$\boxed{\textbf{Fall 2013} \quad \textbf{CIS 515}}$

Fundamentals of Linear Algebra and Optimization
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Homework 3

October 4, 2013; Due October 22, 2013
Beginning of class

Problem B1 (20 pts). Let $U_1, \ldots, U_p$ be any $p \geq 2$ subspaces of some vector space E. Prove that $U_1 + \cdots + U_p$ is a direct sum iff

$$U_i \cap \left(\sum_{j=1}^{i-1} U_j \right) = (0), \quad i = 2, \ldots, p.$$

Problem B2 (50 pts). Given any vector space E, a linear map $f \colon E \to E$ is an *involution* if $f \circ f = \mathrm{id}$.

(1) Prove that an involution f is invertible. What is its inverse?

(2) Let E_1 and E_{-1} be the subspaces of E defined as follows:

$$\begin{aligned} E_1 &= \{u \in E \mid f(u) = u\} \\ E_{-1} &= \{u \in E \mid f(u) = -u\}. \end{aligned}$$

Prove that we have a direct sum

$$E = E_1 \oplus E_{-1}.$$

Hint. For every $u \in E$, write

$$u = \frac{u + f(u)}{2} + \frac{u - f(u)}{2}.$$

(3) If E is finite-dimensional and f is an involution, prove that there is some basis of E over which the matrix of f is of the form

$$I_{k,n-k} = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix (similarly for I_{n-k}) and $k = \dim(E_1)$.

Problem B3 (50 pts). A *rotation* R_θ in the plane $\mathbb{R}^2$ is given by the matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(1) Use `Matlab` to show the action of a rotation R_θ on a simple figure such as a triangle or a rectangle, for various values of θ, including $\theta = \pi/6, \pi/4, \pi/3, \pi/2$.

(2) Prove that R_θ is invertible and that its inverse is $R_{-\theta}$.

(3) For any two rotations R_α and R_β, prove that

$$R_\beta \circ R_\alpha = R_\alpha \circ R_\beta = R_{\alpha+\beta}.$$

Use (2)-(3) to prove that the rotations in the plane form a commutative group denoted $\mathbf{SO}(2)$.

Problem B4 (110 pts). Consider the affine map $R_{\theta,(a_1,a_2)}$ in $\mathbb{R}^2$ given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

(1) Prove that if $\theta \neq k2\pi$, with $k \in \mathbb{Z}$, then $R_{\theta,(a_1,a_2)}$ has a unique fixed point (c_1, c_2), that is, there is a unique point (c_1, c_2) such that

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = R_{\theta,(a_1,a_2)} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

and this fixed point is given by

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{2\sin(\theta/2)} \begin{pmatrix} \cos(\pi/2 - \theta/2) & -\sin(\pi/2 - \theta/2) \\ \sin(\pi/2 - \theta/2) & \cos(\pi/2 - \theta/2) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

(2) In this question, we still assume that $\theta \neq k2\pi$, with $k \in \mathbb{Z}$. By translating the coordinate system with origin $(0,0)$ to the new coordinate system with origin (c_1, c_2), which means that if (x_1, x_2) are the coordinates with respect to the standard origin $(0,0)$ and if (x'_1, x'_2) are the coordinates with respect to the new origin (c_1, c_2), we have

$$\begin{aligned} x_1 &= x'_1 + c_1 \\ x_2 &= x'_2 + c_2 \end{aligned}$$

and similarly for (y_1, y_2) and (y'_1, y'_2), then show that

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = R_{\theta,(a_1,a_2)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

becomes

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = R_\theta \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}.$$

Conclude that with respect to the new origin (c_1, c_2), the affine map $R_{\theta,(a_1,a_2)}$ becomes the rotation R_θ. We say that $R_{\theta,(a_1,a_2)}$ is a *rotation of center* (c_1, c_2).

(3) Use `Matlab` to show the action of the affine map $R_{\theta,(a_1,a_2)}$ on a simple figure such as a triangle or a rectangle, for $\theta = \pi/3$ and various values of (a_1, a_2). Display the center (c_1, c_2) of the rotation.

What kind of transformations correspond to $\theta = k2\pi$, with $k \in \mathbb{Z}$?

(4) Prove that the inverse of $R_{\theta,(a_1,a_2)}$ is of the form $R_{-\theta,(b_1,b_2)}$, and find (b_1, b_2) in terms of θ and (a_1, a_2).

