

Fundamentals of Linear Algebra and Optimization

Jean Gallier

Homework 3

October 2, 2014; Due October 21, 2014

Beginning of class

Problem B1 (20 pts). Let U_1, \dots, U_p be any $p \geq 2$ subspaces of some vector space E . Prove that $U_1 + \dots + U_p$ is a direct sum iff

$$U_i \cap \left(\sum_{j=1}^{i-1} U_j \right) = (0), \quad i = 2, \dots, p.$$

Problem B2 (50 pts). Given any vector space E , a linear map $f: E \rightarrow E$ is an *involution* if $f \circ f = \text{id}$.

- (1) Prove that an involution f is invertible. What is its inverse?
- (2) Let E_1 and E_{-1} be the subspaces of E defined as follows:

$$\begin{aligned} E_1 &= \{u \in E \mid f(u) = u\} \\ E_{-1} &= \{u \in E \mid f(u) = -u\}. \end{aligned}$$

Prove that we have a direct sum

$$E = E_1 \oplus E_{-1}.$$

Hint. For every $u \in E$, write

$$u = \frac{u + f(u)}{2} + \frac{u - f(u)}{2}.$$

(3) If E is finite-dimensional and f is an involution, prove that there is some basis of E over which the matrix of f is of the form

$$I_{k,n-k} = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix (similarly for I_{n-k}) and $k = \dim(E_1)$. Can you give a geometric interpretation of the action of f (especially when $k = n - 1$)?

Problem B3 (50 pts). A rotation R_θ in the plane \mathbb{R}^2 is given by the matrix

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

(1) Use `Matlab` to show the action of a rotation R_θ on a simple figure such as a triangle or a rectangle, for various values of θ , including $\theta = \pi/6, \pi/4, \pi/3, \pi/2$.

(2) Prove that R_θ is invertible and that its inverse is $R_{-\theta}$.

(3) For any two rotations R_α and R_β , prove that

$$R_\beta \circ R_\alpha = R_\alpha \circ R_\beta = R_{\alpha+\beta}.$$

Use (2)-(3) to prove that the rotations in the plane form a commutative group denoted $\text{SO}(2)$.

Problem B4 (110 pts). Consider the affine map $R_{\theta,(a_1,a_2)}$ in \mathbb{R}^2 given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

(1) Prove that if $\theta \neq k2\pi$, with $k \in \mathbb{Z}$, then $R_{\theta,(a_1,a_2)}$ has a unique fixed point (c_1, c_2) , that is, there is a unique point (c_1, c_2) such that

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = R_{\theta,(a_1,a_2)} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

and this fixed point is given by

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{2 \sin(\theta/2)} \begin{pmatrix} \cos(\pi/2 - \theta/2) & -\sin(\pi/2 - \theta/2) \\ \sin(\pi/2 - \theta/2) & \cos(\pi/2 - \theta/2) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

(2) In this question, we still assume that $\theta \neq k2\pi$, with $k \in \mathbb{Z}$. By translating the coordinate system with origin $(0, 0)$ to the new coordinate system with origin (c_1, c_2) , which means that if (x_1, x_2) are the coordinates with respect to the standard origin $(0, 0)$ and if (x'_1, x'_2) are the coordinates with respect to the new origin (c_1, c_2) , we have

$$\begin{aligned} x_1 &= x'_1 + c_1 \\ x_2 &= x'_2 + c_2 \end{aligned}$$

and similarly for (y_1, y_2) and (y'_1, y'_2) , then show that

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = R_{\theta,(a_1,a_2)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

becomes

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = R_\theta \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix}.$$

Conclude that with respect to the new origin (c_1, c_2) , the affine map $R_{\theta, (a_1, a_2)}$ becomes the rotation R_θ . We say that $R_{\theta, (a_1, a_2)}$ is a *rotation of center* (c_1, c_2) .

(3) Use `Matlab` to show the action of the affine map $R_{\theta, (a_1, a_2)}$ on a simple figure such as a triangle or a rectangle, for $\theta = \pi/3$ and various values of (a_1, a_2) . Display the center (c_1, c_2) of the rotation.

What kind of transformations correspond to $\theta = k2\pi$, with $k \in \mathbb{Z}$?

