

Chapter 11

Basics of Hermitian Geometry

11.1 Sesquilinear Forms, Hermitian Forms, Hermitian Spaces, Pre-Hilbert Spaces

In this chapter, we generalize the basic results of Euclidean geometry presented in Chapter 9 to *vector spaces over the complex numbers*.

Some complications arise, due to *complex conjugation*.

Recall that for any complex number $z \in \mathbb{C}$, if $z = x + iy$ where $x, y \in \mathbb{R}$, we let $\Re z = x$, the real part of z , and $\Im z = y$, the imaginary part of z .

We also denote the *conjugate* of $z = x + iy$ as $\bar{z} = x - iy$, and the absolute value (or length, or modulus) of z as $|z|$. Recall that $|z|^2 = z\bar{z} = x^2 + y^2$.

There are many natural situations where a map $\varphi: E \times E \rightarrow \mathbb{C}$ is linear in its first argument and only semilinear in its second argument.

For example, the natural inner product to deal with functions $f: \mathbb{R} \rightarrow \mathbb{C}$, especially Fourier series, is

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx,$$

which is semilinear (but not linear) in g .

Definition 11.1. Given two vector spaces E and F over the complex field \mathbb{C} , a function $f: E \rightarrow F$ is *semilinear* if

$$\begin{aligned} f(u + v) &= f(u) + f(v), \\ f(\lambda u) &= \bar{\lambda} f(u), \end{aligned}$$

for all $u, v \in E$ and all $\lambda \in \mathbb{C}$.

Remark: Instead of defining semilinear maps, we could have defined the vector space \overline{E} as the vector space with the same carrier set E , whose addition is the same as that of E , but whose multiplication by a complex number is given by

$$(\lambda, u) \mapsto \overline{\lambda}u.$$

Then, it is easy to check that a function $f: E \rightarrow \mathbb{C}$ is semilinear iff $f: \overline{E} \rightarrow \mathbb{C}$ is linear.

We can now define sesquilinear forms and Hermitian forms.

Definition 11.2. Given a complex vector space E , a function $\varphi: E \times E \rightarrow \mathbb{C}$ is a *sesquilinear form* iff it is linear in its first argument and semilinear in its second argument, which means that

$$\varphi(u_1 + u_2, v) = \varphi(u_1, v) + \varphi(u_2, v),$$

$$\varphi(u, v_1 + v_2) = \varphi(u, v_1) + \varphi(u, v_2),$$

$$\varphi(\lambda u, v) = \lambda \varphi(u, v),$$

$$\varphi(u, \mu v) = \bar{\mu} \varphi(u, v),$$

for all $u, v, u_1, u_2, v_1, v_2 \in E$, and all $\lambda, \mu \in \mathbb{C}$. A function $\varphi: E \times E \rightarrow \mathbb{C}$ is a *Hermitian form* iff it is sesquilinear and if

$$\varphi(v, u) = \overline{\varphi(u, v)}$$

for all $u, v \in E$.

Obviously, $\varphi(0, v) = \varphi(u, 0) = 0$.

Also note that if $\varphi: E \times E \rightarrow \mathbb{C}$ is sesquilinear, we have

$$\begin{aligned}\varphi(\lambda u + \mu v, \lambda u + \mu v) &= |\lambda|^2 \varphi(u, u) + \lambda \bar{\mu} \varphi(u, v) \\ &\quad + \bar{\lambda} \mu \varphi(v, u) + |\mu|^2 \varphi(v, v),\end{aligned}$$

and if $\varphi: E \times E \rightarrow \mathbb{C}$ is Hermitian, we have

$$\begin{aligned}\varphi(\lambda u + \mu v, \lambda u + \mu v) \\ = |\lambda|^2 \varphi(u, u) + 2\Re(\lambda \bar{\mu} \varphi(u, v)) + |\mu|^2 \varphi(v, v).\end{aligned}$$

Note that restricted to real coefficients, a sesquilinear form is bilinear (we sometimes say \mathbb{R} -bilinear).

The function $\Phi: E \rightarrow \mathbb{C}$ defined such that $\Phi(u) = \varphi(u, u)$ for all $u \in E$ is called the *quadratic form* associated with φ .

The standard example of a Hermitian form on \mathbb{C}^n is the map φ defined such that

$$\varphi((x_1, \dots, x_n), (y_1, \dots, y_n)) = x_1\overline{y_1} + x_2\overline{y_2} + \dots + x_n\overline{y_n}.$$

This map is also positive definite, but before dealing with these issues, we show the following useful proposition.

