$\boxed{\textbf{Spring 2012} \quad \textbf{CIS 515}}$

Fundamentals of Linear Algebra and Optimization
Jean Gallier

Homework 2

February, 9 2012; Due February 28, 2012

Problem B1 (10 pts). Given any $m \times n$ matrix A and any $n \times p$ matrix B, if we denote the columns of A by $A_1, \ldots, A_n$ and the rows of B by $(B_1^\top)^\top, \ldots, (B_n^\top)^\top$, prove that

$$AB = A_1(B_1^\top)^\top + \cdots + A_n(B_n^\top)^\top.$$

Problem B2 (10 pts). Let $f \colon E \to F$ be a linear map which is also a bijection (it is injective and surjective). Prove that the inverse function $f^{-1} \colon F \to E$ is linear.

Problem B3 (40 pts). Let $U_1, \ldots, U_p$ be any $p \geq 2$ subspaces of some vector space E and recall that the linear map

$$a \colon U_1 \times \cdots \times U_p \to E$$

is given by

$$a(u_1, \ldots, u_p) = u_1 + \cdots + u_p,$$

with $u_i \in U_i$ for $i = 1, \ldots, p$.

(1) If we let $Z_i \subseteq U_1 \times \cdots \times U_p$ be given by

$$Z_i = \left\{ \left(u_1, \ldots, u_{i-1}, -\sum_{j=1, j\neq i}^{p} u_j, u_{i+1}, \ldots, u_p\right) \mid u_j \in U_j, j \in \{1, \ldots, p\} - \{i\} \right\}$$

for $i = 1, \ldots, p$, then prove that

$$\operatorname{Ker} a = Z_1 + \cdots + Z_p.$$

(2) Prove that $U_1 + \cdots + U_p$ is a direct sum iff

$$U_i \cap \left(\sum_{j=1, j\neq i}^{p} U_j \right) = (0) \quad 1 \leq i \leq p.$$

(3) Let $f_i \colon E \to E$ be any $p \geq 2$ linear maps such that

$$f_1 + \cdots + f_p = \mathrm{id}_E.$$

Prove that the following properties are equivalent:

(1) $f_i^2 = f_i$, $1 \leq i \leq p$.

(2) $f_j \circ f_i = 0$, for all $i \neq j$, $1 \leq i, j \leq p$.

Problem B4 (20 pts). Prove that for every vector space E, if $f \colon E \to E$ is an idempotent linear map, i.e., $f \circ f = f$, then we have a direct sum

$$E = \mathrm{Ker}\, f \oplus \mathrm{Im}\, f,$$

so that f is the projection onto its image $\mathrm{Im}\, f$.

Problem B5 (10 pts). Given any $n \times n$ matrix A, prove that multiplying A by the elementary matrix $E_{i,j;\beta}$ *on the right* yields the matrix $AE_{i,j;\beta}$ in which β times the ith column is added to the jth column.

Problem B6 (50 pts). (a) Find an upper triangular matrix E such that

$$E \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

(b) What is the effect of the product (on the left) with

$$E_{4,3;-1}E_{3,2;-1}E_{4,3;-1}E_{2,1;-1}E_{3,2;-1}E_{4,3;-1}$$

on the matrix

$$Pa_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}.$$

(c) Find the inverse of the matrix Pa_3.

(d) Consider the $(n+1) \times (n+1)$ Pascal matrix Pa_n whose ith row is given by the binomial coefficients

$$\binom{i-1}{j-1},$$

with $1 \leq i \leq n+1$, $1 \leq j \leq n+1$, and with the usual convention that

$$\binom{0}{0} = 1, \quad \binom{i}{j} = 0 \quad \text{if} \quad j > i.$$

The matrix Pa_3 is shown in question (c) and Pa_4 is shown below:

$$Pa_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}.$$

Find n elementary matrices $E_{i_k,j_k;\beta_k}$ such that

$$E_{i_n,j_n;\beta_n}\cdots E_{i_1,j_1;\beta_1}Pa_n = \begin{pmatrix} 1 & 0 \\ 0 & Pa_{n-1}\end{pmatrix}.$$

Use the above to prove that the inverse of Pa_n is the lower triangular matrix whose ith row is given by the signed binomial coefficients

$$(-1)^{i+j-2}\binom{i-1}{j-1},$$

with $1 \leq i \leq n+1$, $1 \leq j \leq n+1$. For example,

$$Pa_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

Problem B7 (10 pts). Let $f\colon E \to F$ be a linear map with $\dim(E) = n$ and $\dim(F) = m$. Prove that f has rank 1 iff f is represented by an $m \times n$ matrix of the form

$$A = uv^\top$$

with u a nonzero column vector of dimension m and v a nonzero column vector of dimension n.

TOTAL: 150 points.