Chapter 3

Direct Sums, Affine Maps

3.1 Direct Products, Sums, and Direct Sums

There are some useful ways of forming new vector spaces from older ones.

Definition 3.1. Given $p \geq 2$ vector spaces $E_1, \ldots, E_p$, the product $F = E_1 \times \cdots \times E_p$ can be made into a vector space by defining addition and scalar multiplication as follows:

$$\begin{aligned} (u_1, \ldots, u_p) + (v_1, \ldots, v_p) &= (u_1 + v_1, \ldots, u_p + v_p) \\ \lambda(u_1, \ldots, u_p) &= (\lambda u_1, \ldots, \lambda u_p), \end{aligned}$$

for all $u_i, v_i \in E_i$ and all $\lambda \in \mathbb{R}$.

With the above addition and multiplication, the vector space $F = E_1 \times \cdots \times E_p$ is called the *direct product* of the vector spaces $E_1, \ldots, E_p$.

The *projection maps* $pr_i\colon E_1 \times \cdots \times E_p \to E_i$ given by

$$pr_i(u_1, \ldots, u_p) = u_i$$

are clearly linear.

Similarly, the maps $\mathrm{in}_i\colon E_i \to E_1 \times \cdots \times E_p$ given by

$$\mathrm{in}_i(u_i) = (0, \ldots, 0, u_i, 0, \ldots, 0)$$

are injective and linear.

It can be shown (using bases) that

$$\dim(E_1 \times \cdots \times E_p) = \dim(E_1) + \cdots + \dim(E_p).$$

Let us now consider a vector space E and p subspaces $U_1, \ldots, U_p$ of E.

We have a map

$$a\colon U_1 \times \cdots \times U_p \to E$$

given by

$$a(u_1, \ldots, u_p) = u_1 + \cdots + u_p,$$

with $u_i \in U_i$ for $i = 1, \ldots, p$.

It is clear that this map is linear, and so its image is a subspace of E denoted by

$$U_1 + \cdots + U_p$$

and called the *sum* of the subspaces $U_1, \ldots, U_p$.

By definition,

$$U_1 + \cdots + U_p = \{u_1 + \cdots + u_p \mid u_i \in U_i,\ 1 \leq i \leq p\},$$

and it is immediately verified that $U_1 + \cdots + U_p$ is the smallest subspace of E containing $U_1, \ldots, U_p$.

If the map a is injective, then $\mathrm{Ker}\, a = \{(\underbrace{0, \ldots, 0}_{p})\}$, which means that if $u_i \in U_i$ for $i = 1, \ldots, p$ and if

$$u_1 + \cdots + u_p = 0$$

then $u_1 = 0, \ldots, u_p = 0$.

In this case, every $u \in U_1 + \cdots + U_p$ has a *unique* expression as a sum

$$u = u_1 + \cdots + u_p,$$

with $u_i \in U_i$, for $i = 1, \ldots, p$.

It is also clear that for any p nonzero vectors $u_i \in U_i$, $u_1, \ldots, u_p$ are linearly independent.

Definition 3.2. For any vector space E and any $p \geq 2$ subspaces $U_1, \ldots, U_p$ of E, if the map a defined above is injective, then the sum $U_1 + \cdots + U_p$ is called a *direct sum* and it is denoted by

$$U_1 \oplus \cdots \oplus U_p.$$

The space E is the *direct sum* of the subspaces U_i if

$$E = U_1 \oplus \cdots \oplus U_p.$$

Observe that when the map a is injective, then it is a linear isomorphism between $U_1 \times \cdots \times U_p$ and $U_1 \oplus \cdots \oplus U_p$.

The difference is that $U_1 \times \cdots \times U_p$ is defined even if the spaces U_i are not assumed to be subspaces of some common space.

There are natural injections from each U_i to E denoted by $\mathrm{in}_i \colon U_i \to E$.