Proposition 11.1. *Given a complex vector space E , the following properties hold:*

- (1) *A sesquilinear form $\varphi: E \times E \rightarrow \mathbb{C}$ is a Hermitian form iff $\varphi(u, u) \in \mathbb{R}$ for all $u \in E$.*
- (2) *If $\varphi: E \times E \rightarrow \mathbb{C}$ is a sesquilinear form, then*

$$4\varphi(u, v) = \varphi(u + v, u + v) - \varphi(u - v, u - v) \\ + i\varphi(u + iv, u + iv) - i\varphi(u - iv, u - iv),$$

and

$$2\varphi(u, v) = (1 + i)(\varphi(u, u) + \varphi(v, v)) \\ - \varphi(u - v, u - v) - i\varphi(u - iv, u - iv).$$

These are called *polarization identities*.

Proposition 11.1 shows that a sesquilinear form is completely determined by the quadratic form $\Phi(u) = \varphi(u, u)$, even if φ is not Hermitian.

This is false for a real bilinear form, unless it is symmetric.

For example, the bilinear form $\varphi: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ defined such that

$$\varphi((x_1, y_1), (x_2, y_2)) = x_1y_2 - x_2y_1$$

is not identically zero, and yet, it is null on the diagonal.

However, a real symmetric bilinear form is indeed determined by its values on the diagonal, as we saw in Chapter 9.

As in the Euclidean case, Hermitian forms for which $\varphi(u, u) \geq 0$ play an important role.

Definition 11.3. Given a complex vector space E , a Hermitian form $\varphi: E \times E \rightarrow \mathbb{C}$ is *positive* iff $\varphi(u, u) \geq 0$ for all $u \in E$, and *positive definite* iff $\varphi(u, u) > 0$ for all $u \neq 0$. A pair $\langle E, \varphi \rangle$ where E is a complex vector space and φ is a Hermitian form on E is called a *pre-Hilbert space* if φ is positive, and a *Hermitian (or unitary) space* if φ is positive definite.

We warn our readers that some authors, such as Lang [22], define a pre-Hilbert space as what we define to be a Hermitian space.

We prefer following the terminology used in Schwartz [26] and Bourbaki [7].

The quantity $\varphi(u, v)$ is usually called the *Hermitian product* of u and v . We will occasionally call it the inner product of u and v .

Given a pre-Hilbert space $\langle E, \varphi \rangle$, as in the case of a Euclidean space, we also denote $\varphi(u, v)$ as

$$u \cdot v, \quad \text{or} \quad \langle u, v \rangle, \quad \text{or} \quad (u|v),$$

and $\sqrt{\Phi(u)}$ as $\|u\|$.

Example 1. The complex vector space \mathbb{C}^n under the Hermitian form

$$\varphi((x_1, \dots, x_n), (y_1, \dots, y_n)) = x_1 \bar{y}_1 + x_2 \bar{y}_2 + \dots + x_n \bar{y}_n$$

is a Hermitian space.

Example 2. Let l^2 denote the set of all countably infinite sequences $x = (x_i)_{i \in \mathbb{N}}$ of complex numbers such that $\sum_{i=0}^{\infty} |x_i|^2$ is defined (i.e. the sequence $\sum_{i=0}^n |x_i|^2$ converges as $n \rightarrow \infty$).

It can be shown that the map $\varphi: l^2 \times l^2 \rightarrow \mathbb{C}$ defined such that

$$\varphi((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) = \sum_{i=0}^{\infty} x_i \bar{y}_i$$

is well defined, and l^2 is a Hermitian space under φ . Actually, l^2 is even a Hilbert space.

Example 3. Consider the set $\mathcal{C}_{piece}[a, b]$ of piecewise bounded continuous functions $f: [a, b] \rightarrow \mathbb{C}$ under the Hermitian form

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx.$$

It is easy to check that this Hermitian form is positive, but it is not definite. Thus, under this Hermitian form, $\mathcal{C}_{piece}[a, b]$ is only a pre-Hilbert space.

Example 4. Let $\mathcal{C}[a, b]$ be the set of complex-valued continuous functions $f: [a, b] \rightarrow \mathbb{C}$ under the Hermitian form

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx.$$

It is easy to check that this Hermitian form is positive definite. Thus, $\mathcal{C}[a, b]$ is a Hermitian space.

The Cauchy-Schwarz inequality and the Minkowski inequalities extend to pre-Hilbert spaces and to Hermitian spaces.

