Chapter 12

Spectral Theorems in Euclidean and Hermitian Spaces

12.1 Normal Linear Maps

Let $E$ be a real Euclidean space (or a complex Hermitian space) with inner product $u, v \mapsto \langle u, v \rangle$.

In the real Euclidean case, recall that $\langle - , - \rangle$ is bilinear, symmetric and positive definite (i.e., $\langle u, u \rangle > 0$ for all $u \neq 0$).

In the complex Hermitian case, recall that $\langle - , - \rangle$ is sesquilinear, which means that it linear in the first argument, semilinear in the second argument (i.e., $\langle u, \mu v \rangle = \overline{\mu} \langle u, v \rangle$, $\langle v, u \rangle = \overline{\langle u, v \rangle}$, and positive definite (as above).
In both cases we let \( \|u\| = \sqrt{\langle u, u \rangle} \) and the map \( u \mapsto \|u\| \) is a norm.

Recall that every linear map, \( f : E \to E \), has an adjoint \( f^* \) which is a linear map, \( f^* : E \to E \), such that
\[
\langle f(u), v \rangle = \langle u, f^*(v) \rangle,
\]
for all \( u, v \in E \).

Since \( \langle -, - \rangle \) is symmetric, it is obvious that \( f^{**} = f \).

**Definition 12.1.** Given a Euclidean (or Hermitian) space, \( E \), a linear map \( f : E \to E \) is normal iff
\[
f \circ f^* = f^* \circ f.
\]

A linear map \( f : E \to E \) is self-adjoint if \( f = f^* \), skew-self-adjoint if \( f = -f^* \), and orthogonal if \( f \circ f^* = f^* \circ f = \text{id} \).
Our first goal is to show that for every normal linear map $f: E \rightarrow E$ (where $E$ is a Euclidean space), there is an orthonormal basis (w.r.t. $\langle - , - \rangle$) such that the matrix of $f$ over this basis has an especially nice form:

It is a block diagonal matrix in which the blocks are either one-dimensional matrices (i.e., single entries) or two-dimensional matrices of the form

$$
\begin{pmatrix}
\lambda & \mu \\
-\mu & \lambda
\end{pmatrix}
$$

This normal form can be further refined if $f$ is self-adjoint, skew-self-adjoint, or orthogonal.

As a first step, we show that $f$ and $f^*$ have the same kernel when $f$ is normal.

**Proposition 12.1.** Given a Euclidean space $E$, if $f: E \rightarrow E$ is a normal linear map, then $\text{Ker } f = \text{Ker } f^*$. 
The next step is to show that for every linear map \( f : E \to E \), there is some subspace \( W \) of dimension 1 or 2 such that \( f(W) \subseteq W \).

When \( \dim(W) = 1 \), \( W \) is actually an eigenspace for some real eigenvalue of \( f \).

Furthermore, when \( f \) is normal, there is a subspace \( W \) of dimension 1 or 2 such that \( f(W) \subseteq W \) and \( f^*(W) \subseteq W \).

The difficulty is that the eigenvalues of \( f \) are not necessarily real. One way to get around this problem is to \textit{complexify} both the vector space \( E \) and the inner product \( \langle -, - \rangle \).

First, we need to embed a real vector space \( E \) into a complex vector space \( E_\mathbb{C} \).
Definition 12.2. Given a real vector space $E$, let $E_\mathbb{C}$ be the structure $E \times E$ under the addition operation

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2),$$

and multiplication by a complex scalar $z = x + iy$ defined such that

$$(x + iy) \cdot (u, v) = (xu - yv, yu + xv).$$

The space $E_\mathbb{C}$ is called the complexification of $E$.

It is easily shown that the structure $E_\mathbb{C}$ is a complex vector space.

It is also immediate that

$$(0, v) = i(v, 0),$$

and thus, identifying $E$ with the subspace of $E_\mathbb{C}$ consisting of all vectors of the form $(u, 0)$, we can write

$$(u, v) = u + iv.$$ 

Given a vector $w = u + iv$, its conjugate $\overline{w}$ is the vector $\overline{w} = u - iv$. 
Observe that if \((e_1, \ldots, e_n)\) is a basis of \(E\) (a real vector space), then \((e_1, \ldots, e_n)\) is also a basis of \(E_{\mathbb{C}}\) (recall that \(e_i\) is an abbreviation for \((e_i, 0))\).

Given a linear map \(f : E \to E\), the map \(f\) can be extended to a linear map \(f_{\mathbb{C}} : E_{\mathbb{C}} \to E_{\mathbb{C}}\) defined such that

\[
f_{\mathbb{C}}(u + iv) = f(u) + if(v).
\]

For any basis \((e_1, \ldots, e_n)\) of \(E\), the matrix \(M(f)\) representing \(f\) over \((e_1, \ldots, e_n)\) is identical to the matrix \(M(f_{\mathbb{C}})\) representing \(f_{\mathbb{C}}\) over \((e_1, \ldots, e_n)\), where we view \((e_1, \ldots, e_n)\) as a basis of \(E_{\mathbb{C}}\).

