$$\boxed{\textbf{Spring, 2012} \quad \textbf{CIS 515}}$$

Fundamentals of Linear Algebra and Optimization
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Homework 4 + Project 2

March 22, 2012; Due April 2, 2012

Problem B1 (40 pts). Let A be an $n \times n$ matrix which is strictly row diagonally dominant, which means that

$$|a_{i\,i}| > \sum_{j \neq i} |a_{i\,j}|,$$

for $i = 1, \ldots, n$, and let

$$\delta = \min_i \left\{ |a_{i\,i}| - \sum_{j \neq i} |a_{i\,j}| \right\}.$$

The fact that A is is strictly row diagonally dominant is equivalent to the condition $\delta > 0$.

(1) For any nonzero vector v, prove that

$$\|Av\|_\infty \geq \|v\|_\infty \, \delta.$$

Use the above to prove that A is invertible.

(2) Prove that

$$\left\|A^{-1}\right\|_\infty \leq \delta^{-1}.$$

Hint. Prove that

$$\sup_{v \neq 0} \frac{\|A^{-1}v\|_\infty}{\|v\|_\infty} = \sup_{w \neq 0} \frac{\|w\|_\infty}{\|Aw\|_\infty}.$$

Problem B2 (20 pts). Let A be any invertible complex $n \times n$ matrix.

(1) For any vector norm $\| \;\|$ on $\mathbb{C}^n$, prove that the function $\| \;\|_A : \mathbb{C}^n \to \mathbb{R}$ given by

$$\|x\|_A = \|Ax\| \quad \text{for all} \quad x \in \mathbb{C}^n,$$

is a vector norm.

(2) Prove that the operator norm induced by $\| \;\|_A$, also denoted by $\| \;\|_A$, is given by

$$\|B\|_A = \left\|ABA^{-1}\right\| \quad \text{for every } n \times n \text{ matrix} \quad B,$$

where $\|ABA^{-1}\|$ uses the operator norm induced by $\| \;\|$.

Problem B3 (80 pts). (1) Implement the method for converting a rectangular matrix to reduced row echelon fom.

(2) Use the above method to find the inverse of an invertible $n \times n$ matrix A, by applying it to the the $n \times 2n$ matrix $[A\, I]$ obtained by adding the n columns of the identity matrix to A.

(3) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 3 & 4 & 5 & \cdots & n+1 \\ 3 & 4 & 5 & 6 & \cdots & n+2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ n & n+1 & n+2 & n+3 & \cdots & 2n-1 \end{pmatrix}.$$

Using your program, find the row reduced echelon form of A for $n = 4, \ldots, 20$.

Also run the `Matlab rref` function and compare results.

Your program probably disagrees with `rref` even for small values of n. The problem is that some pivots are very small and the normalization step (to make the pivot 1) causes roundoff errors. Use a tolerance parameter to fix this problem.

What can you conjecture about the rank of A?

(4) Prove that the matrix A has the following row reduced form:

$$R = \begin{pmatrix} 1 & 0 & -1 & -2 & \cdots & -(n-2) \\ 0 & 1 & 2 & 3 & \cdots & n-1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Deduce from the above that A has rank 2.

Hint. Some well chosen sequence of row operations.

(5) Use your program to show that if you add any number greater than or equal to $(2/25)n^2$ to every diagonal entry of A you get an invertible matrix! In fact, running the `Matlab` fuction `chol` should tell you that these matrices are SPD (symmetric, positive definite).

Remark: The above phenomenon will be explained in Problem B4. If you have a rigorous and *simple* explanation for this phenomenon, let me know!

Problem B4 (120 pts). The purpose of this problem is to prove that the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 3 & 4 & 5 & \cdots & n+1 \\ 3 & 4 & 5 & 6 & \cdots & n+2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ n & n+1 & n+2 & n+3 & \cdots & 2n-1 \end{pmatrix}$$

is

$$P_A(\lambda) = \lambda^{n-2}\left(\lambda^2 - n^2\lambda - \frac{1}{12}n^2(n^2-1)\right).$$

(1) Prove that the characteristic polynomial $P_A(\lambda)$ is given by

$$P_A(\lambda) = \lambda^{n-2} P(\lambda),$$

with

$$P(\lambda) = \begin{vmatrix} \lambda - 1 & -2 & -3 & -4 & \cdots & -n+3 & -n+2 & -n+1 & -n \\ -\lambda - 1 & \lambda - 1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \end{vmatrix}.$$

