q_k R_k$ is the QR decomposition of $A_k$ given by part (2).

Run the above procedure for various values of $k$ (50, 100, ...) and various real matrices $A$, in particular some symmetric matrices; also run the `Matlab` command `eig` on $A_k$, and compare the diagonal entries of $A_k$ with the eigenvalues given by `eig`$(A_k)$.

What do you observe? How do you explain this behavior?

**TOTAL: 400 + 260 points + 160 extra.**