(4) Prove that the inverse of $R_{\theta, (a_1, a_2)}$ is of the form $R_{-\theta, (b_1, b_2)}$, and find (b_1, b_2) in terms of θ and (a_1, a_2) .

(5) Given two affine maps $R_{\alpha, (a_1, a_2)}$ and $R_{\beta, (b_1, b_2)}$, prove that

$$R_{\beta, (b_1, b_2)} \circ R_{\alpha, (a_1, a_2)} = R_{\alpha + \beta, (t_1, t_2)}$$

for some (t_1, t_2) , and find (t_1, t_2) in terms of β , (a_1, a_2) and (b_1, b_2) .

Even in the case where $(a_1, a_2) = (0, 0)$, prove that in general

$$R_{\beta, (b_1, b_2)} \circ R_\alpha \neq R_\alpha \circ R_{\beta, (b_1, b_2)}.$$

Use (4)-(5) to show that the affine maps of the plane defined in this problem form a nonabelian group denoted $\mathbf{SE}(2)$.

Prove that $R_{\beta, (b_1, b_2)} \circ R_{\alpha, (a_1, a_2)}$ is not a translation (possibly the identity) iff $\alpha + \beta \neq k2\pi$, for all $k \in \mathbb{Z}$. Find its center of rotation when $(a_1, a_2) = (0, 0)$.

If $\alpha + \beta = k2\pi$, then $R_{\beta, (b_1, b_2)} \circ R_{\alpha, (a_1, a_2)}$ is a pure translation. Find the translation vector of $R_{\beta, (b_1, b_2)} \circ R_{\alpha, (a_1, a_2)}$.

Problem B5 (80 pts). A subset \mathcal{A} of \mathbb{R}^n is called an *affine subspace* if either $\mathcal{A} = \emptyset$, or there is some vector $a \in \mathbb{R}^n$ and some subspace U of \mathbb{R}^n such that

$$\mathcal{A} = a + U = \{a + u \mid u \in U\}.$$

We define the dimension $\dim(\mathcal{A})$ of \mathcal{A} as the dimension $\dim(U)$ of U .

(1) If $\mathcal{A} = a + U$, why is $a \in \mathcal{A}$?

What are affine subspaces of dimension 0? What are affine subspaces of dimension 1 (begin with \mathbb{R}^2)? What are affine subspaces of dimension 2 (begin with \mathbb{R}^3)?

Prove that any nonempty affine subspace is closed under affine combinations.

(2) Prove that if $\mathcal{A} = a + U$ is any nonempty affine subspace, then $\mathcal{A} = b + U$ for any $b \in \mathcal{A}$.

(3) Let \mathcal{A} be any nonempty subset of \mathbb{R}^n closed under affine combinations. For any $a \in \mathcal{A}$, prove that

$$U_a = \{x - a \in \mathbb{R}^n \mid x \in \mathcal{A}\}$$

is a (linear) subspace of \mathbb{R}^n such that

$$\mathcal{A} = a + U_a.$$

Prove that U_a does not depend on the choice of $a \in \mathcal{A}$; that is, $U_a = U_b$ for all $a, b \in \mathcal{A}$. In fact, prove that

$$U_a = U = \{y - x \in \mathbb{R}^n \mid x, y \in \mathcal{A}\}, \quad \text{for all } a \in \mathcal{A},$$

and so

$$\mathcal{A} = a + U, \quad \text{for any } a \in \mathcal{A}.$$

Remark: The subspace U is called the *direction* of \mathcal{A} .

(4) Two nonempty affine subspaces \mathcal{A} and \mathcal{B} are said to be *parallel* iff they have the same direction. Prove that if $\mathcal{A} \neq \mathcal{B}$ and \mathcal{A} and \mathcal{B} are parallel, then $\mathcal{A} \cap \mathcal{B} = \emptyset$.

Remark: The above shows that affine subspaces behave quite differently from linear subspaces.