As a consequence, \(\det(zI - M(f)) = \det(zI - M(f_{\mathbb{C}}))\), which means that \(f\) and \(f_{\mathbb{C}}\) have the same characteristic polynomial (which has real coefficients).

We know that every polynomial of degree \(n\) with real (or complex) coefficients always has \(n\) complex roots (counted with their multiplicity), and the roots of \(\det(zI - M(f_{\mathbb{C}}))\) that are real (if any) are the eigenvalues of \(f\).
Next, we need to extend the inner product on $E$ to an inner product on $E_\mathbb{C}$.

The inner product $\langle -, - \rangle$ on a Euclidean space $E$ is extended to the Hermitian positive definite form $\langle -, - \rangle_\mathbb{C}$ on $E_\mathbb{C}$ as follows:

$$\langle u_1 + iv_1, u_2 + iv_2 \rangle_\mathbb{C} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle + i(\langle u_2, v_1 \rangle - \langle u_1, v_2 \rangle).$$

Then, given any linear map $f : E \to E$, it is easily verified that the map $f^*_\mathbb{C}$ defined such that

$$f^*_\mathbb{C}(u + iv) = f^*(u) + if^*(v)$$

for all $u, v \in E$, is the adjoint of $f_\mathbb{C}$ w.r.t. $\langle -, - \rangle_\mathbb{C}$. 

Assuming again that $E$ is a Hermitian space, observe that Proposition 12.1 also holds. We deduce the following corollary.

**Proposition 12.2.** Given a Hermitian space $E$, for any normal linear map $f : E \to E$, we have $\operatorname{Ker}(f) \cap \operatorname{Im}(f) = (0)$.

**Proposition 12.3.** Given a Hermitian space $E$, for any normal linear map $f : E \to E$, a vector $u$ is an eigenvector of $f$ for the eigenvalue $\lambda$ (in $\mathbb{C}$) iff $u$ is an eigenvector of $f^*$ for the eigenvalue $\overline{\lambda}$.

The next proposition shows a very important property of normal linear maps: eigenvectors corresponding to distinct eigenvalues are orthogonal.

**Proposition 12.4.** Given a Hermitian space $E$, for any normal linear map $f : E \to E$, if $u$ and $v$ are eigenvectors of $f$ associated with the eigenvalues $\lambda$ and $\mu$ (in $\mathbb{C}$) where $\lambda \neq \mu$, then $\langle u, v \rangle = 0$. 

We can also show easily that the eigenvalues of a self-adjoint linear map are real.

**Proposition 12.5.** Given a Hermitian space $E$, the eigenvalues of any self-adjoint linear map $f : E \to E$ are real.

There is also a version of Proposition 12.5 for a (real) Euclidean space $E$ and a self-adjoint map $f : E \to E$.

**Proposition 12.6.** Given a Euclidean space $E$, if $f : E \to E$ is any self-adjoint linear map, then every eigenvalue $\lambda$ of $f_\mathbb{C}$ is real and is actually an eigenvalue of $f$ (which means that there is some real eigenvector $u \in E$ such that $f(u) = \lambda u$). Therefore, all the eigenvalues of $f$ are real.

Given any subspace $W$ of a Hermitian space $E$, recall that the *orthogonal* $W^\perp$ of $W$ is the subspace defined such that

$$W^\perp = \{ u \in E \mid \langle u, w \rangle = 0, \text{ for all } w \in W \}. $$
Recall that $E = W \oplus W^\perp$ (construct an orthonormal basis of $E$ using the Gram–Schmidt orthonormalization procedure). The same result also holds for Euclidean spaces.

As a warm up for the proof of Theorem 12.10, let us prove that every self-adjoint map on a Euclidean space can be diagonalized with respect to an orthonormal basis of eigenvectors.

**Theorem 12.7.** Given a Euclidean space $E$ of dimension $n$, for every self-adjoint linear map $f : E \to E$, there is an orthonormal basis $(e_1, \ldots, e_n)$ of eigenvectors of $f$ such that the matrix of $f$ w.r.t. this basis is a diagonal matrix

$$
\begin{pmatrix}
\lambda_1 & & \\
& \lambda_2 & \\
& & \ddots \\
& & & \lambda_n
\end{pmatrix},
$$

with $\lambda_i \in \mathbb{R}$. 
One of the key points in the proof of Theorem 12.7 is that we found a subspace $W$ with the property that $f(W) \subseteq W$ implies that $f(W^\perp) \subseteq W^\perp$.

In general, this does not happen, but normal maps satisfy a stronger property which ensures that such a subspace exists.

