

# Fundamentals of Linear Algebra and Optimization

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## Homework 4

October 24, 2018; Due November 7, 2018, beginning of class

**Problem B1 (50 pts).** (1) Prove that the dimension of the subspace of  $2 \times 2$  matrices  $A$ , such that the sum of the entries of every row is the same (say  $c_1$ ) and the sum of entries of every column is the same (say  $c_2$ ) is 2.

(2) Prove that the dimension of the subspace of  $2 \times 2$  matrices  $A$ , such that the sum of the entries of every row is the same (say  $c_1$ ), the sum of entries of every column is the same (say  $c_2$ ), and  $c_1 = c_2$ , is also 2. Prove that every such matrix is of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

and give a basis for this subspace.

(3) Prove that the dimension of the subspace of  $3 \times 3$  matrices  $A$ , such that the sum of the entries of every row is the same (say  $c_1$ ), the sum of entries of every column is the same (say  $c_2$ ), and  $c_1 = c_2$ , is 5. Begin by showing that the above constraints are given by the set of equations

$$\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{31} \\ a_{32} \\ a_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Prove that every matrix satisfying the above constraints is of the form

$$\begin{pmatrix} a+b-c & -a+c+e & -b+c+d \\ -a-b+c+d+e & a & b \\ c & d & e \end{pmatrix},$$

with  $a, b, c, d, e \in \mathbb{R}$ . Find a basis for this subspace. (Use the method to find a basis for the kernel of a matrix).

**Problem B2 (10 pts).** If  $A$  is an  $n \times n$  symmetric matrix and  $B$  is any  $n \times n$  invertible matrix, prove that  $A$  is positive definite iff  $B^T A B$  is positive definite.

**Problem B3 (100 pts).** (1) Let  $A$  be any invertible  $2 \times 2$  matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Prove that there is an invertible matrix  $S$  such that

$$SA = \begin{pmatrix} 1 & 0 \\ 0 & ad - bc \end{pmatrix},$$

where  $S$  is the product of at most four elementary matrices of the form  $E_{i,j;\beta}$ .

Conclude that every matrix  $A$  in  $\mathbf{SL}(2)$  (the group of invertible  $2 \times 2$  matrices  $A$  with  $\det(A) = +1$ ) is the product of at most four elementary matrices of the form  $E_{i,j;\beta}$ .

For any  $a \neq 0, 1$ , give an explicit factorization as above for

$$A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

What is this decomposition for  $a = -1$ ?

(2) Recall that a rotation matrix  $R$  (a member of the group  $\mathbf{SO}(2)$ ) is a matrix of the form

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Prove that if  $\theta \neq k\pi$  (with  $k \in \mathbb{Z}$ ), any rotation matrix can be written as a product

$$R = ULU,$$

where  $U$  is upper triangular and  $L$  is lower triangular of the form

$$U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}.$$

Therefore, every plane rotation (except a flip about the origin when  $\theta = \pi$ ) can be written as the composition of three shear transformations!

(3) Recall that  $E_{i,d}$  is the diagonal matrix

$$E_{i,d} = \text{diag}(1, \dots, 1, d, 1, \dots, 1),$$

whose diagonal entries are all  $+1$ , except the  $(i, i)$ th entry which is equal to  $d$ .

Given any  $n \times n$  matrix  $A$ , for any pair  $(i, j)$  of distinct row indices ( $1 \leq i, j \leq n$ ), prove that there exist two elementary matrices  $E_1(i, j)$  and  $E_2(i, j)$  of the form  $E_{k,\ell;\beta}$ , such that

$$E_{j,-1}E_1(i, j)E_2(i, j)E_1(i, j)A = P(i, j)A,$$

the matrix obtained from the matrix  $A$  by permuting row  $i$  and row  $j$ . Equivalently, we have

$$E_1(i, j)E_2(i, j)E_1(i, j)A = E_{j,-1}P(i, j)A,$$

the matrix obtained from  $A$  by permuting row  $i$  and row  $j$  and multiplying row  $j$  by  $-1$ .

Prove that for every  $i = 2, \dots, n$ , there exist four elementary matrices  $E_3(i, d), E_4(i, d), E_5(i, d), E_6(i, d)$  of the form  $E_{k,\ell;\beta}$ , such that

$$E_6(i, d)E_5(i, d)E_4(i, d)E_3(i, d)E_{n,d} = E_{i,d}.$$

What happens when  $d = -1$ , that is, what kind of simplifications occur?

Prove that all permutation matrices can be written as products of elementary operations of the form  $E_{k,\ell;\beta}$  and the operation  $E_{n,-1}$ .

(4) Prove that for every invertible  $n \times n$  matrix  $A$ , there is a matrix  $S$  such that

$$SA = \begin{pmatrix} I_{n-1} & 0 \\ 0 & d \end{pmatrix} = E_{n,d},$$

with  $d = \det(A)$ , and where  $S$  is a product of elementary matrices of the form  $E_{k,\ell;\beta}$ .

In particular, every matrix in  $\mathbf{SL}(n)$  (the group of invertible  $n \times n$  matrices  $A$  with  $\det(A) = +1$ ) can be written as a product of elementary matrices of the form  $E_{k,\ell;\beta}$ . Prove that at most  $n(n+1) - 2$  such transformations are needed.

**Extra Credit (20 points).** Prove that every matrix in  $\mathbf{SL}(n)$  can be written as a product of at most  $(n-1)(\max\{n, 3\} + 1)$  elementary matrices of the form  $E_{k,\ell;\beta}$ .

**Problem B4 (50 pts).** A matrix,  $A$ , is called *strictly column diagonally dominant* iff

$$|a_{jj}| > \sum_{i=1, i \neq j}^n |a_{ij}|, \quad \text{for } j = 1, \dots, n$$

Prove that if  $A$  is strictly column diagonally dominant, then Gaussian elimination with partial pivoting does not require pivoting, and  $A$  is invertible.