(2) Prove that the sum of the roots λ_1, λ_2 of the (degree two) polynomial $P(\lambda)$ is

$$\lambda_1 + \lambda_2 = n^2.$$

The problem is thus to compute the product $\lambda_1\lambda_2$ of these roots. Prove that

$$\lambda_1\lambda_2 = P(0).$$

(3) The problem is now to evaluate $d_n = P(0)$, where

$$d_n = \begin{vmatrix} -1 & -2 & -3 & -4 & \cdots & -n+3 & -n+2 & -n+1 & -n \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \end{vmatrix}$$

I suggest the following strategy: cancel out the first entry in row 1 and row 2 by adding a suitable multiple of row 3 to row 1 and row 2, and then subtract row 2 from row 1.

Do this twice.

You will notice that the first two entries on row 1 and the first two entries on row 2 change, but the rest of the matrix looks the same, except that the dimension is reduced.

This suggests setting up a recurrence involving the entries u_k, v_k, x_k, y_k in the determinant

$$D_k = \begin{vmatrix} u_k & x_k & -3 & -4 & \cdots & -n+k-3 & -n+k-2 & -n+k-1 & -n+k \\ v_k & y_k & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \end{vmatrix},$$

starting with $k = 0$, with

$$u_0 = -1, \quad v_0 = -1, \quad x_0 = -2, \quad y_0 = -1,$$

and ending with $k = n-2$, so that

$$d_n = D_{n-2} = \begin{vmatrix} u_{n-3} & x_{n-3} & -3 \\ v_{n-3} & y_{n-3} & -1 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} u_{n-2} & x_{n-2} \\ v_{n-2} & y_{n-2} \end{vmatrix}.$$

Prove that we have the recurrence relations

$$\begin{pmatrix} u_{k+1} \\ v_{k+1} \\ x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 & -1 \\ 0 & 2 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_k \\ v_k \\ x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -2 \\ -1 \end{pmatrix}.$$

These appear to be nasty affine recurrence relations, so we will use the trick to convert this affine map to a linear map.

(4) Consider the linear map given by

$$\begin{pmatrix} u_{k+1} \\ v_{k+1} \\ x_{k+1} \\ y_{k+1} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 & -1 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_k \\ v_k \\ x_k \\ y_k \\ 1 \end{pmatrix},$$

and show that its action on u_k, v_k, x_k, y_k is the same as the affine action of part (3).

Use `Matlab` to find the eigenvalues of the matrix

$$T = \begin{pmatrix} 2 & -2 & 1 & -1 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

You will be stunned!

Let N be the matrix given by

$$N = T - I.$$

Prove that

$$N^4 = 0.$$

Use this to prove that

$$T^k = I + kN + \frac{1}{2}k(k-1)N^2 + \frac{1}{6}k(k-1)(k-2)N^3,$$

for all $k \geq 0$.

(5) Prove that

$$\begin{pmatrix} u_k \\ v_k \\ x_k \\ y_k \\ 1 \end{pmatrix} = T^k \begin{pmatrix} -1 \\ -1 \\ -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 & -1 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^k \begin{pmatrix} -1 \\ -1 \\ -2 \\ -1 \\ 1 \end{pmatrix},$$

for $k \geq 0$.

Prove that

$$T^k = \begin{pmatrix} k+1 & -k(k+1) & k & -k^2 & \frac{1}{6}(k-1)k(2k-7) \\ 0 & k+1 & 0 & k & -\frac{1}{2}(k-1)k \\ -k & k^2 & 1-k & (k-1)k & -\frac{1}{3}k((k-6)k+11) \\ 0 & -k & 0 & 1-k & \frac{1}{2}(k-3)k \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and thus, that

$$\begin{pmatrix} u_k \\ v_k \\ x_k \\ y_k \end{pmatrix} = \begin{pmatrix} \frac{1}{6}(2k^3+3k^2-5k-6) \\ -\frac{1}{2}(k^2+3k+2) \\ \frac{1}{3}(-k^3+k-6) \\ \frac{1}{2}(k^2+k-2) \end{pmatrix},$$

and that

$$\begin{vmatrix} u_k & x_k \\ v_k & y_k \end{vmatrix} = -1 - \frac{7}{3}k - \frac{23}{12}k^2 - \frac{2}{3}k^3 - \frac{1}{12}k^4.$$