The following proposition provides a condition that will allow us to show that a normal linear map can be diagonalized. It actually holds for any linear map.

**Proposition 12.8.** Given a Hermitian space $E$, for any linear map $f: E \to E$ and any subspace $W$ of $E$, if $f(W) \subseteq W$, then $f^*(W^\perp) \subseteq W^\perp$.

Consequently, if $f(W) \subseteq W$ and $f^*(W) \subseteq W$, then $f(W^\perp) \subseteq W^\perp$ and $f^*(W^\perp) \subseteq W^\perp$.

The above Proposition also holds for Euclidean spaces. Although we are ready to prove that for every normal linear map $f$ (over a Hermitian space) there is an orthonormal basis of eigenvectors, we now return to real Euclidean spaces.
If \( f : E \to E \) is a linear map and \( w = u + iv \) is an eigenvector of \( f_\mathbb{C} : E_\mathbb{C} \to E_\mathbb{C} \) for the eigenvalue \( z = \lambda + i\mu \), where \( u, v \in E \) and \( \lambda, \mu \in \mathbb{R} \), since

\[
    f_\mathbb{C}(u + iv) = f(u) + if(v)
\]

and

\[
    f_\mathbb{C}(u + iv) = (\lambda + i\mu)(u + iv) = \lambda u - \mu v + i(\mu u + \lambda v),
\]

we have

\[
    f(u) = \lambda u - \mu v \quad \text{and} \quad f(v) = \mu u + \lambda v,
\]

from which we immediately obtain

\[
    f_\mathbb{C}(u - iv) = (\lambda - i\mu)(u - iv),
\]

which shows that \( \overline{w} = u - iv \) is an eigenvector of \( f_\mathbb{C} \) for \( \overline{z} = \lambda - i\mu \). Using this fact, we can prove the following proposition:
Proposition 12.9. Given a Euclidean space $E$, for any normal linear map $f : E \to E$, if $w = u + iv$ is an eigenvector of $f_\mathbb{C}$ associated with the eigenvalue $z = \lambda + i\mu$ (where $u, v \in E$ and $\lambda, \mu \in \mathbb{R}$), if $\mu \neq 0$ (i.e., $z$ is not real) then $\langle u, v \rangle = 0$ and $\langle u, u \rangle = \langle v, v \rangle$, which implies that $u$ and $v$ are linearly independent, and if $W$ is the subspace spanned by $u$ and $v$, then $f(W) = W$ and $f^*(W) = W$. Furthermore, with respect to the (orthogonal) basis $(u, v)$, the restriction of $f$ to $W$ has the matrix

$$
\begin{pmatrix}
\lambda & \mu \\
-\mu & \lambda
\end{pmatrix}.
$$

If $\mu = 0$, then $\lambda$ is a real eigenvalue of $f$ and either $u$ or $v$ is an eigenvector of $f$ for $\lambda$. If $W$ is the subspace spanned by $u$ if $u \neq 0$, or spanned by $v \neq 0$ if $u = 0$, then $f(W) \subseteq W$ and $f^*(W) \subseteq W$. 
Theorem 12.10. (Main Spectral Theorem) Given a Euclidean space $E$ of dimension $n$, for every normal linear map $f: E \to E$, there is an orthonormal basis $(e_1, \ldots, e_n)$ such that the matrix of $f$ w.r.t. this basis is a block diagonal matrix of the form

$$
\begin{pmatrix}
A_1 & \cdots & \\
& A_2 & \cdots & \\
& & \ddots & \vdots \\
& & & A_p
\end{pmatrix}
$$

such that each block $A_j$ is either a one-dimensional matrix (i.e., a real scalar) or a two-dimensional matrix of the form

$$
A_j = \begin{pmatrix}
\lambda_j & -\mu_j \\
\mu_j & \lambda_j
\end{pmatrix}
$$

where $\lambda_j, \mu_j \in \mathbb{R}$, with $\mu_j > 0$. 
After this relatively hard work, we can easily obtain some nice normal forms for the matrices of self-adjoint, skew-self-adjoint, and orthogonal, linear maps.

However, for the sake of completeness, we state the following theorem.

**Theorem 12.11.** Given a Hermitian space \( E \) of dimension \( n \), for every normal linear map \( f : E \rightarrow E \), there is an orthonormal basis \((e_1, \ldots, e_n)\) of eigenvectors of \( f \) such that the matrix of \( f \) w.r.t. this basis is a diagonal matrix

\[
\begin{pmatrix}
\lambda_1 & \cdots \\
& \lambda_2 & \cdots \\
& & \ddots & \ddots \\
& & & \cdots & \lambda_n
\end{pmatrix}
\]

where \( \lambda_j \in \mathbb{C} \).

