Problem B1 (20 pts). Let $U_1, \ldots, U_p$ be any $p \geq 2$ subspaces of some vector space $E$. Prove that $U_1 + \cdots + U_p$ is a direct sum iff

$$U_i \cap \left( \sum_{j=1}^{i-1} U_j \right) = (0), \quad i = 2, \ldots, p.$$ 

Problem B2 (50 pts). Given any vector space $E$, a linear map $f : E \to E$ is an involution if $f \circ f = \text{id}$. 

(1) Prove that an involution $f$ is invertible. What is its inverse? 

(2) Let $E_1$ and $E_{-1}$ be the subspaces of $E$ defined as follows:

$$E_1 = \{ u \in E \mid f(u) = u \}$$
$$E_{-1} = \{ u \in E \mid f(u) = -u \}.$$

Prove that we have a direct sum

$$E = E_1 \oplus E_{-1}.$$ 

Hint. For every $u \in E$, write

$$u = \frac{u + f(u)}{2} + \frac{u - f(u)}{2}.$$ 

(3) If $E$ is finite-dimensional and $f$ is an involution, prove that there is some basis of $E$ with respect to which the matrix of $f$ is of the form 

$$I_{k,n-k} = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix},$$

where $I_k$ is the $k \times k$ identity matrix (similarly for $I_{n-k}$) and $k = \dim(E_1)$. Can you give a geometric interpretation of the action of $f$ (especially when $k = n - 1$)?
Problem B3 (50 pts). A rotation \( R_\theta \) in the plane \( \mathbb{R}^2 \) is given by the matrix

\[
R_\theta = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.
\]

(1) Use Matlab to show the action of a rotation \( R_\theta \) on a simple figure such as a triangle or a rectangle, for various values of \( \theta \), including \( \theta = \pi/6, \pi/4, \pi/3, \pi/2 \).

(2) Prove that \( R_\theta \) is invertible and that its inverse is \( R_{-\theta} \).

(3) For any two rotations \( R_\alpha \) and \( R_\beta \), prove that

\[ R_\beta \circ R_\alpha = R_\alpha \circ R_\beta = R_{\alpha + \beta}. \]

Use (2)-(3) to prove that the rotations in the plane form a commutative group denoted \( \text{SO}(2) \).

Problem B4 (110 pts). Consider the affine map \( R_{\theta,(a_1,a_2)} \) in \( \mathbb{R}^2 \) given by

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.
\]

(1) Prove that if \( \theta \neq k2\pi \), with \( k \in \mathbb{Z} \), then \( R_{\theta,(a_1,a_2)} \) has a unique fixed point \((c_1, c_2)\), that is, there is a unique point \((c_1, c_2)\) such that

\[
\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = R_{\theta,(a_1,a_2)} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},
\]

and this fixed point is given by

\[
\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{2 \sin(\theta/2)} \begin{pmatrix}
\cos(\pi/2 - \theta/2) & -\sin(\pi/2 - \theta/2) \\
\sin(\pi/2 - \theta/2) & \cos(\pi/2 - \theta/2)
\end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.
\]

(2) In this question, we still assume that \( \theta \neq k2\pi \), with \( k \in \mathbb{Z} \). By translating the coordinate system with origin \((0,0)\) to the new coordinate system with origin \((c_1,c_2)\), which means that if \((x_1,x_2)\) are the coordinates with respect to the standard origin \((0,0)\) and if \((x_1',x_2')\) are the coordinates with respect to the new origin \((c_1,c_2)\), we have

\[
x_1 = x_1' + c_1 \\
x_2 = x_2' + c_2
\]

and similarly for \((y_1,y_2)\) and \((y_1',y_2')\), then show that

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = R_{\theta,(a_1,a_2)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]
becomes
\[
\begin{pmatrix}
y_1' \\ y_2'
\end{pmatrix} = R_\theta \begin{pmatrix}
x_1' \\ x_2'
\end{pmatrix}.
\]

Conclude that with respect to the new origin \((c_1, c_2)\), the affine map \(R_{\theta(a_1, a_2)}\) becomes the rotation \(R_\theta\). We say that \(R_{\theta(a_1, a_2)}\) is a rotation of center \((c_1, c_2)\).

(3) Use Matlab to show the action of the affine map \(R_{\theta(a_1, a_2)}\) on a simple figure such as a triangle or a rectangle, for \(\theta = \pi/3\) and various values of \((a_1, a_2)\). Display the center \((c_1, c_2)\) of the rotation.

What kind of transformations correspond to \(\theta = k2\pi\), with \(k \in \mathbb{Z}\)?

(4) Prove that the inverse of \(R_{\theta(a_1, a_2)}\) is of the form \(R_{-\theta(b_1, b_2)}\), and find \((b_1, b_2)\) in terms of \(\theta\) and \((a_1, a_2)\).

