Spring, 2012 CIS 515

Fundamentals of Linear Algebra and Optimization
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Homework 3

February 28, 2012; Due March 13, 2012

Problem B1 (20 pts). (1) Given two vectors in $\mathbb{R}^2$ of coordinates $(c_1 - a_1, c_2 - a_2)$ and $(b_1 - a_1, b_2 - a_2)$, prove that they are linearly dependent iff

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

(2) Given three vectors in $\mathbb{R}^3$ of coordinates $(d_1-a_1, d_2-a_2, d_3-a_3)$, $(c_1-a_1, c_2-a_2, c_3-a_3)$, and $(b_1 - a_1, b_2 - a_2, b_3 - a_3)$, prove that they are linearly dependent iff

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

Problem B2 (10 pts). If A is an $n \times n$ symmetric matrix and B is any $n \times n$ invertible matrix, prove that A is positive definite iff $B^\top AB$ is positive definite.

Problem B3 (80 pts). (1) Let A be any invertible 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Prove that there is an invertible matrix S such that

$$SA = \begin{pmatrix} 1 & 0 \\ 0 & ad - bc \end{pmatrix},$$

where S is the product of at most four elementary matrices of the form $E_{i,j;\beta}$.

Conclude that every matrix A in $\mathbf{SL}(2)$ (the group of invertible 2×2 matrices A with $\det(A) = +1$) is the product of at most four elementary matrices of the form $E_{i,j;\beta}$.

For any $a \neq 0, 1$, give an explicit factorization as above for

$$A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

What is this decomposition for $a = -1$?

(2) Recall that a rotation matrix R (a member of the group $\mathbf{SO}(2)$) is a matrix of the form

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Prove that if $\theta \neq k\pi$ (with $k \in \mathbb{Z}$), any rotation matrix can be written as a product

$$R = ULU,$$

where U is upper triangular and L is lower triangular of the form

$$U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}.$$

Therefore, every plane rotation (except a flip about the origin when $\theta = \pi$) can be written as the composition of three shear transformations!

(3) Recall that $E_{i,d}$ is the diagonal matrix

$$E_{i,d} = \mathrm{diag}(1, \ldots, 1, d, 1, \ldots, 1),$$

whose diagonal entries are all $+1$, except the (i,i)th entry which is equal to d.

Given any $n \times n$ matrix A, for any pair (i,j) of distinct row indices ($1 \leq i, j \leq n$), prove that there exist two elementary matrices $E_1(i,j)$ and $E_2(i,j)$ of the form $E_{k,\ell;\beta}$, such that

$$E_{j,-1}E_1(i,j)E_2(i,j)E_1(i,j)A = P(i,j)A,$$

the matrix obtained from the matrix A by permuting row i and row j. Equivalently, we have

$$E_1(i,j)E_2(i,j)E_1(i,j)A = E_{j,-1}P(i,j)A,$$

the matrix obtained from A by permuting row i and row j and multiplying row j by -1.

Prove that for every $i = 2, \ldots, n$, there exist four elementary matrices $E_3(i,d), E_4(i,d), E_5(i,d), E_6(i,d)$ of the form $E_{k,\ell;\beta}$, such that

$$E_6(i)E_5(i)E_4(i)E_3(i)E_{n,d} = E_{i,d}.$$

What happens when $d = -1$, that is, what kinds of simplifications occur?

Prove that all permutation matrices can be written as products of elementary operations of the form $E_{k,\ell;\beta}$ and the operation $E_{n,-1}$.

(4) Prove that for every invertible $n \times n$ matrix A, there is a matrix S such that

$$SA = \begin{pmatrix} I_{n-1} & 0 \\ 0 & d \end{pmatrix} = E_{n,d},$$

with $d = \det(A)$, and where S is a product of elementary matrices of the form $E_{k,\ell;\beta}$.

