Chapter 9
The Quaternions and the Spaces S^3, $\mathbf{SU}(2)$, $\mathbf{SO}(3)$, and $\mathbb{RP}^3$

9.1 The Algebra $\mathbb{H}$ of Quaternions

In this chapter, we discuss the representation of rotations of $\mathbb{R}^3$ in terms of quaternions. Such a representation is not only concise and elegant, it also yields a very efficient way of handling composition of rotations. It also tends to be numerically more stable than the representation in terms of orthogonal matrices.

The group of rotations $\mathbf{SO}(2)$ is isomorphic to the group $\mathbf{U}(1)$ of complex numbers $e^{i\theta} = \cos\theta + i\sin\theta$ of unit length. This follows immediately from the fact that the map

$$e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is a group isomorphism. Geometrically, observe that $\mathbf{U}(1)$ is the unit circle S^1. We can identify the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, letting $z = x + iy \in \mathbb{C}$ represent $(x, y) \in \mathbb{R}^2$. Then every plane rotation ρ_θ by an angle θ is represented by multiplication by the complex number $e^{i\theta} \in \mathbf{U}(1)$, in the sense that for all $z, z' \in \mathbb{C}$,

$$z' = \rho_\theta(z) \quad \text{iff} \quad z' = e^{i\theta} z.$$

In some sense, the quaternions generalize the complex numbers in such a way that rotations of $\mathbb{R}^3$ are represented by multiplication by quaternions of unit length. This is basically true with some twists. For instance, quaternion multiplication is not commutative, and a rotation in $\mathbf{SO}(3)$ requires conjugation with a quaternion for its representation. Instead of the unit circle S^1, we need to consider the sphere S^3 in $\mathbb{R}^4$, and $\mathbf{U}(1)$ is replaced by $\mathbf{SU}(2)$.

Recall that the 3-sphere S^3 is the set of points $(x, y, z, t) \in \mathbb{R}^4$ such that

$$x^2 + y^2 + z^2 + t^2 = 1,$$

and that the real projective space $\mathbb{RP}^3$ is the quotient of S^3 modulo the equivalence relation that identifies antipodal points (where (x, y, z, t) and $(-x, -y, -z, -t)$ are antipodal points). The group $\mathbf{SO}(3)$ of rotations of $\mathbb{R}^3$ is intimately related to the 3-sphere S^3 and to the real projective space $\mathbb{RP}^3$. The key to this relationship is the fact that rotations can be represented by quaternions, discovered by Hamilton in 1843. Historically, the quaternions were the first instance of a skew field. As we shall see, quaternions represent rotations in $\mathbb{R}^3$ very concisely.

It will be convenient to define the quaternions as certain 2×2 complex matrices. We write a complex number z as $z = a + \mathrm{i}b$, where $a, b \in \mathbb{R}$, and the *conjugate* $\overline{z}$ of z is $\overline{z} = a - \mathrm{i}b$. Let $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ be the following matrices:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{i} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix},$$
$$\mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{k} = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

Definition 9.1. Let $\mathbb{H}$ be the set of all matrices of the form

$$a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k},$$

where $(a, b, c, d) \in \mathbb{R}^4$. Thus, every matrix in $\mathbb{H}$ is of the form

$$A = \begin{pmatrix} x & y \\ -\overline{y} & \overline{x} \end{pmatrix},$$

where $x = a + \mathrm{i}b$ and $y = c + \mathrm{i}d$. The matrices in $\mathbb{H}$ are called *quaternions*. The null quaternion is denoted by 0 (or $\mathbf{0}$, if confusion may arise). Quaternions of the form $b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ are called *pure quaternions*. The set of pure quaternions is denoted by $\mathbb{H}_p$.

Note that the rows (and columns) of matrices in $\mathbb{H}$ are vectors in $\mathbb{C}^2$ that are orthogonal with respect to the Hermitian inner product of $\mathbb{C}^2$ given by

$$(x_1, y_1) \cdot (x_2, y_2) = x_1\overline{x_2} + y_1\overline{y_2}.$$

Furthermore, their norm is

$$\sqrt{x\overline{x} + y\overline{y}} = \sqrt{a^2 + b^2 + c^2 + d^2},$$

and the determinant of A is $a^2 + b^2 + c^2 + d^2$.

It is easily seen that the following famous identities (discovered by Hamilton) hold:

$$\begin{aligned}
\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} &= -\mathbf{1}, \\
\mathbf{ij} = -\mathbf{ji} &= \mathbf{k}, \\
\mathbf{jk} = -\mathbf{kj} &= \mathbf{i}, \\
\mathbf{ki} = -\mathbf{ik} &= \mathbf{j}.
\end{aligned}$$

Using these identities, it can be verified that $\mathbb{H}$ is a ring (with multiplicative identity $\mathbf{1}$) and a real vector space of dimension 4 with basis $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$. In fact, the quaternions form an associative algebra. For details, see Berger [3], Veblen and Young [22], Dieudonné [5], Bertin [4].

The quaternions $\mathbb{H}$ are often defined as the real algebra generated by the four elements $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and satisfying the identities just stated above. The problem with such a definition is that it is not obvious that the algebraic structure $\mathbb{H}$ actually exists. A rigorous justification requires the notions of freely generated algebra and of quotient of an algebra by an ideal. Our definition in terms of matrices makes the existence of $\mathbb{H}$ trivial (but requires showing that the identities hold, which is an easy matter).

Given any two quaternions $X = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ and $Y = a'\mathbf{1} + b'\mathbf{i} + c'\mathbf{j} + d'\mathbf{k}$, it can be verified that

$$\begin{aligned} XY = {} & (aa' - bb' - cc' - dd')\mathbf{1} + (ab' + ba' + cd' - dc')\mathbf{i} \\ & + (ac' + ca' + db' - bd')\mathbf{j} + (ad' + da' + bc' - cb')\mathbf{k}. \end{aligned}$$

It is worth noting that these formulae were discovered independently by Olinde Rodrigues in 1840, a few years before Hamilton (Veblen and Young [22]). However, Rodrigues was working with a different formalism, homogeneous transformations, and he did not discover the quaternions. The map from $\mathbb{R}$ to $\mathbb{H}$ defined such that $a \mapsto a\mathbf{1}$ is an injection that allows us to view $\mathbb{R}$ as a subring $\mathbb{R}\mathbf{1}$ (in fact, a field) of $\mathbb{H}$. Similarly, the map from $\mathbb{R}^3$ to $\mathbb{H}$ defined such that $(b, c, d) \mapsto b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is an injection that allows us to view $\mathbb{R}^3$ as a subspace of $\mathbb{H}$, in fact, the hyperplane $\mathbb{H}_p$.

Given a quaternion $X = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, we define its *conjugate* $\overline{X}$ as

$$\overline{X} = a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}.$$

It is easily verified that

$$X\overline{X} = (a^2 + b^2 + c^2 + d^2)\mathbf{1}.$$

The quantity $a^2 + b^2 + c^2 + d^2$, also denoted by $N(X)$, is called the *reduced norm* of X.