(5) Given two affine maps $R_{\alpha,(a_1,a_2)}$ and $R_{\beta,(b_1,b_2)}$, prove that

$$R_{\beta,(b_1,b_2)} \circ R_{\alpha,(a_1,a_2)} = R_{\alpha+\beta,(t_1,t_2)}$$

for some (t_1, t_2), and find (t_1, t_2) in terms of β, (a_1, a_2) and (b_1, b_2).

Even in the case where $(a_1, a_2) = (0, 0)$, prove that in general

$$R_{\beta,(b_1,b_2)} \circ R_\alpha \neq R_\alpha \circ R_{\beta,(b_1,b_2)}.$$

Use (4)-(5) to show that the affine maps of the plane defined in this problem form a nonabelian group denoted $\mathbf{SE}(2)$.

Prove that if $R_{\beta,(b_1,b_2)} \circ R_{\alpha,(a_1,a_2)}$ is not the identity, then it is a rotation iff $\alpha + \beta \neq k2\pi$, with $k \in \mathbb{Z}$. Find its center of rotation when $(a_1, a_2) = (0, 0)$.

If $\alpha + \beta = k2\pi$, then $R_{\beta,(b_1,b_2)} \circ R_{\alpha,(a_1,a_2)}$ is a pure translation. Find the translation vector of $R_{\beta,(b_1,b_2)} \circ R_{\alpha,(a_1,a_2)}$.

Problem B5 (50 pts). A subset $\mathcal{A}$ of $\mathbb{R}^n$ is called an *affine subspace* if either $\mathcal{A} = \emptyset$, or there is some vector $a \in \mathbb{R}^n$ and some subspace U of $\mathbb{R}^n$ such that

$$\mathcal{A} = a + U = \{a + u \mid u \in U\}.$$

We define the dimension $\dim(\mathcal{A})$ of $\mathcal{A}$ as the dimension $\dim(U)$ of U.

(1) If $\mathcal{A} = a + U$, why is $a \in \mathcal{A}$?

What are affine subspaces of dimension 0? What are affine subspaces of dimension 1 (begin with $\mathbb{R}^2$)? What are affine subspaces of dimension 2 (begin with $\mathbb{R}^3$)?

Prove that any nonempty affine subspace is closed under affine combinations.

(2) Prove that if $\mathcal{A} = a + U$ is any nonempty affine subspace, then $\mathcal{A} = b + U$ for any $b \in \mathcal{A}$.

(3) Let $\mathcal{A}$ be any nonempty subset of $\mathbb{R}^n$ closed under affine combinations. For any $a \in \mathcal{A}$, prove that

$$U_a = \{x - a \in \mathbb{R}^n \mid x \in \mathcal{A}\}$$

is a (linear) subspace of $\mathbb{R}^n$ such that

$$\mathcal{A} = a + U_a.$$

Prove that U_a does not depend on the choice of $a \in \mathcal{A}$; that is, $U_a = U_b$ for all $a, b \in \mathcal{A}$. In fact, prove that

$$U_a = U = \{y - x \in \mathbb{R}^n \mid x, y \in \mathcal{A}\}, \quad \text{for all } a \in \mathcal{A},$$

and so

$$\mathcal{A} = a + U, \quad \text{for any } a \in \mathcal{A}.$$

Remark: The subspace U is called the *direction* of $\mathcal{A}$.

(4) Two nonempty affine subspaces $\mathcal{A}$ and $\mathcal{B}$ are said to be *parallel* iff they have the same direction. Prove that that if $\mathcal{A} \neq \mathcal{B}$ and $\mathcal{A}$ and $\mathcal{B}$ are parallel, then $\mathcal{A} \cap \mathcal{B} = \emptyset$.

Remark: The above shows that affine subspaces behave quite differently from linear subspaces.