Now, if $p = 2$, it is easy to determine the kernel of the map $a\colon U_1 \times U_2 \to E$. We have

$$a(u_1, u_2) = u_1 + u_2 = 0 \text{ iff } u_1 = -u_2,\; u_1 \in U_1, u_2 \in U_2,$$

which implies that

$$\operatorname{Ker} a = \{(u, -u) \mid u \in U_1 \cap U_2\}.$$

Now, $U_1 \cap U_2$ is a subspace of E and the linear map $u \mapsto (u, -u)$ is clearly an isomorphism between $U_1 \cap U_2$ and $\operatorname{Ker} a$, so $\operatorname{Ker} a$ is isomorphic to $U_1 \cap U_2$.

As a consequence, we get the following result:

Proposition 3.1. *Given any vector space E and any two subspaces U_1 and U_2, the sum $U_1 + U_2$ is a direct sum iff $U_1 \cap U_2 = (0)$.*

Recall that an $n \times n$ matrix $A \in \mathrm{M}_n$ is *symmetric* if $A^\top = A$, *skew -symmetric* if $A^\top = -A$. It is clear that

$$\mathbf{S}(n) = \{A \in \mathrm{M}_n \mid A^\top = A\}$$
$$\mathbf{Skew}(n) = \{A \in \mathrm{M}_n \mid A^\top = -A\}$$

are subspaces of M_n, and that $\mathbf{S}(n) \cap \mathbf{Skew}(n) = (0)$.

Observe that for any matrix $A \in \mathrm{M}_n$, the matrix $H(A) = (A + A^\top)/2$ is symmetric and the matrix $S(A) = (A - A^\top)/2$ is skew-symmetric. Since

$$A = H(A) + S(A) = \frac{A + A^\top}{2} + \frac{A - A^\top}{2},$$

we have the direct sum

$$\mathrm{M}_n = \mathbf{S}(n) \oplus \mathbf{Skew}(n).$$

Proposition 3.2. *Given any vector space E and any $p \geq 2$ subspaces $U_1, \ldots, U_p$, the following properties are equivalent:*

(1) The sum $U_1 + \cdots + U_p$ is a direct sum.

(2) We have

$$U_i \cap \left(\sum_{j=1, j \neq i}^{p} U_j \right) = (0), \quad i = 1, \ldots, p.$$

(3) We have

$$U_i \cap \left(\sum_{j=1}^{i-1} U_j \right) = (0), \quad i = 2, \ldots, p.$$

The isomorphism $U_1 \times \cdots \times U_p \approx U_1 \oplus \cdots \oplus U_p$ implies

Proposition 3.3. *If E is any vector space, for any (finite-dimensional) subspaces $U_1, \ldots, U_p$ of E, we have*

$$\dim(U_1 \oplus \cdots \oplus U_p) = \dim(U_1) + \cdots + \dim(U_p).$$

If E is a direct sum

$$E = U_1 \oplus \cdots \oplus U_p,$$

since every $u \in E$ can be written in a unique way as

$$u = u_1 + \cdots + u_p$$

for some $u_i \in U_i$ for $i = 1 \ldots, p$, we can define the maps $\pi_i \colon E \to U_i$, called *projections*, by

$$\pi_i(u) = \pi_i(u_1 + \cdots + u_p) = u_i.$$

It is easy to check that these maps are linear and satisfy the following properties:

$$\pi_j \circ \pi_i = \begin{cases} \pi_i & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

$$\pi_1 + \cdots + \pi_p = \mathrm{id}_E.$$

For example, in the case of the direct sum

$$\mathrm{M}_n = \mathbf{S}(n) \oplus \mathbf{Skew}(n),$$

the projection onto $\mathbf{S}(n)$ is given by

$$\pi_1(A) = H(A) = \frac{A + A^\top}{2},$$

and the projection onto $\mathbf{Skew}(n)$ is given by

$$\pi_2(A) = S(A) = \frac{A - A^\top}{2}.$$

Clearly, $H(A) + S(A) = A$, $H(H(A)) = H(A)$, $S(S(A)) = S(A)$, and $H(S(A)) = S(H(A)) = 0$.

A function f such that $f \circ f = f$ is said to be *idempotent*. Thus, the projections π_i are idempotent.