As a consequence, prove that amazingly,

$$d_n = D_{n-2} = -\frac{1}{12}n^2(n^2-1).$$

(6) Prove that the characteristic polynomial of A is indeed

$$P_A(\lambda) = \lambda^{n-2}\left(\lambda^2 - n^2\lambda - \frac{1}{12}n^2(n^2-1)\right).$$

Use the above to show that the two nonzero eigenvalues of A are

$$\lambda = \frac{n}{2}\left(n \pm \frac{\sqrt{3}}{3}\sqrt{4n^2-1}\right).$$

The negative eigenvalue λ_1 can also be expressed as

$$\lambda_1 = n^2\frac{(3-2\sqrt{3})}{6}\sqrt{1-\frac{1}{4n^2}}.$$

Use this expression to explain the phenomenon in B3(5): If we add any number greater than or equal to $(2/25)n^2$ to every diagonal entry of A we get an invertible matrix. What about $0.077351n^2$? Try it!

Problem B5 (40 pts). A method for computing the nth root $x^{1/n}$ of a positive real number $x \in \mathbb{R}$, with $n \in \mathbb{N}$ a positive integer $n \geq 2$, proceeds as follows: Define the sequence (x_k), where x_0 is any chosen positive real, and

$$x_{k+1} = \frac{1}{n}\left((n-1)x_k + \frac{x}{x_k^{n-1}}\right), \quad k \geq 0.$$

(1) Implement the above method in `Matlab`, and test it for various input values of x, x_0, and of $n \geq 2$, by running successively your program for $m = 2, 3, \ldots, 100$ iterations. Have your program plot the points (i, x_i) to watch how quickly the sequence converges.

Experiment with various choices of x_0. One of these choices should be $x_0 = x$. Compare your answers with the result of applying the of `Matlab` function $x \mapsto x^{1/n}$.

In some case, when x_0 is small, the number of iterations has to be at least 1000. Exhibit this behavior.

Problem B6 (80 pts). Refer to Problem B5 for the definition of the sequence (x_k).

(1) Define the *relative error* ϵ_k as

$$\epsilon_k = \frac{x_k}{x^{1/n}} - 1, \quad k \geq 0.$$

Prove that

$$\epsilon_{k+1} = \frac{x^{(1-1/n)}}{nx_k^{n-1}}\left(\frac{(n-1)x_k^n}{x} - \frac{nx_k^{n-1}}{x^{(1-1/n)}} + 1\right),$$

and then that

$$\epsilon_{k+1} = \frac{1}{n(\epsilon_k+1)^{n-1}}\left(\epsilon_k(\epsilon_k+1)^{n-2}((n-1)\epsilon_k + (n-2)) + 1 - (\epsilon_k+1)^{n-2}\right),$$

for all $k \geq 0$.

(2) Since

$$\epsilon_k + 1 = \frac{x_k}{x^{1/n}},$$

and since we assumed $x_0, x > 0$, we have $\epsilon_0 + 1 > 0$. We would like to prove that

$$\epsilon_k \geq 0, \quad \text{for all} \quad k \geq 1.$$

For this, consider the variations of the function f given by

$$f(u) = (n-1)u^n - nx^{1/n}u^{n-1} + x,$$

for $u \in \mathbb{R}$.

Use the above to prove that $f(u) \geq 0$ for all $u \geq 0$. Conclude that

$$\epsilon_k \geq 0, \quad \text{for all} \quad k \geq 1.$$

(3) Prove that if $n = 2$, then

$$0 \leq \epsilon_{k+1} = \frac{\epsilon_k^2}{2(\epsilon_k + 1)}, \quad \text{for all} \quad k \geq 0,$$

else if $n \geq 3$, then

$$0 \leq \epsilon_{k+1} \leq \frac{(n-1)}{n}\epsilon_k, \quad \text{for all} \quad k \geq 1.$$

Prove that the sequence (x_k) converges to $x^{1/n}$ for every initial value $x_0 > 0$.