Problem B6 (80 pts). Recall that an $n \times n$ matrix B is *skew symmetric* iff $B^\top = -B$, and denote the vector space of $n \times n$ symmetric matrices by $\mathfrak{so}(n)$. Also, the *orthogonal group* $\mathbf{O}(n)$ is the group of invertible matrices defined by

$$\mathbf{O}(n) = \{A \in \mathbf{M}_n \mid A^\top A = AA^\top = I\},$$

and the *special orthogonal group* $\mathbf{SO}(n)$ is the subgroup of $\mathbf{O}(n)$ (the rotations) given by

$$\mathbf{SO}(n) = \{A \in \mathbf{M}_n \mid A^\top A = AA^\top = I, \ \det(A) = 1\}.$$

Recall that the *special Euclidean group* $\mathbf{SE}(n)$ consists of all invertible affine maps (Q, u), with $Q \in \mathbf{SO}(n)$ and $u \in \mathbb{R}^n$. As usual, we represent an element (Q, u) of $\mathbf{SE}(n)$ by the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} Q & u \\ 0 & 1 \end{pmatrix},$$

with $\mathbb{R}^n$ embedded in $\mathbb{R}^{n+1}$ by adding 1 as $(n+1)$th coordinate.

(1) For any k and n such that $1 \leq k \leq n$, let $I_{k,n-k}$ be the matrix defined in Problem B2(3). Let $\sigma \colon \mathbf{SE}(n) \to \mathbf{SE}(n)$ be the map given by

$$\sigma \begin{pmatrix} Q & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_{k,n-k} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Q & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_{k,n-k} & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} Q & z \\ 0 & 1 \end{pmatrix} \in \mathbf{SE}(n).$$

Prove that $\sigma^2 = \mathrm{id}$, and that σ is a group homomorphism (that is, $\sigma((Q,u)(R,v)) = \sigma(Q,u)\sigma(R,v)$, for all $(Q,u),(R,v) \in \mathbf{SE}(n)$).

(2) The *subgroup* $\mathbf{SE}(n)^\sigma$ *fixed by* σ is defined by

$$\mathbf{SE}(n)^\sigma = \{(Q,u) \in \mathbf{SE}(n) \mid \sigma(Q,u) = (Q,u)\}.$$

Prove that

$$\mathbf{SE}(n)^\sigma = \left\{ \begin{pmatrix} Q & 0 & u \\ 0 & R & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; Q \in \mathbf{O}(k),\ R \in \mathbf{O}(n-k),\ \det(Q)\det(R) = 1,\ u \in \mathbb{R}^k \right\}.$$

(3) Let $\mathfrak{se}(n)$ be the following vector space

$$\mathfrak{se}(n) = \left\{ \begin{pmatrix} S & -A^\top & u \\ A & T & v \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; S \in \mathfrak{so}(k),\ T \in \mathfrak{so}(n-k),\ A \in \mathrm{M}_{n-k,k},\ u \in \mathbb{R}^k,\ v \in \mathbb{R}^{n-k} \right\}.$$

Are the matrices in $\mathfrak{se}(n)$ skew-symmetric? If not, give a necessary and sufficient condition for such matrices to be skew-symmetric.

By analogy with (2), define the map $\theta\colon \mathfrak{se}(n) \to \mathfrak{se}(n)$ by

$$\theta(X) = \begin{pmatrix} I_{k,n-k} & 0 \\ 0 & 1 \end{pmatrix} X \begin{pmatrix} I_{k,n-k} & 0 \\ 0 & 1 \end{pmatrix}, \quad X \in \mathfrak{se}(n).$$

Prove that θ is a linear involution of $\mathfrak{se}(n)$. Prove that the subspaces

$$\mathfrak{h} = \{X \in \mathfrak{se}(n) \mid \theta(X) = X\}$$
$$\mathfrak{m} = \{X \in \mathfrak{se}(n) \mid \theta(X) = -X\}$$

are given by

$$\mathfrak{h} = \left\{ \begin{pmatrix} S & 0 & u \\ 0 & T & 0 \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; S \in \mathfrak{so}(k),\ T \in \mathfrak{so}(n-k),\ u \in \mathbb{R}^k \right\}$$

and

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -A^\top & 0 \\ A & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; A \in \mathrm{M}_{n-k,k},\ v \in \mathbb{R}^{n-k} \right\}.$$

(4) Prove (very quickly) that

$$\mathfrak{se}(n) = \mathfrak{h} \oplus \mathfrak{m}.$$

Prove that $\dim(\mathfrak{m}) = (k+1)(n-k)$.

Remark: This is the dimension of the *affine Grassmannian* $AG(k,n)$, the set of all k-dimensional affine subspaces of $\mathbb{R}^n$.

TOTAL: 360 points.