Clearly, X is nonnull iff $N(X) \neq 0$, in which case $\overline{X}/N(X)$ is the multiplicative inverse of X. Thus, $\mathbb{H}$ is a skew field. Since $X + \overline{X} = 2a\mathbf{1}$, we also call $2a$ the *reduced trace* of X, and we denote it by $\mathrm{Tr}(X)$. A quaternion X is a pure quaternion iff $\overline{X} = -X$ iff $\mathrm{Tr}(X) = 0$.

The following identities can be shown (see Berger [3], Dieudonné [5], Bertin [4]):

$$\begin{aligned}
\overline{XY} &= \overline{Y}\,\overline{X},\\
\mathrm{Tr}(XY) &= \mathrm{Tr}(YX),\\
N(XY) &= N(X)N(Y),\\
\mathrm{Tr}(ZXZ^{-1}) &= \mathrm{Tr}(X),
\end{aligned}$$

whenever $Z \neq 0$.

If $X = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ and $Y = b'\mathbf{i} + c'\mathbf{j} + d'\mathbf{k}$ are pure quaternions, identifying X and Y with the corresponding vectors in $\mathbb{R}^3$, the inner product $X \cdot Y$ and the cross product $X \times Y$ make sense, and letting $[0, X \times Y]$ denote the quaternion whose first component is 0 and whose last three components are those of $X \times Y$, we have the remarkable identity

$$XY = -(X \cdot Y)\mathbf{1} + [0, X \times Y].$$

More generally, given a quaternion $X = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, we can write it as

$$X = [a, (b, c, d)],$$

where a is called the *scalar part* of X and (b, c, d) the *pure part* of X. Then, if $X = [a, U]$ and $Y = [a', U']$, it is easily seen that the quaternion product XY can be expressed as

$$XY = [aa' - U \cdot U',\, aU' + a'U + U \times U'].$$

The above formula for quaternion multiplication allows us to show the following fact. Let $Z \in \mathbb{H}$, and assume that $ZX = XZ$ for all $X \in \mathbb{H}$. We claim that the pure part of Z is null, i.e., $Z = a\mathbf{1}$ for some $a \in \mathbb{R}$. Indeed, writing $Z = [a, U]$, if $U \neq 0$, there is at least one nonnull pure quaternion $X = [0, V]$ such that $U \times V \neq 0$ (for example, take any nonnull vector V in the orthogonal complement of U). Then

$$ZX = [-U \cdot V, aV + U \times V], \quad XZ = [-V \cdot U, aV + V \times U],$$

and since $V \times U = -(U \times V)$ and $U \times V \neq 0$, we have $XZ \neq ZX$, a contradiction. Conversely, it is trivial that if $Z = [a, 0]$, then $XZ = ZX$ for all $X \in \mathbb{H}$. Thus, the set of quaternions that commute with all quaternions is $\mathbb{R}\mathbf{1}$.

Remark: It is easy to check that for arbitrary quaternions $X = [a, U]$ and $Y = [a', U']$,

$$XY - YX = [0, 2(U \times U')],$$

and that for pure quaternions $X, Y \in \mathbb{H}_p$,

$$2(X \cdot Y)\mathbf{1} = -(XY + YX).$$

Since quaternion multiplication is bilinear, for a given X, the map $Y \mapsto XY$ is linear, and similarly for a given Y, the map $X \mapsto XY$ is linear. It is immediate that if the matrix of the first map is L_X and the matrix of the second map is R_Y, then

$$XY = L_X Y = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix}$$

and

$$XY = R_Y X = \begin{pmatrix} a' & -b' & -c' & -d' \\ b' & a' & d' & -c' \\ c' & -d' & a' & b' \\ d' & c' & -b' & a' \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Observe that the columns (and the rows) of the above matrices are orthogonal. Thus, when X and Y are unit quaternions, both L_X and R_Y are orthogonal matrices. Furthermore, it is obvious that $L_{\overline{X}} = L_X^\top$, the transpose of L_X, and similarly, $R_{\overline{Y}} = R_Y^\top$. Since $X\overline{X} = N(X)$, the matrix $L_X L_X^\top$ is the diagonal matrix $N(X)I$ (where I is the identity 4×4 matrix), and similarly the matrix $R_Y R_Y^\top$ is the diagonal matrix $N(Y)I$. Since L_X and $L_X^\top$ have the same determinant, we deduce that $\det(L_X)^2 = N(X)^4$, and thus $\det(L_X) = \pm N(X)^2$. However, it is obvious that one of the terms in $\det(L_X)$ is a^4, and thus

$$\det(L_X) = (a^2 + b^2 + c^2 + d^2)^2.$$

This shows that when X is a unit quaternion, L_X is a rotation matrix, and similarly when Y is a unit quaternion, R_Y is a rotation matrix (see Veblen and Young [22]).

Define the map $\varphi\colon \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ as follows:

$$\varphi(X,Y) = \frac{1}{2}\operatorname{Tr}(X\overline{Y}) = aa' + bb' + cc' + dd'.$$

It is easily verified that φ is bilinear, symmetric, and definite positive. Thus, the quaternions form a Euclidean space under the inner product defined by φ (see Berger [3], Dieudonné [5], Bertin [4]).

It is immediate that under this inner product, the norm of a quaternion X is just $\sqrt{N(X)}$. As a Euclidean space, $\mathbb{H}$ is isomorphic to $\mathbb{E}^4$. It is also immediate that the subspace $\mathbb{H}_p$ of pure quaternions is orthogonal to the space of "real quaternions" $\mathbb{R}\mathbf{1}$. The subspace $\mathbb{H}_p$ of pure quaternions inherits a Euclidean structure, and this subspace is isomorphic to the Euclidean space $\mathbb{E}^3$. Since $\mathbb{H}$ and $\mathbb{E}^4$ are isomorphic Euclidean spaces, their groups of rotations $\mathbf{SO}(\mathbb{H})$ and $\mathbf{SO}(4)$ are isomorphic, and we will identify them. Similarly, we will identify $\mathbf{SO}(\mathbb{H}_p)$ and $\mathbf{SO}(3)$.