**Remark:** There is a converse to Theorem 12.11, namely, if there is an orthonormal basis \((e_1, \ldots, e_n)\) of eigenvectors of \( f \), then \( f \) is normal.
12.2 Self-Adjoint, Skew-Self-Adjoint, and Orthogonal Linear Maps

**Theorem 12.12.** Given a Euclidean space $E$ of dimension $n$, for every self-adjoint linear map $f : E \to E$, there is an orthonormal basis $(e_1, \ldots, e_n)$ of eigenvectors of $f$ such that the matrix of $f$ w.r.t. this basis is a diagonal matrix

$$
\begin{pmatrix}
\lambda_1 & \cdots \\
\vdots & \ddots & \vdots \\
\lambda_n & \cdots & \lambda_n
\end{pmatrix}
$$

where $\lambda_i \in \mathbb{R}$.

Theorem 12.12 implies that if $\lambda_1, \ldots, \lambda_p$ are the distinct real eigenvalues of $f$ and $E_i$ is the eigenspace associated with $\lambda_i$, then

$$E = E_1 \oplus \cdots \oplus E_p,$$

where $E_i$ and $E_j$ are orthogonal for all $i \neq j$. 
Theorem 12.13. Given a Euclidean space $E$ of dimension $n$, for every skew-self-adjoint linear map $f : E \to E$, there is an orthonormal basis $(e_1, \ldots, e_n)$ such that the matrix of $f$ w.r.t. this basis is a block diagonal matrix of the form

$$
\begin{pmatrix}
A_1 & \cdots \\
& A_2 & \cdots \\
& & \ddots & \cdots \\
& & & \cdots & A_p
\end{pmatrix}
$$

such that each block $A_j$ is either 0 or a two-dimensional matrix of the form

$$A_j = \begin{pmatrix} 0 & -\mu_j \\ \mu_j & 0 \end{pmatrix}$$

where $\mu_j \in \mathbb{R}$, with $\mu_j > 0$. In particular, the eigenvalues of $f_C$ are pure imaginary of the form $\pm i\mu_j$, or 0.
**Theorem 12.14.** Given a Euclidean space $E$ of dimension $n$, for every orthogonal linear map $f : E \to E$, there is an orthonormal basis $(e_1, \ldots, e_n)$ such that the matrix of $f$ w.r.t. this basis is a block diagonal matrix of the form

$$
\begin{pmatrix}
A_1 & & \\
& \ddots & \\
& & A_p
\end{pmatrix}
$$

such that each block $A_j$ is either 1, $-1$, or a two-dimensional matrix of the form

$$
A_j = \begin{pmatrix}
\cos \theta_j & -\sin \theta_j \\
\sin \theta_j & \cos \theta_j
\end{pmatrix}
$$

where $0 < \theta_j < \pi$.

In particular, the eigenvalues of $f_C$ are of the form $\cos \theta_j \pm i \sin \theta_j$, or 1, or $-1$. 
It is obvious that we can reorder the orthonormal basis of eigenvectors given by Theorem 12.14, so that the matrix of \( f \) w.r.t. this basis is a block diagonal matrix of the form

\[
\begin{pmatrix}
A_1 & \cdots \\
\vdots & \ddots & \vdots \\
\cdots & & A_r \\
\cdots & & -I_q & I_p
\end{pmatrix}
\]

where each block \( A_j \) is a two-dimensional rotation matrix \( A_j \neq \pm I_2 \) of the form

\[
A_j = \begin{pmatrix}
\cos \theta_j & -\sin \theta_j \\
\sin \theta_j & \cos \theta_j
\end{pmatrix}
\]

with \( 0 < \theta_j < \pi \).

The linear map \( f \) has an eigenspace \( E(1, f) = \ker (f - \text{id}) \) of dimension \( p \) for the eigenvalue 1, and an eigenspace \( E(-1, f) = \ker (f + \text{id}) \) of dimension \( q \) for the eigenvalue \(-1\).
If $\det(f) = +1$ ($f$ is a rotation), the dimension $q$ of $E(-1, f)$ must be even, and the entries in $-I_q$ can be paired to form two-dimensional blocks, if we wish.

**Remark:** Theorem 12.14 can be used to prove a sharper version of the Cartan-Dieudonné Theorem.

**Theorem 12.15.** Let $E$ be a Euclidean space of dimension $n \geq 2$. For every isometry $f \in O(E)$, if $p = \dim(E(1, f)) = \dim(\ker(f - \text{id}))$, then $f$ is the composition of $n - p$ reflections and $n - p$ is minimal.

The theorems of this section and of the previous section can be immediately applied to matrices.
12.3 Normal, Symmetric, Skew-Symmetric, Orthogonal, Hermitian, Skew-Hermitian, and Unitary Matrices

First, we consider real matrices.