Problem B6 (120 pts). (Affine frames and affine maps) For any vector $u \in \mathbb{R}^n$, let $\widehat{u} \in \mathbb{R}^{n+1}$ be the vector defined by

$$\widehat{u}_i = \begin{cases} u_i & \text{if } 1 \leq i \leq n, \\ 1 & \text{if } i = n + 1. \end{cases}$$

(1) For any $m + 1$ vectors (u_0, u_1, \dots, u_m) with $u_i \in \mathbb{R}^n$ and $m \leq n$, prove that if the m vectors $(u_1 - u_0, \dots, u_m - u_0)$ are linearly independent, then the $m + 1$ vectors $(\widehat{u}_0, \dots, \widehat{u}_m)$ are linearly independent.

(2) Prove that if the $m + 1$ vectors $(\widehat{u}_0, \dots, \widehat{u}_m)$ are linearly independent, then for any choice of i , with $0 \leq i \leq m$, the m vectors $u_j - u_i$ for $j \in \{0, \dots, m\}$ with $j - i \neq 0$ are linearly independent.

Any $m + 1$ vectors (u_0, u_1, \dots, u_m) such that the $m + 1$ vectors $(\widehat{u}_0, \dots, \widehat{u}_m)$ are linearly independent are said to be *affinely independent*.

From (1) and (2), the vector (u_0, u_1, \dots, u_m) are affinely independent iff for any any choice of i , with $0 \leq i \leq m$, the m vectors $u_j - u_i$ for $j \in \{0, \dots, m\}$ with $j - i \neq 0$ are linearly independent. If $m = n$, we say that $n + 1$ affinely independent vectors (u_0, u_1, \dots, u_n) form an *affine frame* of \mathbb{R}^n .

(3) if (u_0, u_1, \dots, u_n) is an affine frame of \mathbb{R}^n , then prove that for every vector $v \in \mathbb{R}^n$, there is a unique $(n+1)$ -tuple $(\lambda_0, \lambda_1, \dots, \lambda_n) \in \mathbb{R}^{n+1}$, with $\lambda_0 + \lambda_1 + \dots + \lambda_n = 1$, such that

$$v = \lambda_0 u_0 + \lambda_1 u_1 + \dots + \lambda_n u_n.$$

The scalars $(\lambda_0, \lambda_1, \dots, \lambda_n)$ are called the *barycentric* (or *affine*) *coordinates* of v w.r.t. the affine frame (u_0, u_1, \dots, u_n) .

If we write $e_i = u_i - u_0$, for $i = 1, \dots, n$, then prove that we have

$$v = u_0 + \lambda_1 e_1 + \dots + \lambda_n e_n,$$

and since (e_1, \dots, e_n) is a basis of \mathbb{R}^n (by (1) & (2)), the n -tuple $(\lambda_1, \dots, \lambda_n)$ consists of the standard coordinates of $v - u_0$ over the basis (e_1, \dots, e_n) .

Conversely, for any vector $u_0 \in \mathbb{R}^n$ and for any basis (e_1, \dots, e_n) of \mathbb{R}^n , let $u_i = u_0 + e_i$ for $i = 1, \dots, n$. Prove that (u_0, u_1, \dots, u_n) is an affine frame of \mathbb{R}^n , and for any $v \in \mathbb{R}^n$, if

$$v = u_0 + x_1 e_1 + \dots + x_n e_n,$$

with $(x_1, \dots, x_n) \in \mathbb{R}^n$ (unique), then

$$v = (1 - (x_1 + \dots + x_n))u_0 + x_1 u_1 + \dots + x_n u_n,$$

so that $(1 - (x_1 + \dots + x_n), x_1, \dots, x_n)$, are the barycentric coordinates of v w.r.t. the affine frame (u_0, u_1, \dots, u_n) .

The above shows that there is a one-to-one correspondence between affine frames (u_0, \dots, u_n) and pairs $(u_0, (e_1, \dots, e_n))$, with (e_1, \dots, e_n) a basis. Given an affine frame (u_0, \dots, u_n) , we obtain the basis (e_1, \dots, e_n) with $e_i = u_i - u_0$, for $i = 1, \dots, n$; given the pair $(u_0, (e_1, \dots, e_n))$ where (e_1, \dots, e_n) is a basis, we obtain the affine frame (u_0, \dots, u_n) , with $u_i = u_0 + e_i$, for $i = 1, \dots, n$. There is also a one-to-one correspondence between barycentric coordinates w.r.t. the affine frame (u_0, \dots, u_n) and standard coordinates w.r.t. the basis (e_1, \dots, e_n) . The barycentric coordinates $(\lambda_0, \lambda_1, \dots, \lambda_n)$ of v (with $\lambda_0 + \lambda_1 + \dots + \lambda_n = 1$) yield the standard coordinates $(\lambda_1, \dots, \lambda_n)$ of $v - u_0$; the standard coordinates (x_1, \dots, x_n) of $v - u_0$ yield the barycentric coordinates $(1 - (x_1 + \dots + x_n), x_1, \dots, x_n)$ of v .