(5) Given two affine maps \(R_{\alpha(a_1, a_2)}\) and \(R_{\beta(b_1, b_2)}\), prove that
\[
R_{\beta(b_1, b_2)} \circ R_{\alpha(a_1, a_2)} = R_{\alpha + \beta(t_1, t_2)}
\]
for some \((t_1, t_2)\), and find \((t_1, t_2)\) in terms of \(\beta\), \((a_1, a_2)\) and \((b_1, b_2)\).

Even in the case where \((a_1, a_2) = (0, 0)\), prove that in general
\[
R_{\beta(b_1, b_2)} \circ R_{\alpha} \neq R_{\alpha} \circ R_{\beta(b_1, b_2)}.
\]

Use (4)-(5) to show that the affine maps of the plane defined in this problem form a nonabelian group denoted \(\text{SE}(2)\).

Prove that \(R_{\beta(b_1, b_2)} \circ R_{\alpha(a_1, a_2)}\) is not a translation (possibly the identity) iff \(\alpha + \beta \neq k2\pi\), for all \(k \in \mathbb{Z}\). Find its center of rotation when \((a_1, a_2) = (0, 0)\).

If \(\alpha + \beta = k2\pi\), then \(R_{\beta(b_1, b_2)} \circ R_{\alpha(a_1, a_2)}\) is a pure translation. Find the translation vector of \(R_{\beta(b_1, b_2)} \circ R_{\alpha(a_1, a_2)}\).

**Problem B5 (80 pts).** A subset \(\mathcal{A}\) of \(\mathbb{R}^n\) is called an affine subspace if either \(\mathcal{A} = \emptyset\), or there is some vector \(a \in \mathbb{R}^n\) and some subspace \(U\) of \(\mathbb{R}^n\) such that
\[
\mathcal{A} = a + U = \{a + u \mid u \in U\}.
\]

We define the dimension \(\dim(\mathcal{A})\) of \(\mathcal{A}\) as the dimension \(\dim(U)\) of \(U\).

(1) If \(\mathcal{A} = a + U\), why is \(a \in \mathcal{A}\)?

What are affine subspaces of dimension 0? What are affine subspaces of dimension 1 (begin with \(\mathbb{R}^2\))? What are affine subspaces of dimension 2 (begin with \(\mathbb{R}^3\))? Prove that any nonempty affine subspace is closed under affine combinations.

(2) Prove that if \(\mathcal{A} = a + U\) is any nonempty affine subspace, then \(\mathcal{A} = b + U\) for any \(b \in \mathcal{A}\).
(3) Let \( \mathcal{A} \) be any nonempty subset of \( \mathbb{R}^n \) closed under affine combinations. For any \( a \in \mathcal{A} \), prove that

\[
U_a = \{ x - a \in \mathbb{R}^n \mid x \in \mathcal{A} \}
\]

is a (linear) subspace of \( \mathbb{R}^n \) such that

\[
\mathcal{A} = a + U_a.
\]

Prove that \( U_a \) does not depend on the choice of \( a \in \mathcal{A} \); that is, \( U_a = U_b \) for all \( a, b \in \mathcal{A} \). In fact, prove that

\[
U_a = U = \{ y - x \in \mathbb{R}^n \mid x, y \in \mathcal{A} \}, \quad \text{for all } a \in \mathcal{A},
\]

and so

\[
\mathcal{A} = a + U, \quad \text{for any } a \in \mathcal{A}.
\]

Remark: The subspace \( U \) is called the direction of \( \mathcal{A} \).

(4) Two nonempty affine subspaces \( \mathcal{A} \) and \( \mathcal{B} \) are said to be parallel iff they have the same direction. Prove that if \( \mathcal{A} \neq \mathcal{B} \) and \( \mathcal{A} \) and \( \mathcal{B} \) are parallel, then \( \mathcal{A} \cap \mathcal{B} = \emptyset \).

Remark: The above shows that affine subspaces behave quite differently from linear subspaces.

Problem B6 (120 pts). (Affine frames and affine maps) For any vector \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \), let \( \hat{v} \in \mathbb{R}^{n+1} \) be the vector \( \hat{v} = (v_1, \ldots, v_n, 1) \). Equivalently, \( \hat{v} = (\hat{v}_1, \ldots, \hat{v}_{n+1}) \in \mathbb{R}^{n+1} \) is the vector defined by

\[
\hat{v}_i = \begin{cases} 
v_i & \text{if } 1 \leq i \leq n, \\
1 & \text{if } i = n + 1.
\end{cases}
\]

(1) For any \( m + 1 \) vectors \( (u_0, u_1, \ldots, u_m) \) with \( u_i \in \mathbb{R}^n \) and \( m \leq n \), prove that if the \( m \) vectors \( (u_1 - u_0, \ldots, u_m - u_0) \) are linearly independent, then the \( m + 1 \) vectors \( (\hat{u}_0, \ldots, \hat{u}_m) \) are linearly independent.