**Definition 12.3.** Given a real $m \times n$ matrix $A$, the **transpose** $A^\top$ of $A$ is the $n \times m$ matrix $A^\top = (a_{i,j}^\top)$ defined such that

$$a_{i,j}^\top = a_{j,i}$$

for all $i, j$, $1 \leq i \leq m$, $1 \leq j \leq n$. A real $n \times n$ matrix $A$ is

1. **normal** iff

   $$AA^\top = A^\top A,$$

2. **symmetric** iff

   $$A^\top = A,$$

3. **skew-symmetric** iff

   $$A^\top = -A,$$

4. **orthogonal** iff

   $$AA^\top = A^\top A = I_n.$$
Theorem 12.16. For every normal matrix $A$, there is an orthogonal matrix $P$ and a block diagonal matrix $D$ such that $A = PD P^T$, where $D$ is of the form

$$D = \begin{pmatrix} D_1 & \cdots \\ & D_2 & \cdots \\ & & \ddots & \cdots \\ & & & \cdots & D_p \end{pmatrix}$$

such that each block $D_j$ is either a one-dimensional matrix (i.e., a real scalar) or a two-dimensional matrix of the form

$$D_j = \begin{pmatrix} \lambda_j & -\mu_j \\ \mu_j & \lambda_j \end{pmatrix}$$

where $\lambda_j, \mu_j \in \mathbb{R}$, with $\mu_j > 0$. 
Theorem 12.17. For every symmetric matrix $A$, there is an orthogonal matrix $P$ and a diagonal matrix $D$ such that $A = PDP^\top$, where $D$ is of the form

$$D = \begin{pmatrix} \lambda_1 & \cdots \\
                    & \lambda_2 \\
                    & \vdots & \ddots & \vdots \\
                    & \cdots & \cdots & \lambda_m \end{pmatrix}$$

where $\lambda_i \in \mathbb{R}$.
Theorem 12.18. For every skew-symmetric matrix $A$, there is an orthogonal matrix $P$ and a block diagonal matrix $D$ such that $A = PD P^\top$, where $D$ is of the form

$$D = \begin{pmatrix} D_1 & \cdots \\ & D_2 & \cdots \\ & & \ddots & \cdots \\ & & & \cdots & D_p \end{pmatrix}$$

such that each block $D_j$ is either 0 or a two-dimensional matrix of the form

$$D_j = \begin{pmatrix} 0 & -\mu_j \\ \mu_j & 0 \end{pmatrix}$$

where $\mu_j \in \mathbb{R}$, with $\mu_j > 0$. In particular, the eigenvalues of $A$ are pure imaginary of the form $\pm i\mu_j$, or 0.
**Theorem 12.19.** For every orthogonal matrix $A$, there is an orthogonal matrix $P$ and a block diagonal matrix $D$ such that $A = PD P^T$, where $D$ is of the form

$$
D = \begin{pmatrix}
D_1 & \cdots \\
& D_2 & \cdots \\
& & \ddots & \cdots \\
& & & \ddots & \cdots \\
& & & & \cdots & D_p
\end{pmatrix}
$$

such that each block $D_j$ is either 1, $-1$, or a two-dimensional matrix of the form

$$
D_j = \begin{pmatrix}
\cos \theta_j & -\sin \theta_j \\
\sin \theta_j & \cos \theta_j
\end{pmatrix}
$$

where $0 < \theta_j < \pi$.

In particular, the eigenvalues of $A$ are of the form $\cos \theta_j \pm i \sin \theta_j$, or 1, or $-1$.

We now consider complex matrices.
Definition 12.4. Given a complex $m \times n$ matrix $A$, the \textit{transpose} $A^\top$ of $A$ is the $n \times m$ matrix $A^\top = (a_{i,j}^\top)$ defined such that

$$a_{i,j}^\top = a_{j,i}$$

for all $i, j, 1 \leq i \leq m, 1 \leq j \leq n$. The \textit{conjugate} $\overline{A}$ of $A$ is the $m \times n$ matrix $\overline{A} = (b_{i,j})$ defined such that

$$b_{i,j} = \overline{a_{i,j}}$$

for all $i, j, 1 \leq i \leq m, 1 \leq j \leq n$. Given an $n \times n$ complex matrix $A$, the \textit{adjoint} $A^*$ of $A$ is the matrix defined such that

$$A^* = (\overline{A^\top}) = (\overline{A})^\top.$$ 

A complex $n \times n$ matrix $A$ is

1. \textit{normal} iff

$$AA^* = A^*A,$$

2. \textit{Hermitian} iff

$$A^* = A,$$

3. \textit{skew-Hermitian} iff

$$A^* = -A,$$

4. \textit{unitary} iff

$$AA^* = A^*A = I_n.$$
Theorem 12.11 can be restated in terms of matrices as follows. We can also say a little more about eigenvalues (easy exercise left to the reader).