Conversely, the following proposition can be shown:

Proposition 3.4. *Let E be a vector space. For any $p \geq 2$ linear maps $f_i\colon E \to E$, if*

$$f_j \circ f_i = \begin{cases} f_i & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

$$f_1 + \cdots + f_p = \mathrm{id}_E,$$

then if we let $U_i = f_i(E)$, we have a direct sum

$$E = U_1 \oplus \cdots \oplus U_p.$$

We also have the following proposition characterizing idempotent linear maps whose proof is also left as an exercise.

Proposition 3.5. *For every vector space E, if $f\colon E \to E$ is an idempotent linear map, i.e., $f \circ f = f$, then we have a direct sum*

$$E = \mathrm{Ker}\, f \oplus \mathrm{Im}\, f,$$

so that f is the projection onto its image $\mathrm{Im}\, f$.

We are now ready to prove a very crucial result relating the rank and the dimension of the kernel of a linear map.

3.2 The Rank-Nullity Theorem; Grassmann's Relations

Theorem 3.6. *Let $f\colon E \to F$ be a linear map. For any choice of a basis $(f_1, \ldots, f_r)$ of $\mathrm{Im}\, f$, let $(u_1, \ldots, u_r)$ be any vectors in E such that $f_i = f(u_i)$, for $i = 1, \ldots, r$. If $s\colon \mathrm{Im}\, f \to E$ is the unique linear map defined by $s(f_i) = u_i$, for $i = 1, \ldots, r$, then s is injective, $f \circ s = \mathrm{id}$, and we have a direct sum*

$$E = \mathrm{Ker}\, f \oplus \mathrm{Im}\, s$$

as illustrated by the following diagram:

$$\mathrm{Ker}\, f \longrightarrow E = \mathrm{Ker}\, f \oplus \mathrm{Im}\, s \underset{s}{\overset{f}{\rightleftarrows}} \mathrm{Im}\, f \subseteq F.$$

As a consequence,

$$\dim(E) = \dim(\mathrm{Ker}\, f) + \dim(\mathrm{Im}\, f) = \dim(\mathrm{Ker}\, f) + \mathrm{rk}(f).$$

Remark: The dimension $\dim(\mathrm{Ker}\, f)$ of the kernel of a linear map f is often called the *nullity* of f.

We now derive some important results using Theorem 3.6.

Proposition 3.7. *Given a vector space E, if U and V are any two subspaces of E, then*

$$\dim(U) + \dim(V) = \dim(U + V) + \dim(U \cap V),$$

an equation known as Grassmann's relation.

The Grassmann relation can be very useful to figure out whether two subspace have a nontrivial intersection in spaces of dimension > 3.

For example, it is easy to see that in $\mathbb{R}^5$, there are subspaces U and V with $\dim(U) = 3$ and $\dim(V) = 2$ such that $U \cap V = (0)$.

However, we can show that if $\dim(U) = 3$ and $\dim(V) = 3$, then $\dim(U \cap V) \geq 1$.

As another consequence of Proposition 3.7, if U and V are two hyperplanes in a vector space of dimension n, so that $\dim(U) = n - 1$ and $\dim(V) = n - 1$, we have

$$\dim(U \cap V) \geq n - 2,$$

and so, if $U \neq V$, then

$$\dim(U \cap V) = n - 2.$$

Proposition 3.8. *If $U_1, \ldots, U_p$ are any subspaces of a finite dimensional vector space E, then*

$$\dim(U_1 + \cdots + U_p) \leq \dim(U_1) + \cdots + \dim(U_p),$$

and

$$\dim(U_1 + \cdots + U_p) = \dim(U_1) + \cdots + \dim(U_p)$$

iff the U_is form a direct sum $U_1 \oplus \cdots \oplus U_p$.

Another important corollary of Theorem 3.6 is the following result:

Proposition 3.9. *Let E and F be two vector spaces with the same finite dimension $\dim(E) = \dim(F) = n$. For every linear map $f\colon E \to F$, the following properties are equivalent:*

(a) f is bijective.

(b) f is surjective.

(c) f is injective.