9.2 Quaternions and Rotations in $\mathbf{SO}(3)$

We have just observed that for any nonnull quaternion X, both maps $Y \mapsto XY$ and $Y \mapsto YX$ (where $Y \in \mathbb{H}$) are linear maps, and that when $N(X) = 1$, these linear maps are in $\mathbf{SO}(4)$. This suggests looking at maps $\rho_{Y,Z}\colon \mathbb{H} \to \mathbb{H}$ of the form $X \mapsto YXZ$, where $Y,Z \in \mathbb{H}$ are any two fixed nonnull quaternions such that $N(Y)N(Z) = 1$. Since $N(Y)N(Z) = 1$, in view of the identity $N(UV) = N(U)N(V)$ for all $U,V \in \mathbb{H}$, we have

$$\begin{aligned}\rho_{Y,Z}(X) &= YXZ = (\sqrt{N(Y)}(Y/\sqrt{N(Y)}))X(\sqrt{N(Z)}(Z/\sqrt{N(Z)}))\\ &= \sqrt{N(Y)N(Z)}(Y/\sqrt{N(Y)})X(Z/\sqrt{N(Z)}) = (Y/\sqrt{N(Y)})X(Z/\sqrt{N(Z)}),\end{aligned}$$

so

$$\rho_{Y,Z} = (\rho_{Y/\sqrt{N(Y)},\mathbf{1}}) \circ (\rho_{\mathbf{1},Z/\sqrt{N(Z)}}).$$

Since $\rho_{Y/\sqrt{N(Y)},\mathbf{1}}$ is the map $X \mapsto (Y/\sqrt{N(Y)})X$ and $\rho_{\mathbf{1},Z/\sqrt{N(Z)}}$ is the map $X \mapsto X(Z/\sqrt{N(Z)}$, which are both rotations since $Y/\sqrt{N(Y)}$ and $Z/\sqrt{N(Z)}$ are unit quaternions, $\rho_{Y,Z}$ itself is a rotation, i.e., $\rho_{Y,Z} \in \mathbf{SO}(4)$. We will prove that every rotation in $\mathbf{SO}(4)$ arises in this fashion.

When $Z = Y^{-1}$, the map $\rho_{Y,Y^{-1}}$ is denoted more simply by ρ_Y. In this case, it is easy to check that ρ_Y is the identity on $\mathbf{1}\mathbb{R}$, and maps $\mathbb{H}_p$ into itself. Indeed (renaming Y as Z), observe that

$$\rho_Z(X+Y) = \rho_Z(X) + \rho_Z(Y).$$

It is also easy to check that

$$\rho_Z(\overline{X}) = \overline{\rho_Z(X)}.$$

Then we have

$$\rho_Z(X+\overline{X}) = \rho_Z(X) + \rho_Z(\overline{X}) = \rho_Z(X) + \overline{\rho_Z(X)},$$

and since if $X = [a,U]$, then $X + \overline{X} = 2a\mathbf{1}$, where a is the real part of X, if X is pure, i.e., $X + \overline{X} = 0$, then $\rho_Z(X) + \overline{\rho_Z(X)} = 0$, i.e., $\rho_Z(X)$ is also pure. Thus, $\rho_Z \in \mathbf{SO}(3)$, i.e., ρ_Z is a rotation of $\mathbb{E}^3$. We will prove that every rotation in $\mathbf{SO}(3)$ arises in this fashion.

Remark: If a bijective map $\rho\colon \mathbb{H} \to \mathbb{H}$ satisfies the three conditions

$$\begin{aligned}\rho(X+Y) &= \rho(X) + \rho(Y),\\ \rho(\lambda X) &= \lambda\rho(X),\\ \rho(XY) &= \rho(X)\rho(Y),\end{aligned}$$

for all quaternions $X,Y \in \mathbb{H}$ and all $\lambda \in \mathbb{R}$, i.e., ρ is a linear automorphism of $\mathbb{H}$, it can be shown that $\rho(\overline{X}) = \overline{\rho(X)}$ and $N(\rho(X)) = N(X)$. In fact, ρ must be of the form ρ_Z for some nonnull $Z \in \mathbb{H}$.

The quaternions of norm 1, also called *unit quaternions*, are in bijection with points of the real 3-sphere S^3. It is easy to verify that the unit quaternions form a subgroup of the multiplicative group $\mathbb{H}^*$ of nonnull quaternions. In terms of complex matrices, the unit quaternions correspond to the group of unitary complex 2×2

matrices of determinant 1 (i.e., $x\overline{x}+y\overline{y}=1$),

$$A=\begin{pmatrix} x & y \\ -\overline{y} & \overline{x} \end{pmatrix},$$

with respect to the Hermitian inner product in $\mathbb{C}^2$. This group is denoted by $\mathbf{SU}(2)$. The obvious bijection between $\mathbf{SU}(2)$ and S^3 is in fact a homeomorphism, and it can be used to transfer the group structure on $\mathbf{SU}(2)$ to S^3, which becomes a topological group isomorphic to the topological group $\mathbf{SU}(2)$ of unit quaternions. Incidentally, it is easy to see that the group $\mathbf{U}(2)$ of all unitary complex 2×2 matrices consists of all matrices of the form

$$A=\begin{pmatrix} \lambda x & y \\ -\lambda\overline{y} & \overline{x} \end{pmatrix},$$

with $x\overline{x}+y\overline{y}=1$, and where λ is a complex number of modulus 1 ($\lambda\overline{\lambda}=1$). It should also be noted that the fact that the sphere S^3 has a group structure is quite exceptional. As a matter of fact, the only spheres for which a continuous group structure is definable are S^1 and S^3. The algebraic structure of the groups $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$, and their relationship to S^3, is explained very clearly in Chapter 8 of Artin [1], which we highly recommend as a general reference on algebra.

One of the most important properties of the quaternions is that they can be used to represent rotations of $\mathbb{R}^3$, as stated in the following lemma. Our proof is inspired by Berger [3], Dieudonné [5], and Bertin [4].

Lemma 9.1. *For every quaternion $Z\neq 0$, the map*

$$\rho_Z\colon X\mapsto ZXZ^{-1}$$

(where $X\in\mathbb{H}$) is a rotation in $\mathbf{SO}(\mathbb{H})=\mathbf{SO}(4)$ whose restriction to the space $\mathbb{H}_p$ of pure quaternions is a rotation in $\mathbf{SO}(\mathbb{H}_p)=\mathbf{SO}(3)$. Conversely, every rotation in $\mathbf{SO}(3)$ is of the form

$$\rho_Z\colon X\mapsto ZXZ^{-1},$$

for some quaternion $Z\neq 0$ and for all $X\in\mathbb{H}_p$. Furthermore, if two nonnull quaternions Z and Z' represent the same rotation, then $Z'=\lambda Z$ for some $\lambda\neq 0$ in $\mathbb{R}$.

Proof. We have already observed that $\rho_Z\in\mathbf{SO}(3)$. We have to prove that every rotation is of the form ρ_Z. First, it is easily seen that

$$\rho_{YX}=\rho_Y\circ\rho_X.$$

By Theorem 8.1, every rotation that is not the identity is the composition of an even number of reflections (in the three-dimensional case, two reflections), and thus it is enough to show that for every reflection σ of $\mathbb{H}_p$ about a plane H, there is some pure quaternion $Z\neq 0$ such that $\sigma(X)=-ZXZ^{-1}$ for all $X\in\mathbb{H}_p$. If Z is a pure quaternion orthogonal to the plane H, we know that

$$\sigma(X)=X-2\frac{(X\cdot Z)}{(Z\cdot Z)}Z$$

for all $X \in \mathbb{H}_p$. However, for pure quaternions $Y, Z \in \mathbb{H}_p$, we have

$$2(Y \cdot Z)\mathbf{1} = -(YZ + ZY).$$

Then $(Z \cdot Z)\mathbf{1} = -Z^2$, and we have

$$\begin{aligned}\sigma(X) &= X - 2\frac{(X \cdot Z)}{(Z \cdot Z)}Z = X + 2(X \cdot Z)Z^{-1} \\ &= X - (XZ + ZX)Z^{-1} = -ZXZ^{-1},\end{aligned}$$

which shows that $\sigma(X) = -ZXZ^{-1}$ for all $X \in \mathbb{H}_p$, as desired.