**Theorem 12.20.** For every complex normal matrix $A$, there is a unitary matrix $U$ and a diagonal matrix $D$ such that $A = UDU^*$. Furthermore, if $A$ is Hermitian, $D$ is a real matrix, if $A$ is skew-Hermitian, then the entries in $D$ are pure imaginary or null, and if $A$ is unitary, then the entries in $D$ have absolute value 1.
12.4 Conditioning of Eigenvalue Problems

The following $n \times n$ matrix

$$A = \begin{pmatrix} 0 & & & & & 1 & 0 \\ 1 & 0 & & & & & \\ & 1 & 0 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & \ddots \\ 1 & 0 & & & & \end{pmatrix}$$

has the eigenvalue 0 with multiplicity $n$.

However, if we perturb the top rightmost entry of $A$ by $\epsilon$, it is easy to see that the characteristic polynomial of the matrix

$$A(\epsilon) = \begin{pmatrix} 0 & & & & & \epsilon \\ 1 & 0 & & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots \\ 1 & 0 & & & & \end{pmatrix}$$

is $X^n - \epsilon$. 
It follows that if \( n = 40 \) and \( \epsilon = 10^{-40} \), \( A(10^{-40}) \) has the eigenvalues \( e^{k2\pi i/40}10^{-1} \) with \( k = 1, \ldots, 40 \).

Thus, we see that a very small change (\( \epsilon = 10^{-40} \)) to the matrix \( A \) causes a significant change to the eigenvalues of \( A \) (from 0 to \( e^{k2\pi i/40}10^{-1} \)).

Indeed, the relative error is \( 10^{-39} \).

Worse, due to machine precision, since very small numbers are treated as 0, the error on the computation of eigenvalues (for example, of the matrix \( A(10^{-40}) \)) can be very large.

This phenomenon is similar to the phenomenon discussed in Section 6.3 where we studied the effect of a small perturbation of the coefficients of a linear system \( Ax = b \) on its solution.
In Section 6.3, we saw that the behavior of a linear system under small perturbations is governed by the condition number $\text{cond}(A)$ of the matrix $A$.

In the case of the eigenvalue problem (finding the eigenvalues of a matrix), we will see that the conditioning of the problem depends on the condition number of the change of basis matrix $P$ used in reducing the matrix $A$ to its diagonal form $D = P^{-1}AP$, rather than on the condition number of $A$ itself.

The following proposition in which we assume that $A$ is diagonalizable and that the matrix norm $\| \|$ satisfies a special condition (satisfied by the operator norms $\| \|_p$ for $p = 1, 2, \infty$), is due to Bauer and Fike (1960).
Proposition 12.21. Let $A \in \mathbb{M}_n(\mathbb{C})$ be a diagonalizable matrix, $P$ be an invertible matrix and, $D$ be a diagonal matrix $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ such that

$$A = PDP^{-1},$$

and let $\|\|$ be a matrix norm such that

$$\|\text{diag}(\alpha_1, \ldots, \alpha_n)\| = \max_{1 \leq i \leq n} |\alpha_i|,$$

for every diagonal matrix. Then, for every perturbation matrix $\delta A$, if we write

$$B_i = \{ z \in \mathbb{C} \mid |z - \lambda_i| \leq \text{cond}(P) \|\delta A\| \},$$

for every eigenvalue $\lambda$ of $A + \delta A$, we have

$$\lambda \in \bigcup_{k=1}^{n} B_k.$$
Proposition 12.21 implies that for any diagonalizable matrix $A$, if we define $\Gamma(A)$ by

$$\Gamma(A) = \inf \{\text{cond}(P) \mid P^{-1}AP = D\},$$

then for every eigenvalue $\lambda$ of $A + \delta A$, we have

$$\lambda \in \bigcup_{k=1}^{n} \{z \in \mathbb{C}^n \mid |z - \lambda_k| \leq \Gamma(A) \|\delta A\|\}.$$

The number $\Gamma(A)$ is called the *conditioning of $A$ relative to the eigenvalue problem*.

If $A$ is a normal matrix, since by Theorem 12.20, $A$ can be diagonalized with respect to a unitary matrix $U$, and since for the spectral norm $\|U\|_2 = 1$, we see that $\Gamma(A) = 1$. 

Therefore, normal matrices are very well conditionned w.r.t. the eigenvalue problem. In fact, for every eigenvalue $\lambda$ of $A + \delta A$ (with $A$ normal), we have

$$\lambda \in \bigcup_{k=1}^{n} \{ z \in \mathbb{C}^n \mid |z - \lambda_k| \leq \|\delta A\|_2 \}.$$ 

If $A$ and $A + \delta A$ are both symmetric (or Hermitian), there are sharper results; see Proposition 12.27.