(d) $\mathrm{Ker}\, f = (0)$.

One should be warned that Proposition 3.9 fails in infinite dimension.

We also have the following basic proposition about injective or surjective linear maps.

Proposition 3.10. *Let E and F be vector spaces, and let $f\colon E \to F$ be a linear map. If $f\colon E \to F$ is injective, then there is a surjective linear map $r\colon F \to E$ called a retraction, such that $r \circ f = \mathrm{id}_E$. If $f\colon E \to F$ is surjective, then there is an injective linear map $s\colon F \to E$ called a section, such that $f \circ s = \mathrm{id}_F$.*

The notion of rank of a linear map or of a matrix important, both theoretically and practically, since it is the key to the solvability of linear equations.

Proposition 3.11. *Given a linear map $f\colon E \to F$, the following properties hold:*

(i) $\mathrm{rk}(f) + \dim(\mathrm{Ker}\, f) = \dim(E)$.

(ii) $\mathrm{rk}(f) \leq \min(\dim(E), \dim(F))$.

The rank of a matrix is defined as follows.

Definition 3.3. Given a $m \times n$-matrix $A = (a_{i\,j})$, the *rank* $\mathrm{rk}(A)$ of the matrix A is the maximum number of linearly independent columns of A (viewed as vectors in $\mathbb{R}^m$).

In view of Proposition 1.4, the rank of a matrix A is the dimension of the subspace of $\mathbb{R}^m$ generated by the columns of A.

Let E and F be two vector spaces, and let $(u_1, \ldots, u_n)$ be a basis of E, and $(v_1, \ldots, v_m)$ a basis of F. Let $f \colon E \to F$ be a linear map, and let $M(f)$ be its matrix w.r.t. the bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_m)$.

Since the rank $\mathrm{rk}(f)$ of f is the dimension of $\mathrm{Im}\, f$, which is generated by $(f(u_1), \ldots, f(u_n))$, the rank of f is the maximum number of linearly independent vectors in $(f(u_1), \ldots, f(u_n))$, which is equal to the number of linearly independent columns of $M(f)$, since F and $\mathbb{R}^m$ are isomorphic.

Thus, we have $\mathrm{rk}(f) = \mathrm{rk}(M(f))$, for every matrix representing f.

We will see later, using duality, that the rank of a matrix A is also equal to the maximal number of linearly independent rows of A.