(4) When $n = 2$, we saw in B6(3) that

$$0 \leq \epsilon_{k+1} = \frac{\epsilon_k^2}{2(\epsilon_k + 1)}, \quad \text{for all} \quad k \geq 0.$$

For $n = 3$, prove that

$$\epsilon_{k+1} = \frac{2\epsilon_k^2(3/2 + \epsilon_k)}{3(\epsilon_k + 1)^2}, \quad \text{for all} \quad k \geq 0,$$

and for $n = 4$, prove that

$$\epsilon_{k+1} = \frac{3\epsilon_k^2}{4(\epsilon_k + 1)^3}\left(2 + (8/3)\epsilon_k + \epsilon_k^2\right), \quad \text{for all} \quad k \geq 0.$$

Let μ_3 and μ_4 be the functions given by

$$\mu_3(a) = \frac{3}{2} + a$$
$$\mu_4(a) = 2 + \frac{8}{3}a + a^2,$$

so that if $n = 3$, then

$$\epsilon_{k+1} = \frac{2\epsilon_k^2\mu_3(\epsilon_k)}{3(\epsilon_k + 1)^2}, \quad \text{for all} \quad k \geq 0,$$

and if $n = 4$, then

$$\epsilon_{k+1} = \frac{3\epsilon_k^2\mu_4(\epsilon_k)}{4(\epsilon_k + 1)^3}, \quad \text{for all} \quad k \geq 0.$$

Prove that

$$a\mu_3(a) \leq (a+1)^2 - 1, \quad \text{for all} \quad a \geq 0,$$

and

$$a\mu_4(a) \leq (a+1)^3 - 1, \quad \text{for all} \quad a \geq 0.$$

Let $\eta_{3,k} = \mu_3(\epsilon_1)\epsilon_k$ when $n = 3$, and $\eta_{4,k} = \mu_4(\epsilon_1)\epsilon_k$ when $n = 4$. Prove that

$$\eta_{3,k+1} \leq \frac{2}{3}\eta_{3,k}^2, \quad \text{for all} \quad k \geq 1,$$

and

$$\eta_{4,k+1} \leq \frac{3}{4}\eta_{4,k}^2, \quad \text{for all} \quad k \geq 1.$$

Deduce from the above that the rate of convergence of $\eta_{i,k}$ is very fast, for $i = 3, 4$ (and $k \geq 1$).

Remark: If we let $\mu_2(a) = a$ for all a and $\eta_{2,k} = \epsilon_k$, then we proved that

$$\eta_{2,k+1} \leq \frac{1}{2}\eta_{2,k}^2, \quad \text{for all} \quad k \geq 1.$$

Extra Credit (150 pt)

(5) Prove that for all $n \geq 2$, we have

$$\epsilon_{k+1} = \left(\frac{n-1}{n}\right)\frac{\epsilon_k^2\mu_n(\epsilon_k)}{(\epsilon_k+1)^{n-1}}, \quad \text{for all} \quad k \geq 0,$$

where μ_n is given by

$$\mu_n(a) = \frac{1}{2}n + \sum_{j=1}^{n-4}\frac{1}{n-1}\left((n-1)\binom{n-2}{j} + (n-2)\binom{n-2}{j+1} - \binom{n-2}{j+2}\right)a^j + \frac{n(n-2)}{n-1}a^{n-3} + a^{n-2}.$$

Furthermore, prove that μ_n can be expressed as

$$\mu_n(a) = \frac{1}{2}n + \frac{n(n-2)}{3}a + \sum_{j=2}^{n-4}\frac{(j+1)n}{(j+2)(n-1)}\binom{n-1}{j+1}a^j + \frac{n(n-2)}{n-1}a^{n-3} + a^{n-2}.$$

(6) Prove that for every j, with $1 \leq j \leq n-1$, the coefficient of a^j in $a\mu_n(a)$ is less than or equal to the coefficient of a^j in $(a+1)^{n-1} - 1$, and thus

$$a\mu_n(a) \leq (a+1)^{n-1} - 1, \quad \text{for all} \quad a \geq 0,$$

with strict inequality if $n \geq 3$. In fact, prove that if $n \geq 3$, then for every j, with $3 \leq j \leq n-2$, the coefficient of a^j in $a\mu_n(a)$ is strictly less than the coefficient of a^j in $(a+1)^{n-1} - 1$, and if $n \geq 4$, this also holds for $j = 2$.

Let $\eta_{n,k} = \mu_n(\epsilon_1)\epsilon_k$ $(n \geq 2)$. Prove that

$$\eta_{n,k+1} \leq \left(\frac{n-1}{n}\right)\eta_{n,k}^2, \quad \text{for all} \quad k \geq 1.$$

TOTAL: 380 + 150 points.