If $\rho_{Z_1} = \rho_{Z_2}$, then

$$Z_1 X Z_1^{-1} = Z_2 X Z_2^{-1}$$

for all $X \in \mathbb{H}$, which is equivalent to

$$Z_2^{-1} Z_1 X = X Z_2^{-1} Z_1$$

for all $X \in \mathbb{H}$. However, we showed earlier that $Z_2^{-1} Z_1 = a\mathbf{1}$ for some $a \in \mathbb{R}$, and since Z_1 and Z_2 are nonnull, we get $Z_2 = (1/a)Z_1$, where $a \neq 0$. $\square$

As a corollary of

$$\rho_{YX} = \rho_Y \circ \rho_X,$$

it is easy to show that the map $\rho\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ defined such that $\rho(Z) = \rho_Z$ is a surjective and continuous homomorphism whose kernel is $\{\mathbf{1}, -\mathbf{1}\}$. Since $\mathbf{SU}(2)$ and S^3 are homeomorphic as topological spaces, this shows that $\mathbf{SO}(3)$ is homeomorphic to the quotient of the sphere S^3 modulo the antipodal map. But the real projective space $\mathbb{RP}^3$ is defined precisely this way in terms of the antipodal map $\pi\colon S^3 \to \mathbb{RP}^3$, and thus $\mathbf{SO}(3)$ and $\mathbb{RP}^3$ are homeomorphic. This homeomorphism can then be used to transfer the group structure on $\mathbf{SO}(3)$ to $\mathbb{RP}^3$, which becomes a topological group. Moreover, it can be shown that $\mathbf{SO}(3)$ and $\mathbb{RP}^3$ are diffeomorphic manifolds (see Marsden and Ratiu [15]). Thus, $\mathbf{SO}(3)$ and $\mathbb{RP}^3$ are at the same time groups, topological spaces, and manifolds, and in fact they are Lie groups (see Marsden and Ratiu [15] or Bryant [6]).

The axis and the angle of a rotation can also be extracted from a quaternion representing that rotation. The proof of the following lemma is adapted from Berger [3] and Dieudonné [5].

Lemma 9.2. *For every quaternion $Z = a\mathbf{1} + t$ where t is a pure quaternion, $\rho_Z = I$ iff $t = 0$, otherwise the axis of the rotation ρ_Z associated with Z is determined by the vector in $\mathbb{R}^3$ corresponding to t, and the angle of rotation θ is equal to π when $a = 0$, or when $a \neq 0$, given the orientation of the plane orthogonal to the axis of rotation described below, the angle is given by*

$$\tan\frac{\theta}{2} = \frac{\sqrt{N(t)}}{a},$$

with $\theta \neq \pi$ and $0 < \theta < 2\pi$. If $t \neq 0$, the plane orthogonal to t is oriented by choosing a basis (w_1, w_2) in it such that (w_1, w_2, t) is positively oriented; that is, $\det(w_1, w_2, t) > 0$.

Proof. A simple calculation shows that the line of direction t is invariant under the rotation ρ_Z, and thus it is the axis of rotation. Note that for any two nonnull vectors $X, Y \in \mathbb{R}^3$ such that $N(X) = N(Y)$, there is some rotation ρ such that $\rho(X) = Y$. If $X = Y$, we use the identity, and if $X \neq Y$, we use the rotation of axis determined by $X \times Y$ rotating X to Y in the plane containing X and Y. Thus, given any two nonnull pure quaternions X, Y such that $N(X) = N(Y)$, there is some nonnull quaternion W such that $Y = WXW^{-1}$. Furthermore, given any two nonnull quaternions Z, W, we claim that the angle of the rotation ρ_Z is the same as the angle of the rotation $\rho_{WZW^{-1}}$. This can be shown as follows. First, letting $Z = a\mathbf{1} + t$ where t is a pure nonnull quaternion, we show that the axis of the rotation $\rho_{WZW^{-1}}$ is $WtW^{-1} = \rho_W(t)$. Indeed, it is easily checked that WtW^{-1} is pure, and

$$WZW^{-1} = W(a\mathbf{1} + t)W^{-1} = Wa\mathbf{1}W^{-1} + WtW^{-1} = a\mathbf{1} + WtW^{-1}.$$

Second, given any pure nonnull quaternion X orthogonal to t, the angle of the rotation Z is the angle between X and $\rho_Z(X)$. Since rotations preserve orientation (since they preserve the cross product), the angle θ between two vectors X and Y is preserved under rotation. Since rotations preserve the inner product, if $X \cdot t = 0$, we have $\rho_W(X) \cdot \rho_W(t) = 0$, and the angle of the rotation $\rho_{WZW^{-1}} = \rho_W \circ \rho_Z \circ (\rho_W)^{-1}$ is the angle between the two vectors $\rho_W(X)$ and $\rho_{WZW^{-1}}(\rho_W(X))$. Since

$$\begin{aligned} \rho_{WZW^{-1}}(\rho_W(X)) &= (\rho_W \circ \rho_Z \circ (\rho_W)^{-1} \circ \rho_W)(X) \\ &= (\rho_W \circ \rho_Z)(X) = \rho_W(\rho_Z(X)), \end{aligned}$$

the angle of the rotation $\rho_{WZW^{-1}}$ is the angle between the two vectors $\rho_W(X)$ and $\rho_W(\rho_Z(X))$. Since rotations preserve angles, this is also the angle between the two vectors X and $\rho_Z(X)$, which is the angle of the rotation ρ_Z, as claimed. Thus, given any quaternion $Z = a\mathbf{1} + t$, where t is a nonnull pure quaternion, since there is some nonnull quaternion W such that $WtW^{-1} = \sqrt{N(t)}\,\mathbf{i}$ and $WZW^{-1} = a\mathbf{1} + \sqrt{N(t)}\,\mathbf{i}$, it is enough to figure out the angle of rotation for a quaternion Z of the form $a\mathbf{1} + b\mathbf{i}$ with $b > 0$ (a rotation of axis e_1). It suffices to find the angle between $\mathbf{j}$ and $\rho_Z(\mathbf{j})$, assuming that the plane orthogonal to be_1 (with $b > 0$) is oriented such that (e_2, e_3, be_1) has positive orientation, equivalently, (e_1, e_2, e_3) has positive orientation. Since

$$\rho_Z(\mathbf{j}) = (a\mathbf{1} + b\mathbf{i})\mathbf{j}(a\mathbf{1} + b\mathbf{i})^{-1},$$

we get

$$\rho_Z(\mathbf{j}) = \frac{1}{a^2 + b^2}(a\mathbf{1} + b\mathbf{i})\mathbf{j}(a\mathbf{1} - b\mathbf{i}) = \frac{a^2 - b^2}{a^2 + b^2}\,\mathbf{j} + \frac{2ab}{a^2 + b^2}\,\mathbf{k}.$$