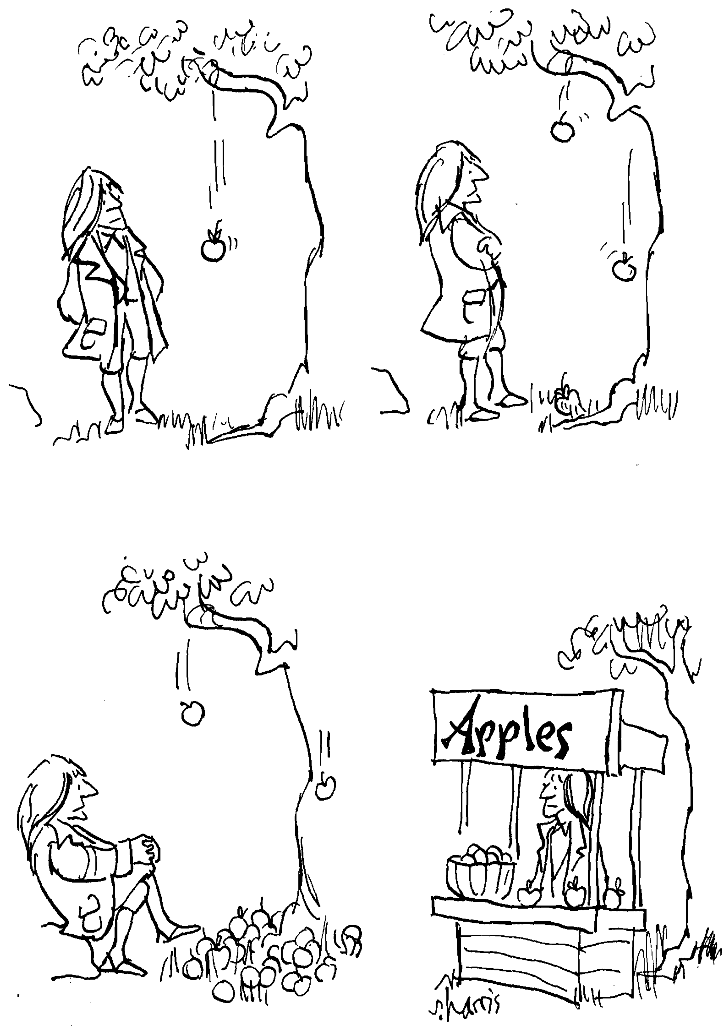

Figure 3.1: How did Newton start a business

3.3 Affine Maps

We showed in Section 1.6 that every linear map f must send the zero vector to the zero vector, that is,

$$f(0) = 0.$$

Yet, for any fixed nonzero vector $u \in E$ (where E is any vector space), the function t_u given by

$$t_u(x) = x + u, \quad \text{for all} \quad x \in E$$

shows up in pratice (for example, in robotics).

Functions of this type are called *translations*. They are *not* linear for $u \neq 0$, since $t_u(0) = 0 + u = u$.

More generally, functions combining linear maps and translations occur naturally in many applications (robotics, computer vision, etc.), so it is necessary to understand some basic properties of these functions.

For this, the notion of affine combination turns out to play a key role.

Recall from Section 1.6 that for any vector space E, given any family $(u_i)_{i\in I}$ of vectors $u_i \in E$, an *affine combination* of the family $(u_i)_{i\in I}$ is an expression of the form

$$\sum_{i\in I} \lambda_i u_i \quad \text{with} \quad \sum_{i\in I} \lambda_i = 1,$$

where $(\lambda_i)_{i\in I}$ is a family of scalars.

A linear combination places no restriction on the scalars involved, but an affine combination is a linear combination, *with the restriction that the scalars λ_i must add up to 1*. Nevertheless, a linear combination can always be viewed as an affine combination, using 0 with the coefficient $1 - \sum_{i\in I} \lambda_i$.

Affine combinations are also called *barycentric combinations*.

Although this is not obvious at first glance, the condition that the scalars λ_i add up to 1 ensures that affine combinations are preserved under translations.

To make this precise, consider functions $f \colon E \to F$, where E and F are two vector spaces, such that there is some *linear map* $h \colon E \to F$ and some fixed vector $b \in F$ (a *translation vector*), such that

$$f(x) = h(x) + b, \quad \text{for all} \quad x \in E.$$

The map f given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 8/5 & -6/5 \\ 3/10 & 2/5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an example of the composition of a linear map with a translation.

We claim that functions of this type preserve affine combinations.

Proposition 3.12. *For any two vector spaces E and F, given any function $f\colon E \to F$ defined such that*

$$f(x) = h(x) + b, \quad \text{for all} \quad x \in E,$$

where $h\colon E \to F$ is a linear map and b is some fixed vector in F, for every affine combination $\sum_{i \in I} \lambda_i u_i$ (with $\sum_{i \in I} \lambda_i = 1$), we have

$$f\left(\sum_{i \in I} \lambda_i u_i\right) = \sum_{i \in I} \lambda_i f(u_i).$$

In other words, f preserves affine combinations.

Surprisingly, the converse of Proposition 3.12 also holds.

Proposition 3.13. *For any two vector spaces E and F, let $f \colon E \to F$ be any function that preserves affine combinations, i.e., for every affine combination $\sum_{i \in I} \lambda_i u_i$ (with $\sum_{i \in I} \lambda_i = 1$), we have*

$$f\left(\sum_{i \in I} \lambda_i u_i\right) = \sum_{i \in I} \lambda_i f(u_i).$$

Then, for any $a \in E$, the function $h \colon E \to F$ given by

$$h(x) = f(a + x) - f(a)$$

is a linear map independent of a, and

$$f(a + x) = f(a) + h(x), \quad \text{for all} \quad x \in E.$$

In particular, for $a = 0$, if we let $c = f(0)$, then

$$f(x) = h(x) + c, \quad \text{for all} \quad x \in E.$$

We should think of a as a *chosen origin* in E.