Note that the matrix $A(\epsilon)$ from the beginning of the section is not normal.
12.5 Rayleigh Ratios and the Courant-Fischer Theorem

A fact that is used frequently in optimization problems is that the eigenvalues of a symmetric matrix are characterized in terms of what is known as the Rayleigh ratio, defined by

$$R(A)(x) = \frac{x^\top Ax}{x^\top x}, \quad x \in \mathbb{R}^n, x \neq 0.$$ 

The following proposition is often used to prove the correctness of various optimization or approximation problems (for example PCA).
Proposition 12.22. (Rayleigh–Ritz) If $A$ is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and if $(u_1, \ldots, u_n)$ is any orthonormal basis of eigenvectors of $A$, where $u_i$ is a unit eigenvector associated with $\lambda_i$, then

$$\max_{x \neq 0} \frac{x^\top Ax}{x^\top x} = \lambda_n$$

(with the maximum attained for $x = u_n$), and

$$\max_{x \neq 0, x \in \{u_{n-k+1}, \ldots, u_n\} \perp} \frac{x^\top Ax}{x^\top x} = \lambda_{n-k}$$

(with the maximum attained for $x = u_{n-k}$), where $1 \leq k \leq n - 1$. Equivalently, if $V_k$ is the subspace spanned by $(u_1, \ldots, u_k)$, then

$$\lambda_k = \max_{x \neq 0, x \in V_k} \frac{x^\top Ax}{x^\top x}, \quad k = 1, \ldots, n.$$
For our purposes, we need the version of Proposition 12.22 applying to \( \min \) instead of \( \max \), whose proof is obtained by a trivial modification of the proof of Proposition 12.22.

**Proposition 12.23.** *(Rayleigh–Ritz)* If \( A \) is a symmetric \( n \times n \) matrix with eigenvalues \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) and if \((u_1, \ldots, u_n)\) is any orthonormal basis of eigenvectors of \( A \), where \( u_i \) is a unit eigenvector associated with \( \lambda_i \), then

\[
\min_{x \neq 0} \frac{x^\top Ax}{x^\top x} = \lambda_1
\]

(with the minimum attained for \( x = u_1 \)), and

\[
\min_{x \neq 0, x \in \{u_1, \ldots, u_{i-1}\}^\perp} \frac{x^\top Ax}{x^\top x} = \lambda_i
\]

(with the minimum attained for \( x = u_i \)), where \( 2 \leq i \leq n \). Equivalently, if \( W_k = V_{k-1}^\perp \) denotes the subspace spanned by \((u_k, \ldots, u_n)\) (with \( V_0 = (0) \)), then

\[
\lambda_k = \min_{x \neq 0, x \in W_k} \frac{x^\top Ax}{x^\top x} = \min_{x \neq 0, x \in V_{k-1}^\perp} \frac{x^\top Ax}{x^\top x}, \quad k = 1, \ldots, n.
\]
Propositions 12.22 and 12.23 together are known the *Rayleigh–Ritz theorem*.

As an application of Propositions 12.22 and 12.23, we prove a proposition which allows us to compare the eigenvalues of two symmetric matrices $A$ and $B = R^TAR$, where $R$ is a rectangular matrix satisfying the equation $R^TR = I$.

First, we need a definition. Given an $n \times n$ symmetric matrix $A$ and an $m \times m$ symmetric $B$, with $m \leq n$, if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $A$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$ are the eigenvalues of $B$, then we say that the eigenvalues of $B$ *interlace* the eigenvalues of $A$ if

$$\lambda_i \leq \mu_i \leq \lambda_{n-m+i}, \quad i = 1, \ldots, m.$$
Proposition 12.24. Let $A$ be an $n \times n$ symmetric matrix, $R$ be an $n \times m$ matrix such that $R^\top R = I$ (with $m \leq n$), and let $B = R^\top A R$ (an $m \times m$ matrix). The following properties hold:

(a) The eigenvalues of $B$ interlace the eigenvalues of $A$.

(b) If $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $A$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$ are the eigenvalues of $B$, and if $\lambda_i = \mu_i$, then there is an eigenvector $v$ of $B$ with eigenvalue $\mu_i$ such that $Rv$ is an eigenvector of $A$ with eigenvalue $\lambda_i$.

Proposition 12.24 immediately implies the Poincaré separation theorem. It can be used in situations, such as in quantum mechanics, where one has information about the inner products $u_i^\top A u_j$. 
Proposition 12.25. (Poincaré separation theorem)
Let \( A \) be a \( n \times n \) symmetric (or Hermitian) matrix, let \( r \) be some integer with \( 1 \leq r \leq n \), and let \( (u_1, \ldots, u_r) \) be \( r \) orthonormal vectors. Let \( B = (u_i^\top A u_j) \) (an \( r \times r \) matrix), let \( \lambda_1(A) \leq \ldots \leq \lambda_n(A) \) be the eigenvalues of \( A \) and \( \lambda_1(B) \leq \ldots \leq \lambda_r(B) \) be the eigenvalues of \( B \); then we have

\[
\lambda_k(A) \leq \lambda_k(B) \leq \lambda_{k+n-r}(A), \quad k = 1, \ldots, r.
\]

Observe that Proposition 12.24 implies that

\[
\lambda_1 + \cdots + \lambda_m \leq \text{tr}(R^\top A R) \leq \lambda_{n-m+1} + \cdots + \lambda_n.
\]
If $P_1$ is the the $n \times (n - 1)$ matrix obtained from the identity matrix by dropping its last column, we have $P_1^\top P_1 = I$, and the matrix $B = P_1^\top A P_1$ is the matrix obtained from $A$ by deleting its last row and its last column. In this case, the interlacing result is

$$
\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-2} \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n,
$$

a genuine interlacing.