(2) Prove that if the \( m + 1 \) vectors \( (\hat{u}_0, \ldots, \hat{u}_m) \) are linearly independent, then for any choice of \( i \), with \( 0 \leq i \leq m \), the \( m \) vectors \( u_j - u_i \) for \( j \in \{0, \ldots, m\} \) with \( j - i \neq 0 \) are linearly independent.

Any \( m + 1 \) vectors \( (u_0, u_1, \ldots, u_m) \) such that the \( m + 1 \) vectors \( (\hat{u}_0, \ldots, \hat{u}_m) \) are linearly independent are said to be affinely independent.

From (1) and (2), the vector \( (u_0, u_1, \ldots, u_m) \) are affinely independent iff for any any choice of \( i \), with \( 0 \leq i \leq m \), the \( m \) vectors \( u_j - u_i \) for \( j \in \{0, \ldots, m\} \) with \( j - i \neq 0 \) are linearly independent. If \( m = n \), we say that \( n + 1 \) affinely independent vectors \( (u_0, u_1, \ldots, u_n) \) form an affine frame of \( \mathbb{R}^n \).
(3) if \((u_0, u_1, \ldots, u_n)\) is an affine frame of \(\mathbb{R}^n\), then prove that for every vector \(v \in \mathbb{R}^n\), there is a unique \((n+1)\)-tuple \((\lambda_0, \lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{n+1}\), with \(\lambda_0 + \lambda_1 + \cdots + \lambda_n = 1\), such that \[v = \lambda_0 u_0 + \lambda_1 u_1 + \cdots + \lambda_n u_n.\]

The scalars \((\lambda_0, \lambda_1, \ldots, \lambda_n)\) are called the barycentric (or affine) coordinates of \(v\) w.r.t. the affine frame \((u_0, u_1, \ldots, u_n)\).

If we write \(e_i = u_i - u_0\), for \(i = 1, \ldots, n\), then prove that we have \[v = u_0 + \lambda_1 e_1 + \cdots + \lambda_n e_n,\]
and since \((e_1, \ldots, e_n)\) is a basis of \(\mathbb{R}^n\) (by (1) & (2)), the \(n\)-tuple \((\lambda_1, \ldots, \lambda_n)\) consists of the standard coordinates of \(v - u_0\) over the basis \((e_1, \ldots, e_n)\).

Conversely, for any vector \(u_0 \in \mathbb{R}^n\) and for any basis \((e_1, \ldots, e_n)\) of \(\mathbb{R}^n\), let \(u_i = u_0 + e_i\) for \(i = 1, \ldots, n\). Prove that \((u_0, u_1, \ldots, u_n)\) is an affine frame of \(\mathbb{R}^n\), and for any \(v \in \mathbb{R}^n\), if \[v = u_0 + x_1 e_1 + \cdots + x_n e_n,\]
with \((x_1, \ldots, x_n) \in \mathbb{R}^n\) (unique), then
\[v = (1 - (x_1 + \cdots + x_n))u_0 + x_1 u_1 + \cdots + x_n u_n,\]
so that \((1 - (x_1 + \cdots + x_n)), x_1, \ldots, x_n)\), are the barycentric coordinates of \(v\) w.r.t. the affine frame \((u_0, u_1, \ldots, u_n)\).

The above shows that there is a one-to-one correspondence between affine frames \((u_0, \ldots, u_n)\) and pairs \((u_0, (e_1, \ldots, e_n))\), with \((e_1, \ldots, e_n)\) a basis. Given an affine frame \((u_0, \ldots, u_n)\), we obtain the basis \((e_1, \ldots, e_n)\) with \(e_i = u_i - u_0\), for \(i = 1, \ldots, n\); given the pair \((u_0, (e_1, \ldots, e_n))\) where \((e_1, \ldots, e_n)\) is a basis, we obtain the affine frame \((u_0, \ldots, u_n)\), with \(u_i = u_0 + e_i\), for \(i = 1, \ldots, n\). There is also a one-to-one correspondence between barycentric coordinates w.r.t. the affine frame \((u_0, \ldots, u_n)\) and standard coordinates w.r.t. the basis \((e_1, \ldots, e_n)\). The barycentric coordinates \((\lambda_0, \lambda_1, \ldots, \lambda_n)\) of \(v\) (with \(\lambda_0 + \lambda_1 + \cdots + \lambda_n = 1\)) yield the standard coordinates \((\lambda_1, \ldots, \lambda_n)\) of \(v - u_0\); the standard coordinates \((x_1, \ldots, x_n)\) of \(v - u_0\) yield the barycentric coordinates \((1 - (x_1 + \cdots + x_n)), x_1, \ldots, x_n)\) of \(v\).