The function f maps the origin a in E to the origin $f(a)$ in F.

Proposition 3.13 shows that the definition of h does not depend on the origin chosen in E. Also, since

$$f(x) = h(x) + c, \quad \text{for all} \quad x \in E$$

for some fixed vector $c \in F$, we see that f is the composition of the linear map h with the translation t_c (in F).

The unique linear map h as above is called the *linear map associated with f* and it is sometimes denoted by $\overrightarrow{f}$.

Observe that the linear map associated with a pure translation is the identity.

In view of Propositions 3.12 and 3.13, it is natural to make the following definition.

Definition 3.4. For any two vector spaces E and F, a function $f\colon E \to F$ is an *affine map* if f preserves affine combinations, *i.e.*, for every affine combination $\sum_{i\in I} \lambda_i u_i$ (with $\sum_{i\in I} \lambda_i = 1$), we have

$$f\left(\sum_{i\in I} \lambda_i u_i\right) = \sum_{i\in I} \lambda_i f(u_i).$$

Equivalently, a function $f\colon E \to F$ is an *affine map* if there is some linear map $h\colon E \to F$ (also denoted by $\overrightarrow{f}$) and some fixed vector $c \in F$ such that

$$f(x) = h(x) + c, \quad \text{for all} \quad x \in E.$$

Note that a linear map always maps the standard origin 0 in E to the standard origin 0 in F.

However an affine map usually maps 0 to a nonzero vector $c = f(0)$. This is the "translation component" of the affine map.

When we deal with affine maps, it is often fruitful to think of the elements of E and F not only as vectors but also as *points*.

In this point of view, *points can only be combined using affine combinations*, but vectors can be combined in an unrestricted fashion using linear combinations.

We can also think of $u + v$ as the *result of translating the point u by the translation t_v*.

These ideas lead to the definition of *affine spaces*, but this would lead us to far afield, and for our purposes, it is enough to stick to vector spaces.

Still, one should be aware that affine combinations really apply to points, and that points are not vectors!

If E and F are finite dimensional vector spaces, with $\dim(E) = n$ and $\dim(F) = m$, then it is useful to represent an affine map with respect to bases in E in F.

However, the translation part c of the affine map must be somewhow incorporated.

There is a standard trick to do this which amounts to viewing an affine map as a linear map between spaces of dimension $n+1$ and $m+1$.

We also have the extra flexibility of choosing origins, $a \in E$ and $b \in F$.

Let $(u_1, \ldots, u_n)$ be a basis of E, $(v_1, \ldots, v_m)$ be a basis of F, and let $a \in E$ and $b \in F$ be any two fixed vectors viewed as *origins*.

Our affine map f has the property that if $v = f(u)$, then

$$v - b = f(a + u - a) - b = f(a) - b + h(u - a).$$

So, if we let $y = v - b$, $x = u - a$, and $d = f(a) - b$, then

$$y = h(x) + d, \quad x \in E.$$

Over the basis $\mathcal{U} = (u_1, \ldots, u_n)$, we write

$$x = x_1 u_1 + \cdots + x_n u_n,$$

and over the basis $\mathcal{V} = (v_1, \ldots, v_m)$, we write

$$\begin{aligned} y &= y_1 v_1 + \cdots + y_m v_m, \\ d &= d_1 v_1 + \cdots + d_m v_m. \end{aligned}$$

Then, since

$$y = h(x) + d,$$

if we let A be the $m \times n$ matrix representing the linear map h, that is, the jth column of A consists of the coordinates of $h(u_j)$ over the basis $(v_1, \ldots, v_m)$, then we can write

$$y_{\mathcal{V}} = A x_{\mathcal{U}} + d_{\mathcal{V}}.$$

where $x_{\mathcal{U}} = (x_1, \ldots, x_n)^\top$, $y_{\mathcal{V}} = (y_1, \ldots, y_m)^\top$, and $d_{\mathcal{V}} = (d_1, \ldots, d_m)^\top$.