We obtain similar results with the matrix $P_{n-r}$ obtained by dropping the last $n - r$ columns of the identity matrix and setting $B = P_{n-r}^\top A P_{n-r}$ ($B$ is the $r \times r$ matrix obtained from $A$ by deleting its last $n-r$ rows and columns).

In this case, we have the following interlacing inequalities known as *Cauchy interlacing theorem*:

$$
\lambda_k \leq \mu_k \leq \lambda_{k+n-r}, \quad k = 1, \ldots, r.
$$

\((*)\)
Another useful tool to prove eigenvalue equalities is the Courant–Fischer characterization of the eigenvalues of a symmetric matrix, also known as the Min-max (and Max-min) theorem.

**Theorem 12.26. (Courant–Fischer)** Let $A$ be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. If $\mathcal{V}_k$ denotes the set of subspaces of $\mathbb{R}^n$ of dimension $k$, then

$$
\lambda_k = \max_{W \in \mathcal{V}_{n-k+1}} \min_{x \in W, x \neq 0} \frac{x^\top Ax}{x^\top x}
$$

$$
\lambda_k = \min_{W \in \mathcal{V}_k} \max_{x \in W, x \neq 0} \frac{x^\top Ax}{x^\top x}.
$$

The Courant–Fischer theorem yields the following useful result about perturbing the eigenvalues of a symmetric matrix due to Hermann Weyl.
Proposition 12.27. Given two $n \times n$ symmetric matrices $A$ and $B = A + \delta A$, if $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ are the eigenvalues of $A$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$ are the eigenvalues of $B$, then

$$|\alpha_k - \beta_k| \leq \rho(\delta A) \leq \|\delta A\|_2, \quad k = 1, \ldots, n.$$  

Proposition 12.27 also holds for Hermitian matrices.

A pretty result of Wielandt and Hoffman asserts that

$$\sum_{k=1}^{n} (\alpha_k - \beta_k)^2 \leq \|\delta A\|_F^2,$$

where $\| \cdot \|_F$ is the Frobenius norm. However, the proof is significantly harder than the above proof; see Lax [25].

The Courant–Fischer theorem can also be used to prove some famous inequalities due to Hermann Weyl.
Given two symmetric (or Hermitian) matrices $A$ and $B$, let $\lambda_i(A)$, $\lambda_i(B)$, and $\lambda_i(A+B)$ denote the $i$th eigenvalue of $A$, $B$, and $A+B$, respectively, arranged in nondecreasing order.

**Proposition 12.28.** (Weyl) Given two symmetric (or Hermitian) $n \times n$ matrices $A$ and $B$, the following inequalities hold: For all $i, j, k$ with $1 \leq i, j, k \leq n$:

1. If $i + j = k + 1$, then
   \[ \lambda_i(A) + \lambda_j(B) \leq \lambda_k(A + B). \]

2. If $i + j = k + n$, then
   \[ \lambda_k(A + B) \leq \lambda_i(A) + \lambda_j(B). \]

In the special case $i = j = k$, we obtain

$\lambda_1(A) + \lambda_1(B) \leq \lambda_1(A + B)$, \hspace{1cm} $\lambda_n(A + B) \leq \lambda_n(A) + \lambda_n(B)$.

It follows that $\lambda_1$ is concave, while $\lambda_n$ is convex.
If \( i = 1 \) and \( j = k \), we obtain
\[
\lambda_1(A) + \lambda_k(B) \leq \lambda_k(A + B),
\]
and if \( i = k \) and \( j = n \), we obtain
\[
\lambda_k(A + B) \leq \lambda_k(A) + \lambda_n(B),
\]
and combining them, we get
\[
\lambda_1(A) + \lambda_k(B) \leq \lambda_k(A + B) \leq \lambda_k(A) + \lambda_n(B).
\]

In particular, if \( B \) is positive semidefinite, since its eigenvalues are nonnegative, we obtain the following inequality known as the \textit{monotonicity theorem} for symmetric (or Hermitian) matrices:

if \( A \) and \( B \) are symmetric (or Hermitian) and \( B \) is positive semidefinite, then
\[
\lambda_k(A) \leq \lambda_k(A + B) \quad k = 1, \ldots, n.
\]