Proposition 11.2. *Let $\langle E, \varphi \rangle$ be a pre-Hilbert space with associated quadratic form Φ . For all $u, v \in E$, we have the **Cauchy-Schwarz inequality**:*

$$|\varphi(u, v)| \leq \sqrt{\Phi(u)}\sqrt{\Phi(v)}.$$

Furthermore, if $\langle E, \varphi \rangle$ is a Hermitian space, the equality holds iff u and v are linearly dependent.

*We also have the **Minkovski inequality**:*

$$\sqrt{\Phi(u + v)} \leq \sqrt{\Phi(u)} + \sqrt{\Phi(v)}.$$

Furthermore, if $\langle E, \varphi \rangle$ is a Hermitian space, the equality holds iff u and v are linearly dependent, where in addition, if $u \neq 0$ and $v \neq 0$, then $u = \lambda v$ for some real λ such that $\lambda > 0$.

As in the Euclidean case, if $\langle E, \varphi \rangle$ is a Hermitian space, the Minkovski inequality

$$\sqrt{\Phi(u + v)} \leq \sqrt{\Phi(u)} + \sqrt{\Phi(v)}$$

shows that the map $u \mapsto \sqrt{\Phi(u)}$ is a **norm** on E .

The norm induced by φ is called the *Hermitian norm induced by φ* .

We usually denote $\sqrt{\Phi(u)}$ as $\|u\|$, and the Cauchy-Schwarz inequality is written as

$$|u \cdot v| \leq \|u\| \|v\| .$$

Since a Hermitian space is a normed vector space, it is a topological space under the topology induced by the norm (a basis for this topology is given by the open balls $B_0(u, \rho)$ of center u and radius $\rho > 0$, where

$$B_0(u, \rho) = \{v \in E \mid \|v - u\| < \rho\}.$$

If E has finite dimension, every linear map is continuous.

The Cauchy-Schwarz inequality

$$|u \cdot v| \leq \|u\| \|v\|$$

shows that $\varphi: E \times E \rightarrow \mathbb{C}$ is continuous, and thus, that $\|\cdot\|$ is continuous.

If $\langle E, \varphi \rangle$ is only pre-Hilbertian, $\|u\|$ is called a *semi-norm*.

In this case, the condition

$$\|u\| = 0 \quad \text{implies} \quad u = 0$$

is not necessarily true.

However, the Cauchy-Schwarz inequality shows that if $\|u\| = 0$, then $u \cdot v = 0$ for all $v \in E$.

We will now basically mirror the presentation of Euclidean geometry given in Chapter 9 rather quickly, leaving out most proofs, except when they need to be seriously amended.

11.2 Orthogonality, Duality, Adjoint of A Linear Map

In this section, we assume that we are dealing with Hermitian spaces. We denote the Hermitian inner product as $u \cdot v$ or $\langle u, v \rangle$.

The concepts of orthogonality, orthogonal family of vectors, orthonormal family of vectors, and orthogonal complement of a set of vectors, are unchanged from the Euclidean case (Definition 9.2).

For example, the set $\mathcal{C}[-\pi, \pi]$ of continuous functions $f: [-\pi, \pi] \rightarrow \mathbb{C}$ is a Hermitian space under the product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx,$$

and the family $(e^{ikx})_{k \in \mathbb{Z}}$ is orthogonal.

Proposition 9.2 and 9.3 hold without any changes.

It is easy to show that

$$\left\| \sum_{i=1}^n u_i \right\|^2 = \sum_{i=1}^n \|u_i\|^2 + \sum_{1 \leq i < j \leq n} 2\Re(u_i \cdot u_j).$$

Analogously to the case of Euclidean spaces of finite dimension, the Hermitian product induces a *canonical bijection* (i.e., independent of the choice of bases) between the vector space E and the space E^* .

This is one of the places where conjugation shows up, but in this case, troubles are minor.

Given a Hermitian space E , for any vector $u \in E$, let $\varphi_u^l: E \rightarrow \mathbb{C}$ be the map defined such that

$$\varphi_u^l(v) = \overline{u \cdot v}, \quad \text{for all } v \in E.$$

Similarly, for any vector $v \in E$, let $\varphi_v^r: E \rightarrow \mathbb{C}$ be the map defined such that

$$\varphi_v^r(u) = u \cdot v, \quad \text{for all } u \in E.$$

Since the Hermitian product is linear in its first argument u , the map φ_v^r is a linear form in E^* , and since it is semilinear in its second argument v , the map φ_u^l is also a linear form in E^* .