This is the *matrix representation* of our affine map f.

The reason for using the origins a and b is that it gives us more flexibility.

In particular, when $E = F$, if there is some $a \in E$ such that $f(a) = a$ (a is a *fixed point* of f), then we can pick $b = a$.

Then, because $f(a) = a$, we get

$$v = f(u) = f(a+u-a) = f(a)+h(u-a) = a+h(u-a),$$

that is

$$v - a = h(u - a).$$

With respect to the new origin a, if we define x and y by

$$\begin{aligned} x &= u - a \\ y &= v - a, \end{aligned}$$

then we get

$$y = h(x).$$

Then, f really behaves like a linear map, but *with respect to the new origin a* (not the *standard origin* 0). This is the case of a rotation around an axis that does not pass through the origin.

Remark: A pair $(a, (u_1, \ldots, u_n))$ where $(u_1, \ldots, u_n)$ is a basis of E and a is an origin chosen in E is called an *affine frame*.

We now describe the trick which allows us to incorporate the translation part d into the matrix A.

We define the $(m+1)\times(n+1)$ matrix A' obtained by first adding d as the $(n+1)$th column, and then $(\underbrace{0, \ldots, 0}_{n}, 1)$ as the $(m+1)$th row:

$$A' = \begin{pmatrix} A & d \\ 0_n & 1 \end{pmatrix}.$$

Then, it is clear that

$$\begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} A & d \\ 0_n & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}$$

iff

$$y = Ax + d.$$

This amounts to considering a point $x \in \mathbb{R}^n$ as a point $(x, 1)$ *in the (affine) hyperplane H_{n+1} in $\mathbb{R}^{n+1}$ of equation $x_{n+1} = 1$*.

Then, an affine map is the restriction to the hyperplane H_{n+1} of the linear map $\widehat{f}$ from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{m+1}$ corresponding to the matrix A', which maps H_{n+1} into H_{m+1} $(\widehat{f}(H_{n+1}) \subseteq H_{m+1})$.

Figure 3.2 illustrates this process for $n = 2$.

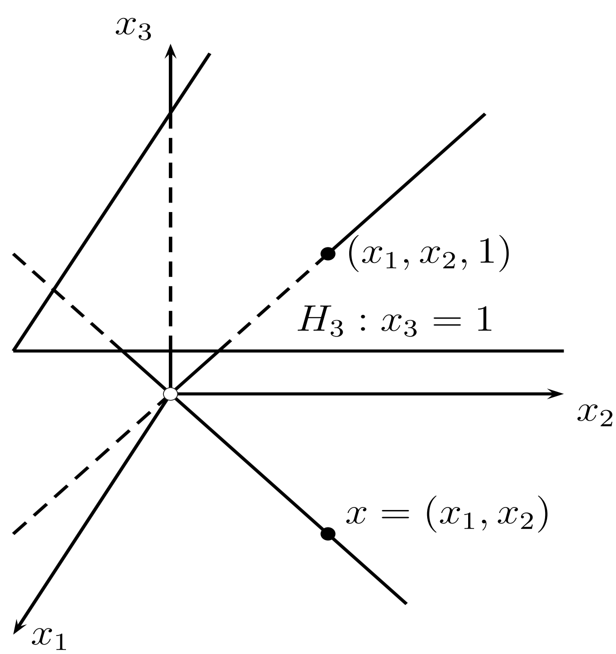

Figure 3.2: Viewing $\mathbb{R}^n$ as a hyperplane in $\mathbb{R}^{n+1}$ $(n = 2)$

For example, the map

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

defines an affine map f which is represented in $\mathbb{R}^3$ by

$$\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}.$$

It is easy to check that the point $a = (6, -3)$ is fixed by f, which means that $f(a) = a$, so by translating the coordinate frame to the origin a, the affine map behaves like a linear map.

The idea of considering $\mathbb{R}^n$ as an hyperplane in $\mathbb{R}^{n+1}$ can be used to define *projective maps*.

Figure 3.3: Dog Logic