Then we must have

$$\cos\theta = \frac{a^2 - b^2}{a^2 + b^2}, \quad \sin\theta = \frac{2ab}{a^2 + b^2}.$$

If $a \neq 0$, we have $\cos\theta \neq -1$, that is, $\theta \neq \pi$, so $\cos(\theta/2) \neq 0$ (recall that $0 < \theta < 2\pi$). Then, using the fact that $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$ and $\cos\theta = 2\cos^2(\theta/2) - 1$, we have

$$\frac{\sin\theta}{\cos\theta + 1} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2) - 1 + 1} = \frac{\sin(\theta/2)}{\cos(\theta/2)} = \tan(\theta/2).$$

Therefore, since

$$\cos\theta + 1 = \frac{a^2 - b^2}{a^2 + b^2} + 1 = \frac{2a^2}{a^2 + b^2}$$

and $a \neq 0$, we get

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{\cos\theta + 1} = \frac{2ab}{a^2 + b^2}\frac{a^2 + b^2}{2a^2} = \frac{b}{a} = \frac{\sqrt{N(t)}}{a}.$$

If $a = 0$, we get

$$\rho_Z(\mathbf{j}) = -\mathbf{j},$$

and $\theta = \pi$. In terms of the original quaternion $Z = a\mathbf{1} + t$ where $t \neq 0$ is arbitrary, the plane orthogonal to t is oriented by choosing a basis (w_1, w_2) in it such that (w_1, w_2, t) is positively oriented; that is, $\det(w_1, w_2, t) > 0$. $\square$

Note that if Z is a unit quaternion, then since

$$\cos\theta = \frac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)}$$

and $a^2 + N(t) = N(Z) = 1$, we get $\cos\theta = a^2 - N(t) = 2a^2 - 1$, and since $\cos\theta = 2\cos^2(\theta/2) - 1$, under the orientation defined above, we have

$$\cos\frac{\theta}{2} = a.$$

Now, since $a^2 + N(t) = N(Z) = 1$, we can write the unit quaternion Z as

$$Z = \left[\cos\frac{\theta}{2}, \sin\frac{\theta}{2}V\right],$$

where V is the unit vector $\frac{t}{\sqrt{N(t)}}$ (with $0 \leq \theta \leq 2\pi$). Also note that $VV = -\mathbf{1}$, and thus, formally, every unit quaternion looks like a complex number $\cos\varphi + i\sin\varphi$, except that i is replaced by a unit vector, and multiplication is quaternion multiplication.

In order to explain the homomorphism $\rho\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ more concretely, we now derive the formula for the rotation matrix of a rotation ρ whose axis D is determined by the nonnull vector w and whose angle of rotation is θ. For simplicity, we may assume that w is a unit vector. Letting $W = (b, c, d)$ be the column vector representing w and H be the plane orthogonal to w, recall from the discussion just before Lemma 8.1 that the matrices representing the projections p_D and p_H are

$$WW^\top \quad \text{and} \quad I - WW^\top.$$

Given any vector $u \in \mathbb{R}^3$, the vector $\rho(u)$ can be expressed in terms of the vectors $p_D(u)$, $p_H(u)$, and $w \times p_H(u)$ as

$$\rho(u) = p_D(u) + \cos\theta\, p_H(u) + \sin\theta\, w \times p_H(u).$$

However, it is obvious that

$$w \times p_H(u) = w \times u,$$

so that

$$\begin{aligned}\rho(u) &= p_D(u) + \cos\theta\, p_H(u) + \sin\theta\, w \times u,\\ \rho(u) &= (u \cdot w)w + \cos\theta\,(u - (u \cdot w)w) + \sin\theta\, w \times u,\end{aligned}$$

and we know from Section 8.9 that the cross product $w \times u$ can be expressed in terms of the multiplication on the left by the matrix

$$A = \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}.$$

Then, letting

$$B = WW^\top = \begin{pmatrix} b^2 & bc & bd \\ bc & c^2 & cd \\ bd & cd & d^2 \end{pmatrix},$$

the matrix R representing the rotation ρ is

$$\begin{aligned}R &= WW^\top + \cos\theta(I - WW^\top) + \sin\theta A,\\ &= \cos\theta\, I + \sin\theta A + (1 - \cos\theta)WW^\top,\\ &= \cos\theta\, I + \sin\theta A + (1 - \cos\theta)B.\end{aligned}$$

It is immediately verified that

$$A^2 = B - I,$$

and thus R is also given by

$$R = I + \sin\theta A + (1 - \cos\theta)A^2.$$

Then the nonnull unit quaternion

$$Z = \left[\cos\frac{\theta}{2}, \sin\frac{\theta}{2} V\right],$$

where $V = (b,c,d)$ is a unit vector, corresponds to the rotation ρ_Z of matrix

$$R = I + \sin\theta A + (1-\cos\theta)A^2.$$

Remark: A related formula known as Rodrigues's formula (1840) gives an expression for a rotation matrix in terms of the exponential of a matrix (the exponential map). Indeed, given $(b,c,d) \in \mathbb{R}^3$, letting $\theta = \sqrt{b^2+c^2+d^2}$, we have

$$\mathrm{e}^A = \cos\theta\, I + \frac{\sin\theta}{\theta}A + \frac{(1-\cos\theta)}{\theta^2}B,$$

with A and B as above, but (b,c,d) not necessarily a unit vector. We will study exponential maps later on.

Using the matrices L_X and R_Y introduced earlier, since $XY = L_X Y = R_Y X$, from $Y = ZXZ^{-1} = ZX\overline{Z}/N(Z)$, we get

$$Y = \frac{1}{N(Z)} L_Z R_{\overline{Z}} X.$$

Thus, if we want to see the effect of the rotation specified by the quaternion Z in terms of matrices, we simply have to compute the matrix

$$R(Z) = \frac{1}{N(Z)} L_Z R_{\overline{Z}} = \nu \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix},$$

where

$$N(Z) = a^2+b^2+c^2+d^2 \quad \text{and} \quad \nu = \frac{1}{N(Z)},$$

which yields

$$\nu \begin{pmatrix} N(Z) & 0 & 0 & 0 \\ 0 & a^2+b^2-c^2-d^2 & 2bc-2ad & 2ac+2bd \\ 0 & 2bc+2ad & a^2-b^2+c^2-d^2 & -2ab+2cd \\ 0 & -2ac+2bd & 2ab+2cd & a^2-b^2-c^2+d^2 \end{pmatrix}.$$

But since every pure quaternion X is a vector whose first component is 0, we see that the rotation matrix $R(Z)$ associated with the quaternion Z is

$$\frac{1}{N(Z)} \begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2ac+2bd \\ 2bc+2ad & a^2-b^2+c^2-d^2 & -2ab+2cd \\ -2ac+2bd & 2ab+2cd & a^2-b^2-c^2+d^2 \end{pmatrix}.$$

This expression for a rotation matrix is due to Euler (see Veblen and Young [22]). It is quite remarkable that this matrix contains only quadratic polynomials in a,b,c,d. This makes it possible to compute easily a quaternion from a rotation matrix.