(4) Let (u_0, \dots, u_n) be any affine frame in \mathbb{R}^n and let (v_0, \dots, v_n) be any vectors in \mathbb{R}^m . Prove that there is a *unique* affine map $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$f(u_i) = v_i, \quad i = 0, \dots, n.$$

(5) Let (a_0, \dots, a_n) be any affine frame in \mathbb{R}^n and let (b_0, \dots, b_n) be any $n+1$ points in \mathbb{R}^n . Prove that the $(n+1) \times (n+1)$ matrix A corresponding to the unique affine map f such that

$$f(a_i) = b_i, \quad i = 0, \dots, n,$$

is given by

$$A = \begin{pmatrix} \widehat{b}_0 & \widehat{b}_1 & \cdots & \widehat{b}_n \end{pmatrix} \begin{pmatrix} \widehat{a}_0 & \widehat{a}_1 & \cdots & \widehat{a}_n \end{pmatrix}^{-1}.$$

In the special case where (a_0, \dots, a_n) is the canonical affine frame with $a_i = e_{i+1}$ for $i = 0, \dots, n-1$ and $a_n = (0, \dots, 0)$ (where e_i is the i th canonical basis vector), show that

$$\begin{pmatrix} \widehat{a}_0 & \widehat{a}_1 & \cdots & \widehat{a}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \widehat{a}_0 & \widehat{a}_1 & \cdots & \widehat{a}_n \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ -1 & -1 & \cdots & -1 & 1 \end{pmatrix}.$$

For example, when $n = 2$, if we write $b_i = (x_i, y_i)$, then we have

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} x_1 - x_3 & x_2 - x_3 & x_3 \\ y_1 - y_3 & y_2 - y_3 & y_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

(6) Recall that a nonempty affine subspace \mathcal{A} of \mathbb{R}^n is any nonempty subset of \mathbb{R}^n closed under affine combinations. For any affine map $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$, for any affine subspace U of \mathbb{R}^n , and any affine subspace V of \mathbb{R}^m , prove that $f(U)$ is an affine subspace of \mathbb{R}^m , and that $f^{-1}(V)$ is an affine subspace of \mathbb{R}^n .

Problem B7 (30 pts). Let A be any $n \times k$ matrix

(1) Prove that the $k \times k$ matrix $A^T A$ and the matrix A have the same nullspace. Use this to prove that $\text{rank}(A^T A) = \text{rank}(A)$. Similarly, prove that the $n \times n$ matrix AA^T and the matrix A^T have the same nullspace, and conclude that $\text{rank}(AA^T) = \text{rank}(A^T)$.

We will prove later that $\text{rank}(A^T) = \text{rank}(A)$.

(2) Let a_1, \dots, a_k be k linearly independent vectors in \mathbb{R}^n ($1 \leq k \leq n$), and let A be the $n \times k$ matrix whose i th column is a_i . Prove that $A^T A$ has rank k , and that it is invertible. Let $P = A(A^T A)^{-1} A^T$ (an $n \times n$ matrix). Prove that

$$\begin{aligned} P^2 &= P \\ P^T &= P. \end{aligned}$$

What is the matrix P when $k = 1$?

(3) Prove that the image of P is the subspace V spanned by a_1, \dots, a_k , or equivalently the set of all vectors in \mathbb{R}^n of the form Ax , with $x \in \mathbb{R}^k$. Prove that the nullspace U of P is the set of vectors $u \in \mathbb{R}^n$ such that $A^\top u = 0$. Can you give a geometric interpretation of U ?

Conclude that P is a projection of \mathbb{R}^n onto the subspace V spanned by a_1, \dots, a_k , and that

$$\mathbb{R}^n = U \oplus V.$$

Hint. You may use results from HW2.

TOTAL: 460 points.