Thus, we have two maps $\flat^l: E \rightarrow E^*$ and $\flat^r: E \rightarrow E^*$, defined such that

$$\flat^l(u) = \varphi_u^l, \quad \text{and} \quad \flat^r(v) = \varphi_v^r.$$

Actually, it is easy to show that $\varphi_u^l = \varphi_u^r$ and $\flat^l = \flat^r$.

Therefore, we use the notation φ_u for both φ_u^l and φ_u^r , and \flat for both \flat^l and \flat^r .

Theorem 11.3. *let E be a Hermitian space E . The map $\flat: E \rightarrow E^*$ defined such that*

$$\flat(u) = \varphi_u^l = \varphi_u^r \quad \text{for all } u \in E$$

is semilinear and injective. When E is also of finite dimension, the map $\flat: \overline{E} \rightarrow E^$ is a canonical isomorphism.*

The inverse of the isomorphism $\flat: \overline{E} \rightarrow E^*$ is denoted by $\sharp: E^* \rightarrow \overline{E}$.

As a corollary of the isomorphism $\flat: \overline{E} \rightarrow E^*$, if E is a Hermitian space of finite dimension, then every linear form $f \in E^*$ corresponds to a unique $v \in E$, such that

$$f(u) = u \cdot v, \quad \text{for every } u \in E.$$

In particular, if f is not the null form, the kernel of f , which is a hyperplane H , is precisely the set of vectors that are orthogonal to v .

Remark. The “musical map” $\flat: \overline{E} \rightarrow E^*$ is not surjective when E has infinite dimension.

This result will be salvaged by restricting our attention to continuous linear maps, and by assuming that the vector space E is a *Hilbert space*.

The existence of the isomorphism $\flat: \overline{E} \rightarrow E^*$ is crucial to the existence of adjoint maps.

Indeed, Theorem 11.3 allows us to define the adjoint of a linear map on a Hermitian space.

Let E be a Hermitian space of finite dimension n , and let $f: E \rightarrow E$ be a linear map.

For every $u \in E$, the map

$$v \mapsto \overline{u \cdot f(v)}$$

is clearly a linear form in E^* , and by Theorem 11.3, there is a unique vector in E denoted by $f^*(u)$, such that

$$\overline{f^*(u) \cdot v} = \overline{u \cdot f(v)},$$

that is,

$$f^*(u) \cdot v = u \cdot f(v), \quad \text{for every } v \in E.$$

Proposition 11.4. *Given a Hermitian space E of finite dimension, for every linear map $f: E \rightarrow E$, there is a unique linear map $f^*: E \rightarrow E$, such that*

$$f^*(u) \cdot v = u \cdot f(v),$$

for all $u, v \in E$. The map f^ is called the **adjoint of f** (w.r.t. to the Hermitian product).*

The fact that

$$v \cdot u = \overline{u \cdot v}$$

implies that the adjoint f^* of f is also characterized by

$$f(u) \cdot v = u \cdot f^*(v),$$

for all $u, v \in E$. It is also obvious that $f^{**} = f$.

Given two Hermitian spaces E and F , where the Hermitian product on E is denoted as $\langle -, - \rangle_1$ and the Hermitian product on F is denoted as $\langle -, - \rangle_2$, given any linear map $f: E \rightarrow F$, it is immediately verified that the proof of Proposition 11.4 can be adapted to show that there is a unique linear map $f^*: F \rightarrow E$ such that

$$\langle f(u), v \rangle_2 = \langle u, f^*(v) \rangle_1$$

for all $u \in E$ and all $v \in F$. The linear map f^* is also called the *adjoint* of f .

As in the Euclidean case, Theorem 11.3 can be used to show that any Hermitian space of finite dimension has an orthonormal basis. The proof is unchanged.

Proposition 11.5. *Given any nontrivial Hermitian space E of finite dimension $n \geq 1$, there is an orthonormal basis (u_1, \dots, u_n) for E .*

The *Gram–Schmidt orthonormalization procedure* also applies to Hermitian spaces of finite dimension, without any changes from the Euclidean case!

Proposition 11.6. *Given any nontrivial Hermitian space E of finite dimension $n \geq 1$, from any basis (e_1, \dots, e_n) for E , we can construct an orthonormal basis*

(u_1, \dots, u_n) for E , with the property that for every k , $1 \leq k \leq n$, the families (e_1, \dots, e_k) and (u_1, \dots, u_k) generate the same subspace.

Remarks: The remarks made after Proposition 9.7 also apply here, except that in the QR -decomposition, Q is a unitary matrix.