From a computational point of view, it is worth noting that computing the composition of two rotations ρ_Y and ρ_Z specified by two quaternions Y,Z using quaternion multiplication (i.e., $\rho_Y \circ \rho_Z = \rho_{YZ}$) is cheaper than using rotation matrices and matrix multiplication. On the other hand, computing the image of a point X under a rotation ρ_Z is more expensive in terms of quaternions (it requires computing ZXZ^{-1}) than it is in terms of rotation matrices (where only AX needs to be computed, where A is a rotation matrix). Thus, if many points need to be rotated and the rotation is specified by a quaternion, it is advantageous to precompute the Euler matrix.

9.3 Quaternions and Rotations in $\mathbf{SO}(4)$

For every nonnull quaternion Z, the map $X \mapsto ZXZ^{-1}$ (where X is a pure quaternion) defines a rotation of $\mathbb{H}_p$, and conversely, every rotation of $\mathbb{H}_p$ is of the above form. What happens if we consider a map of the form

$$X \mapsto YXZ,$$

where $X \in \mathbb{H}$ and $N(Y)N(Z) = 1$? Remarkably, it turns out that we get all the rotations of $\mathbb{H}$. The proof of the following lemma is inspired by Berger [3], Dieudonné [5], and Tisseron [21].

Lemma 9.3. *For every pair (Y,Z) of quaternions such that $N(Y)N(Z) = 1$, the map*

$$\rho_{Y,Z} \colon X \mapsto YXZ$$

(where $X \in \mathbb{H}$) is a rotation in $\mathbf{SO}(\mathbb{H}) = \mathbf{SO}(4)$. Conversely, every rotation in $\mathbf{SO}(4)$ is of the form

$$\rho_{Y,Z} \colon X \mapsto YXZ,$$

for some quaternions Y, Z such that $N(Y)N(Z) = 1$. Furthermore, if two nonnull pairs of quaternions (Y,Z) and (Y',Z') represent the same rotation, then $Y' = \lambda Y$ and $Z' = \lambda^{-1}Z$, for some $\lambda \neq 0$ in $\mathbb{R}$.

Proof. We have already shown that $\rho_{Y,Z} \in \mathbf{SO}(4)$. It remains to prove that every rotation in $\mathbf{SO}(4)$ is of this form.

It is easily seen that

$$\rho_{(Y'Y,ZZ')} = \rho_{Y',Z'} \circ \rho_{Y,Z}.$$

Let $\rho \in \mathbf{SO}(4)$ be a rotation, and let $Z_0 = \rho(\mathbf{1})$ and $g = \rho_{Z_0^{-1},\mathbf{1}}$. Since ρ is an isometry, $Z_0 = \rho(\mathbf{1})$ is a unit quaternion, and thus $g \in \mathbf{SO}(4)$. Observe that

$$g(\rho(\mathbf{1})) = \mathbf{1},$$

which implies that $F = \mathbb{R}\mathbf{1}$ is invariant under $g \circ \rho$. Since $F^\perp = \mathbb{H}_p$, by Lemma 8.2, $g \circ \rho(\mathbb{H}_p) \subseteq \mathbb{H}_p$, which shows that the restriction of $g \circ \rho$ to $\mathbb{H}_p$ is a rotation. By Lemma 9.1, there is some nonnull quaternion Z such that $g \circ \rho = \rho_Z$ on $\mathbb{H}_p$, but since both $g \circ \rho$ and ρ_Z are the identity on $\mathbb{R}\mathbf{1}$, we must have $g \circ \rho = \rho_Z$ on $\mathbb{H}$. Finally, a trivial calculation shows that

$$\rho = g^{-1} \circ \rho_Z = \rho_{Z_0,\mathbf{1}} \rho_Z = \rho_{Z_0,\mathbf{1}} \rho_{Z,Z^{-1}} = \rho_{Z_0 Z, Z^{-1}}.$$

If $\rho_{Y,Z} = \rho_{Y',Z'}$, then

$$YXZ = Y'XZ'$$

for all $X \in \mathbb{H}$, that is,

$$Y^{-1}Y'XZ'Z^{-1} = X$$

for all $X \in \mathbb{H}$. Letting $X = (Y^{-1}Y')^{-1}$, we get $Z'Z^{-1} = (Y^{-1}Y')^{-1}$. From

$$Y^{-1}Y'X(Y^{-1}Y')^{-1} = X$$

for all $Z \in \mathbb{H}$, by a previous remark, we must have $Y^{-1}Y' = \lambda\mathbf{1}$ for some $\lambda \neq 0$ in $\mathbb{R}$, so that $Y' = \lambda Y$, and since $Z'Z^{-1} = (Y^{-1}Y')^{-1}$, we get $Z'Z^{-1} = \lambda^{-1}\mathbf{1}$, i.e. $Z' = \lambda^{-1}Z$. $\square$

Since

$$\rho_{(Y'Y, ZZ')} = \rho_{Y',Z'} \circ \rho_{Y,Z},$$

it is easy to show that the map $\eta \colon S^3 \times S^3 \to \mathbf{SO}(4)$ defined by $\eta(Y,Z) = \rho_{Y,\overline{Z}}$ is a surjective homomorphism whose kernel is $\{(\mathbf{1},\mathbf{1}), (-\mathbf{1},-\mathbf{1})\}$.

Remark: Note that it is necessary to define $\eta \colon S^3 \times S^3 \to \mathbf{SO}(4)$ such that

$$\eta(Y,Z)(X) = YX\overline{Z},$$

where the conjugate $\overline{Z}$ of Z is used rather than Z, to compensate for the switch between Z and Z' in

$$\rho_{(Y'Y, ZZ')} = \rho_{Y',Z'} \circ \rho_{Y,Z}.$$

Otherwise, η would not be a homomorphism from the product group $S^3 \times S^3$ to $\mathbf{SO}(4)$.

We conclude this section on the quaternions with a mention of the exponential map, since it has applications to quaternion interpolation, which, in turn, has applications to motion interpolation.