**Problem B5 (40 pts).** Let  $(\alpha_1, \dots, \alpha_{m+1})$  be a sequence of pairwise distinct scalars in  $\mathbb{R}$  and let  $(\beta_1, \dots, \beta_{m+1})$  be any sequence of scalars in  $\mathbb{R}$ , not necessarily distinct.

(1) Prove that there is a unique polynomial  $P$  of degree at most  $m$  such that

$$P(\alpha_i) = \beta_i, \quad 1 \leq i \leq m+1.$$

*Hint.* Remember Vandermonde!

(2) Let  $L_i(X)$  be the polynomial of degree  $m$  given by

$$L_i(X) = \frac{(X - \alpha_1) \cdots (X - \alpha_{i-1})(X - \alpha_{i+1}) \cdots (X - \alpha_{m+1})}{(\alpha_i - \alpha_1) \cdots (\alpha_i - \alpha_{i-1})(\alpha_i - \alpha_{i+1}) \cdots (\alpha_i - \alpha_{m+1})}, \quad 1 \leq i \leq m + 1.$$

The polynomials  $L_i(X)$  are known as *Lagrange polynomial interpolants*. Prove that

$$L_i(\alpha_j) = \delta_{ij} \quad 1 \leq i, j \leq m + 1.$$

Prove that

$$P(X) = \beta_1 L_1(X) + \cdots + \beta_{m+1} L_{m+1}(X)$$

is the unique polynomial of degree at most  $m$  such that

$$P(\alpha_i) = \beta_i, \quad 1 \leq i \leq m + 1.$$

(3) Prove that  $L_1(X), \dots, L_{m+1}(X)$  are linearly independent, and that they form a basis of all polynomials of degree at most  $m$ .

How is 1 (the constant polynomial 1) expressed over the basis  $(L_1(X), \dots, L_{m+1}(X))$ ?

Give the expression of every polynomial  $P(X)$  of degree at most  $m$  over the basis  $(L_1(X), \dots, L_{m+1}(X))$ .

(4) Prove that the dual basis  $(L_1^*, \dots, L_{m+1}^*)$  of the basis  $(L_1(X), \dots, L_{m+1}(X))$  consists of the linear forms  $L_i^*$  given by

$$L_i^*(P) = P(\alpha_i),$$

for every polynomial  $P$  of degree at most  $m$ ; this is simply *evaluation at  $\alpha_i$* .

**Problem B6 (60 pts).** (a) Find a lower triangular matrix  $E$  such that

$$E \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

(b) What is the effect of the product (on the left) with

$$E_{4,3;-1} E_{3,2;-1} E_{4,3;-1} E_{2,1;-1} E_{3,2;-1} E_{4,3;-1}$$

on the matrix

$$Pa_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}.$$

(c) Find the inverse of the matrix  $Pa_3$ .

(d) Consider the  $(n + 1) \times (n + 1)$  Pascal matrix  $Pa_n$  whose  $i$ th row is given by the binomial coefficients

$$\binom{i-1}{j-1},$$

with  $1 \leq i \leq n + 1$ ,  $1 \leq j \leq n + 1$ , and with the usual convention that

$$\binom{0}{0} = 1, \quad \binom{i}{j} = 0 \quad \text{if } j > i.$$

The matrix  $Pa_3$  is shown in question (c) and  $Pa_4$  is shown below:

$$Pa_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}.$$

Find  $n$  elementary matrices  $E_{i_k, j_k; \beta_k}$  such that

$$E_{i_n, j_n; \beta_n} \cdots E_{i_1, j_1; \beta_1} Pa_n = \begin{pmatrix} 1 & 0 \\ 0 & Pa_{n-1} \end{pmatrix}.$$

Use the above to prove that the inverse of  $Pa_n$  is the lower triangular matrix whose  $i$ th row is given by the signed binomial coefficients

$$(-1)^{i+j-2} \binom{i-1}{j-1},$$

with  $1 \leq i \leq n + 1$ ,  $1 \leq j \leq n + 1$ . For example,

$$Pa_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

*Hint.* Given any  $n \times n$  matrix  $A$ , multiplying  $A$  by the elementary matrix  $E_{i, j; \beta}$  on the right yields the matrix  $AE_{i, j; \beta}$  in which  $\beta$  times the  $i$ th column is added to the  $j$ th column.

**Problem B7 (30 pts).** Consider the  $n \times n$  symmetric matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & \dots & 0 & 0 \\ 2 & 5 & 2 & 0 & \dots & 0 & 0 \\ 0 & 2 & 5 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2 & 5 & 2 & 0 \\ 0 & 0 & \dots & 0 & 2 & 5 & 2 \\ 0 & 0 & \dots & 0 & 0 & 2 & 5 \end{pmatrix}.$$

- (1) Find an upper-triangular matrix  $R$  such that  $A = R^T R$ .
- (2) Prove that  $\det(A) = 1$ .
- (3) Consider the sequence

$$\begin{aligned} p_0(\lambda) &= 1 \\ p_1(\lambda) &= 1 - \lambda \\ p_k(\lambda) &= (5 - \lambda)p_{k-1}(\lambda) - 4p_{k-2}(\lambda) \quad 2 \leq k \leq n. \end{aligned}$$

Prove that

$$\det(A - \lambda I) = p_n(\lambda).$$

**Remark:** It can be shown that  $p_n(\lambda)$  has  $n$  distinct (real) roots and that the roots of  $p_k(\lambda)$  separate the roots of  $p_{k+1}(\lambda)$ .

**TOTAL: 340 + 20 points.**