(4) Let \((u_0, \ldots, u_n)\) be any affine frame in \(\mathbb{R}^n\) and let \((v_0, \ldots, v_n)\) be any vectors in \(\mathbb{R}^m\). Prove that there is a unique affine map \(f: \mathbb{R}^n \to \mathbb{R}^m\) such that \[f(u_i) = v_i, \quad i = 0, \ldots, n.\]

(5) Let \((a_0, \ldots, a_n)\) be any affine frame in \(\mathbb{R}^n\) and let \((b_0, \ldots, b_n)\) be any \(n + 1\) points in \(\mathbb{R}^n\). Prove that the \((n + 1) \times (n + 1)\) matrix \(A\) corresponding to the unique affine map \(f\) such that \[f(a_i) = b_i, \quad i = 0, \ldots, n,\]...
is given by
\[ A = \left( \hat{b}_0 \hat{b}_1 \cdots \hat{b}_n \right) \left( \hat{a}_0 \hat{a}_1 \cdots \hat{a}_n \right)^{-1}. \]

In the special case where \((a_0, \ldots, a_n)\) is the canonical affine frame with \(a_i = e_{i+1}\) for \(i = 0, \ldots, n-1\) and \(a_n = (0, \ldots, 0)\) (where \(e_i\) is the \(i\)th canonical basis vector), show that
\[
(\hat{a}_0 \hat{a}_1 \cdots \hat{a}_n) = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & 0 & 0 \\
0 & 0 & \cdots & 1 & 0 \\
1 & 1 & \cdots & 1 & 1 \\
\end{pmatrix}
\]
and
\[
(\hat{a}_0 \hat{a}_1 \cdots \hat{a}_n)^{-1} = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & 0 & 0 \\
0 & 0 & \cdots & 1 & 0 \\
-1 & -1 & \cdots & -1 & 1 \\
\end{pmatrix}
\]

For example, when \(n = 2\), if we write \(b_i = (x_i, y_i)\), then we have
\[
A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} x_1 \bar{x} & x_2 \bar{x} - x_3 \bar{x} & x_3 \bar{x} \\ y_1 \bar{y} & y_2 \bar{y} - y_3 \bar{y} & y_3 \bar{y} \\ 0 & 0 & 1 \end{pmatrix}
\]

(6) Recall that a nonempty affine subspace \(A\) of \(\mathbb{R}^n\) is any nonempty subset of \(\mathbb{R}^n\) closed under affine combinations. For any affine map \(f : \mathbb{R}^n \to \mathbb{R}^m\), for any affine subspace \(A\) of \(\mathbb{R}^n\), and any affine subspace \(B\) of \(\mathbb{R}^m\), prove that \(f(A)\) is an affine subspace of \(\mathbb{R}^m\), and that \(f^{-1}(B)\) is an affine subspace of \(\mathbb{R}^n\).

**Problem B7 (30 pts).** Let \(A\) be any \(n \times k\) matrix

(1) Prove that the \(k \times k\) matrix \(A^\top A\) and the matrix \(A\) have the same nullspace. Use this to prove that \(\text{rank}(A^\top A) = \text{rank}(A)\). Similarly, prove that the \(n \times n\) matrix \(AA^\top\) and the matrix \(A^\top\) have the same nullspace, and conclude that \(\text{rank}(AA^\top) = \text{rank}(A^\top)\).

We will prove later that \(\text{rank}(A^\top) = \text{rank}(A)\).

(2) Let \(a_1, \ldots, a_k\) be \(k\) linearly independent vectors in \(\mathbb{R}^n\) \((1 \leq k \leq n)\), and let \(A\) be the \(n \times k\) matrix whose \(i\)th column is \(a_i\). Prove that \(A^\top A\) has rank \(k\), and that it is invertible. Let \(P = A(A^\top A)^{-1}A^\top\) (an \(n \times n\) matrix). Prove that
\[
P^2 = P \quad P^\top = P.
\]

What is the matrix \(P\) when \(k = 1\)?
(3) Prove that the image of $P$ is the subspace $V$ spanned by $a_1, \ldots, a_k$, or equivalently the set of all vectors in $\mathbb{R}^n$ of the form $Ax$, with $x \in \mathbb{R}^k$. Prove that the nullspace $U$ of $P$ is the set of vectors $u \in \mathbb{R}^n$ such that $A^\top u = 0$. Can you give a geometric interpretation of $U$?

Conclude that $P$ is a projection of $\mathbb{R}^n$ onto the subspace $V$ spanned by $a_1, \ldots, a_k$, and that

$$\mathbb{R}^n = U \oplus V.$$ 

*Hint.* You may use results from HW2.

**TOTAL:** 460 points.