In particular, every matrix in $\mathbf{SL}(n)$ (the group of invertible $n \times n$ matrices A with $\det(A) = +1$) can be written as a product of elementary matrices of the form $E_{k,\ell;\beta}$. Prove that at most $n(n+1) - 2$ such transformations are needed.

Problem B4 (40 pts). A matrix, A, is called *strictly column diagonally dominant* iff

$$|a_{j\,j}| > \sum_{i=1,\, i \neq j}^{n} |a_{i\,j}|, \quad \text{for } j = 1, \ldots, n$$

Prove that if A is strictly column diagonally dominant, then Gaussian elimination does not require pivoting and A is invertible.

Problem B5 (40 pts). Let $(\alpha_1, \ldots, \alpha_{m+1})$ be a sequence of pairwise distinct scalars in $\mathbb{R}$ and let $(\beta_1, \ldots, \beta_{m+1})$ be any sequence of scalars in $\mathbb{R}$, not necessarily distinct.

(1) Prove that there is a unique polynomial P of degree at most m such that

$$P(\alpha_i) = \beta_i, \quad 1 \leq i \leq m+1.$$

Hint. Remember Vandermonde!

(2) Let $L_i(X)$ be the polynomial of degree m given by

$$L_i(X) = \frac{(X - \alpha_1) \cdots (X - \alpha_{i-1})(X - \alpha_{i+1}) \cdots (X - \alpha_{m+1})}{(\alpha_i - \alpha_1) \cdots (\alpha_i - \alpha_{i-1})(\alpha_i - \alpha_{i+1}) \cdots (\alpha_i - \alpha_{m+1})}, \quad 1 \leq i \leq m+1.$$

The polynomials $L_i(X)$ are known as *Lagrange polynomial interpolants*. Prove that

$$L_i(\alpha_j) = \delta_{i\,j} \quad 1 \leq i, j \leq m+1.$$

Prove that

$$P(X) = \beta_1 L_1(X) + \cdots + \beta_{m+1} L_{m+1}(X)$$

is the unique polynomial of degree at most m such that

$$P(\alpha_i) = \beta_i, \quad 1 \leq i \leq m+1.$$

(3) Prove that $L_1(X), \ldots, L_{m+1}(X)$ are lineary independent, and that they form a basis of all polynomials of degree at most m.

How is 1 (the constant polynomial 1) expressed over the basis $(L_1(X), \ldots, L_{m+1}(X))$?

Give the expression of every polynomial $P(X)$ of degree at most m over the basis $(L_1(X), \ldots, L_{m+1}(X))$.

(4) Prove that the dual basis $(L_1^*, \ldots, L_{m+1}^*)$ of the basis $(L_1(X), \ldots, L_{m+1}(X))$ consists of the linear forms L_i^* given by

$$L_i^*(P) = P(\alpha_i),$$

for every polynomial P of degree at most m; this is simply *evaluation at* α_i.

Problem B6 (30 pts). Consider the $n \times n$ symmetric matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & \dots & 0 & 0 \\ 2 & 5 & 2 & 0 & \dots & 0 & 0 \\ 0 & 2 & 5 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2 & 5 & 2 & 0 \\ 0 & 0 & \dots & 0 & 2 & 5 & 2 \\ 0 & 0 & \dots & 0 & 0 & 2 & 5 \end{pmatrix}.$$

(1) Find an upper-triangular matrix R such that $A = R^\top R$.

(2) Prove that $\det(A) = 1$.

(3) Consider the sequence

$$\begin{aligned} &p_0(\lambda) = 1 \\ &p_1(\lambda) = 1 - \lambda \\ &p_k(\lambda) = (5 - \lambda)p_{k-1}(\lambda) - 4p_{k-2}(\lambda) \quad 2 \le k \le n. \end{aligned}$$

Prove that

$$\det(A - \lambda I) = p_n(\lambda).$$

Remark: It can be shown that $p_n(\lambda)$ has n distinct (real) roots and that the roots of $p_k(\lambda)$ separate the roots of $p_{k+1}(\lambda)$.

TOTAL: 220 points.