Observe that the quaternions $\mathbf{i}, \mathbf{j}, \mathbf{k}$ can also be written as

$$\mathbf{i} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = \mathrm{i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{i} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$
$$\mathbf{k} = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \mathrm{i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so that if we define the matrices $\sigma_1, \sigma_2, \sigma_3$ such that

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we can write

$$Z = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = a\mathbf{1} + \mathrm{i}(d\sigma_1 + c\sigma_2 + b\sigma_3).$$

The matrices $\sigma_1, \sigma_2, \sigma_3$ are called the *Pauli spin matrices*. Note that their traces are null and that they are Hermitian (recall that a complex matrix is Hermitian if it is equal to the transpose of its conjugate, i.e., $A^* = A$). The somewhat unfortunate order reversal of b, c, d has to do with the traditional convention for listing the Pauli matrices. If we let $e_0 = a$, $e_1 = d$, $e_2 = c$, and $e_3 = b$, then Z can be written as

$$Z = e_0\mathbf{1} + \mathrm{i}(e_1\sigma_1 + e_2\sigma_2 + e_3\sigma_3),$$

and e_0, e_1, e_2, e_3 are called the *Euler parameters* of the rotation specified by Z. If $N(Z) = 1$, then we can also write

$$Z = \cos\frac{\theta}{2}\,\mathbf{1} + \mathrm{i}\sin\frac{\theta}{2}\,(\beta\sigma_3 + \gamma\sigma_2 + \delta\sigma_1),$$

where

$$(\beta, \gamma, \delta) = \frac{1}{\sin\frac{\theta}{2}}\,(b, c, d).$$

Letting $A = \beta\sigma_3 + \gamma\sigma_2 + \delta\sigma_1$, it can be shown that

$$\mathrm{e}^{\mathrm{i}\theta A} = \cos\theta\,\mathbf{1} + \mathrm{i}\sin\theta\,A,$$

where the exponential is the usual exponential of matrices, i.e., for a square $n \times n$ matrix M,

$$\exp(M) = I_n + \sum_{k \geq 1} \frac{M^k}{k!}.$$

Note that since A is Hermitian of null trace, $\mathrm{i}A$ is skew Hermitian of null trace.

The above formula turns out to define the exponential map from the Lie algebra of $\mathbf{SU}(2)$ to $\mathbf{SU}(2)$. The Lie algebra of $\mathbf{SU}(2)$ is a real vector space having $\mathrm{i}\sigma_1$, $\mathrm{i}\sigma_2$, and $\mathrm{i}\sigma_3$ as a basis. Now, the vector space $\mathbb{R}^3$ is a Lie algebra if we define the Lie bracket on $\mathbb{R}^3$ as the usual cross product $u \times v$ of vectors. Then the Lie algebra of

$\mathbf{SU}(2)$ is isomorphic to $(\mathbb{R}^3, \times)$, and the exponential map can be viewed as a map $\exp\colon (\mathbb{R}^3, \times) \to \mathbf{SU}(2)$ given by the formula

$$\exp(\theta v) = \left[\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\, v\right],$$

for every vector θv, where v is a unit vector in $\mathbb{R}^3$ and $\theta \in \mathbb{R}$.

The exponential map can be used for quaternion interpolation. Given two unit quaternions X, Y, suppose we want to find a quaternion Z "interpolating" between X and Y. Of course, we have to clarify what this means. Since $\mathbf{SU}(2)$ is topologically the same as the sphere S^3, we define an *interpolant* of X and Y as a quaternion Z on the great circle (on the sphere S^3) determined by the intersection of S^3 with the (2-)plane defined by the two points X and Y (viewed as points on S^3) and the origin $(0,0,0,0)$.

Then the points (quaternions) on this great circle can be defined by first rotating X and Y so that X goes to $\mathbf{1}$ and Y goes to $X^{-1}Y$, by multiplying (on the left) by X^{-1}. Letting

$$X^{-1}Y = [\cos\Omega, \sin\Omega\, w],$$

where $-\pi < \Omega \leq \pi$, the points on the great circle from $\mathbf{1}$ to $X^{-1}Y$ are given by the quaternions

$$(X^{-1}Y)^{\lambda} = [\cos\lambda\Omega, \sin\lambda\Omega\, w],$$

where $\lambda \in \mathbb{R}$. This is because $X^{-1}Y = \exp(2\Omega w)$, and since an interpolant between $(0,0,0)$ and $2\Omega w$ is $2\lambda\Omega w$ in the Lie algebra of $\mathbf{SU}(2)$, the corresponding quaternion is indeed

$$\exp(2\lambda\Omega) = [\cos\lambda\Omega, \sin\lambda\Omega\, w].$$

We cannot justify all this here, but it is indeed correct.

If $\Omega \neq \pi$, then the shortest arc between X and Y is unique, and it corresponds to those λ such that $0 \leq \lambda \leq 1$ (it is a geodesic arc). However, if $\Omega = \pi$, then X and Y are antipodal, and there are infinitely many half circles from X to Y. In this case, w can be chosen arbitrarily.

Finally, having the arc of great circle between $\mathbf{1}$ and $X^{-1}Y$ (assuming $\Omega \neq \pi$), we get the arc of interpolants $Z(\lambda)$ between X and Y by performing the inverse rotation from $\mathbf{1}$ to X and from $X^{-1}Y$ to Y, i.e., by multiplying (on the left) by X, and we get

$$Z(\lambda) = X(X^{-1}Y)^{\lambda}.$$

Note how the geometric reasoning immediately shows that

$$Z(\lambda) = X(X^{-1}Y)^{\lambda} = (YX^{-1})^{\lambda}X.$$

It is remarkable that a closed-form formula for $Z(\lambda)$ can be given, as shown by Shoemake [19, 20]. If $X = [\cos\theta, \sin\theta\, u]$ and $Y = [\cos\varphi, \sin\varphi\, v]$ (where u and v are unit vectors in $\mathbb{R}^3$), letting

$$\cos\Omega = \cos\theta\cos\varphi + \sin\theta\sin\varphi\,(u \cdot v)$$

be the inner product of X and Y viewed as vectors in $\mathbb{R}^4$, it is a bit laborious to show that

$$Z(\lambda) = \frac{\sin(1-\lambda)\Omega}{\sin\Omega}X + \frac{\sin\lambda\Omega}{\sin\Omega}Y.$$

The above formula is quite remarkable, since if $X = \cos\theta + i\sin\theta$ and $Y = \cos\varphi + i\sin\varphi$ are two points on the unit circle S^1 (given as complex numbers of unit length), letting $\Omega = \varphi - \theta$, the interpolating point $\cos((1-\lambda)\theta + \lambda\varphi) + i\sin((1-\lambda)\theta + \lambda\varphi)$ on S^1 is given by the same formula

$$\cos((1-\lambda)\theta + \lambda\varphi) + i\sin((1-\lambda)\theta + \lambda\varphi) = \frac{\sin(1-\lambda)\Omega}{\sin\Omega}X + \frac{\sin\lambda\Omega}{\sin\Omega}Y.$$

9.4 Applications of Euclidean Geometry to Motion Interpolation

Euclidean geometry has a number applications including computer vision, computer graphics, kinematics, and robotics. The motion of a rigid body in space can be described using rigid motions. Given a fixed Euclidean frame $(O,(e_1,e_2,e_3))$, we can assume that some moving frame $(C,(u_1,u_2,u_3))$ is attached (say glued) to a rigid body B (for example, at the center of gravity of B) so that the position and orientation of B in space are completely (and uniquely) determined by some rigid motion (R,U), where U specifies the position of C w.r.t. O, and R is a rotation matrix specifying the orientation of B w.r.t. the fixed frame $(O,(e_1,e_2,e_3))$. For simplicity, we can separate the motion of the center of gravity C of B from the rotation of B around its center of gravity. Then a motion of B in space corresponds to two curves: The trajectory of the center of gravity and a curve in $\mathbf{SO}(3)$ representing the various orientations of B. Given a sequence of "snapshots" of B, say $B_0,B_1,\ldots,B_m$, we may want to find an interpolating motion passing through the given snapshots. Furthermore, in most cases, it desirable that the curve be invariant with respect to a change of coordinates and to rescaling. Often, one looks for an energy minimizing motion. The problem is not as simple as it looks, because the space of rotations $\mathbf{SO}(3)$ is topologically rather complex, and in particular, it is curved.

The problem of motion interpolation has been studied quite extensively both in the robotics and computer graphics communities. Since rotations in $\mathbf{SO}(3)$ can be represented by quaternions (see Chapter 9), the problem of quaternion interpolation has been investigated, an approach apparently initiated by Shoemake [19, 20], who extended the de Casteljau algorithm to the 3-sphere. Related work was done by Barr, Currin, Gabriel, and Hughes [2]. Kim, M.-J., Kim, M.-S. and Shin [12, 13] corrected bugs in Shoemake and introduced various kinds of splines on S^3, using the exponential map. Motion interpolation and rational motions have been investigated by Jüttler [8, 9], Jüttler and Wagner [10, 11], Horsch and Jüttler [7], and Röschel [18]. Park and Ravani [16, 17] also investigated Bézier curves on Riemannian manifolds and Lie groups, $\mathbf{SO}(3)$ in particular. More generally, the problem of interpolating curves on surfaces or higher-dimensional manifolds in an efficient way remains an open problem. A very interesting book on the quaternions and their applications to a number of engineering problems, including aerospace systems, is the book by Kuipers [14], which we highly recommend.

9.5 Problems

9.1. Prove the following identities about quaternion multiplication (discovered by Hamilton):

$$\begin{aligned}\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} &= -\mathbf{1},\\ \mathbf{ij} = -\mathbf{ji} &= \mathbf{k},\\ \mathbf{jk} = -\mathbf{kj} &= \mathbf{i},\\ \mathbf{ki} = -\mathbf{ik} &= \mathbf{j}.\end{aligned}$$

9.2. Given any two quaternions $X = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ and $Y = a'\mathbf{1} + b'\mathbf{i} + c'\mathbf{j} + d'\mathbf{k}$, prove that

$$\begin{aligned}XY = {} & (aa' - bb' - cc' - dd')\mathbf{1} + (ab' + ba' + cd' - dc')\mathbf{i}\\ & + (ac' + ca' + db' - bd')\mathbf{j} + (ad' + da' + bc' - cb')\mathbf{k}.\end{aligned}$$

Also prove that if $X = [a, U]$ and $Y = [a', U']$, the quaternion product XY can be expressed as

$$XY = [aa' - U \cdot U',\, aU' + a'U + U \times U'].$$

9.3. Show that there is a very simple method for producing an orthonormal frame in $\mathbb{R}^4$ whose first vector is any given nonnull vector (a, b, c, d).

9.4. Prove that

$$\begin{aligned}\rho_Z(XY) &= \rho_Z(X)\rho_Z(Y),\\ \rho_Z(X+Y) &= \rho_Z(X) + \rho_Z(Y),\end{aligned}$$

for any nonnull quaternion Z and any two quaternions X, Y (i.e., ρ_Z is an automorphism of $\mathbb{H}$), and that

$$XY - YX = [0, 2(U \times U')]$$

for arbitrary quaternions $X = [a, U]$ and $Y = [a', U']$.

9.5. Give an algorithm to find a quaternion Z corresponding to a rotation matrix R using the Euler form of a rotation matrix $R(Z)$:

$$\frac{1}{N(Z)}\begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2ac+2bd\\ 2bc+2ad & a^2-b^2+c^2-d^2 & -2ab+2cd\\ -2ac+2bd & 2ab+2cd & a^2-b^2-c^2+d^2\end{pmatrix}.$$

What about the choice of the sign of Z?

9.6. Let i, j, and k, be the unit vectors of coordinates $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ in $\mathbb{R}^3$.

(i) Describe geometrically the rotations defined by the following quaternions:

$$p = (0,i), \quad q = (0,j).$$

Prove that the interpolant $Z(\lambda) = p(p^{-1}q)^{\lambda}$ is given by

$$Z(\lambda) = (0, \cos(\lambda\pi/2)i + \sin(\lambda\pi/2)j).$$

Describe geometrically what this rotation is.

(ii) Repeat question (i) with the rotations defined by the quaternions

$$p = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}i\right), \quad q = (0,j).$$

Prove that the interpolant $Z(\lambda)$ is given by

$$Z(\lambda) = \left(\frac{1}{2}\cos(\lambda\pi/2), \frac{\sqrt{3}}{2}\cos(\lambda\pi/2)i + \sin(\lambda\pi/2)j\right).$$

Describe geometrically what this rotation is.

(iii) Repeat question (i) with the rotations defined by the quaternions

$$p = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}i\right), \quad q = \left(0, \frac{1}{\sqrt{2}}(i+j)\right).$$

Prove that the interpolant $Z(\lambda)$ is given by

$$Z(\lambda) = \left(\frac{1}{\sqrt{2}}\cos(\lambda\pi/3) - \frac{1}{\sqrt{6}}\sin(\lambda\pi/3),\right.$$
$$\left.(1/\sqrt{2}\cos(\lambda\pi/3) + 1/\sqrt{6}\sin(\lambda\pi/3))i + \frac{2}{\sqrt{6}}\sin(\lambda\pi/3)j\right).$$

(iv) Prove that

$$w \times (u \times v) = (w \cdot v)u - (u \cdot w)v.$$

Conclude that

$$u \times (u \times v) = (u \cdot v)u - (u \cdot u)v.$$

(v) Let

$$p = (\cos\theta, \sin\theta u), \quad q = (\cos\varphi, \sin\varphi v),$$

where u and v are unit vectors in $\mathbb{R}^3$. If

$$\cos\Omega = \cos\theta\cos\varphi + \sin\theta\sin\varphi\,(u \cdot v)$$

is the inner product of X and Y viewed as vectors in $\mathbb{R}^4$, assuming that $\Omega \neq k\pi$, prove that

$$Z(\lambda) = \frac{\sin(1-\lambda)\Omega}{\sin\Omega}\, p + \frac{\sin\lambda\Omega}{\sin\Omega}\, q.$$

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