As a consequence of Proposition 9.6 (or Proposition 11.6), given any Hermitian space of finite dimension n , if (e_1, \dots, e_n) is an orthonormal basis for E , then for any two vectors $u = u_1e_1 + \dots + u_ne_n$ and $v = v_1e_1 + \dots + v_ne_n$, the Hermitian product $u \cdot v$ is expressed as

$$u \cdot v = (u_1e_1 + \dots + u_ne_n) \cdot (v_1e_1 + \dots + v_ne_n) = \sum_{i=1}^n u_i \bar{v}_i,$$

and the norm $\|u\|$ as

$$\|u\| = \|u_1e_1 + \dots + u_ne_n\| = \sqrt{\sum_{i=1}^n |u_i|^2}.$$

Proposition 9.8 also holds unchanged.

Proposition 11.7. *Given any nontrivial Hermitian space E of finite dimension $n \geq 1$, for any subspace F of dimension k , the orthogonal complement F^\perp of F has dimension $n - k$, and $E = F \oplus F^\perp$. Furthermore, we have $F^{\perp\perp} = F$.*

11.3 Linear Isometries (also called Unitary Transformations)

In this section, we consider linear maps between Hermitian spaces that preserve the Hermitian norm.

All definitions given for Euclidean spaces in Section 9.3 extend to Hermitian spaces, *except that orthogonal transformations are called unitary transformation*, but Proposition 9.9 only extends with a modified condition (2).

Indeed, the old proof that (2) implies (3) does not work, and the implication is in fact false! It can be repaired by strengthening condition (2). For the sake of completeness, we state the Hermitian version of Definition 9.3.

Definition 11.4. Given any two nontrivial Hermitian spaces E and F of the same finite dimension n , a function $f: E \rightarrow F$ is *a unitary transformation, or a linear isometry* iff it is linear and

$$\|f(u)\| = \|u\|,$$

for all $u \in E$.

Proposition 9.9 can be salvaged by strengthening condition (2).

Proposition 11.8. *Given any two nontrivial Hermitian space E and F of the same finite dimension n , for every function $f: E \rightarrow F$, the following properties are equivalent:*

- (1) *f is a linear map and $\|f(u)\| = \|u\|$, for all $u \in E$;*
- (2) *$\|f(v) - f(u)\| = \|v - u\|$ and $f(iu) = if(u)$, for all $u, v \in E$;*
- (3) *$f(u) \cdot f(v) = u \cdot v$, for all $u, v \in E$.*

Furthermore, such a map is bijective.

Observe that from $f(iu) = if(u)$, for $u = 0$, we get $f(0) = if(0)$, which implies that $f(0) = 0$.

Remarks: (i) In the Euclidean case, we proved that the assumption

$$(2') \quad \|f(v) - f(u)\| = \|v - u\|, \text{ for all } u, v \in E, \text{ and } f(0) = 0;$$

implies (3). For this, we used the polarization identity

$$2u \cdot v = \|u\|^2 + \|v\|^2 - \|u - v\|^2.$$

In the Hermitian case, the polarization identity involves the complex number i .

In fact, the implication (2') implies (3) is *false* in the Hermitian case! Conjugation $z \mapsto \bar{z}$ satisfies (2') since

$$|\overline{z_2} - \overline{z_1}| = |\overline{z_2 - z_1}| = |z_2 - z_1|,$$

and yet, it is not linear!

(ii) If we modify (2) by changing the second condition by now requiring that there is some $\tau \in E$ such that

$$f(\tau + iu) = f(\tau) + i(f(\tau + u) - f(\tau))$$

for all $u \in E$, then the function $g: E \rightarrow E$ defined such that

$$g(u) = f(\tau + u) - f(\tau)$$

satisfies the old conditions of (2), and the implications (2) \rightarrow (3) and (3) \rightarrow (1) prove that g is linear, and thus that f is affine.

In view of the first remark, some condition involving i is needed on f , in addition to the fact that f is distance-preserving.

We are now going to take a closer look at the isometries $f: E \rightarrow E$ of a Hermitian space of finite dimension.

11.4 The Unitary Group, Unitary Matrices

In this section, as a mirror image of our treatment of the isometries of a Euclidean space, we explore some of the fundamental properties of the unitary group and of unitary matrices.

Definition 11.5. Given a complex $m \times n$ matrix A , the *transpose* A^\top of A is the $n \times m$ matrix $A^\top = (a_{ij}^\top)$ defined such that

$$a_{ij}^\top = a_{ji}$$

and the *conjugate* \bar{A} of A is the $m \times n$ matrix $\bar{A} = (b_{ij})$ defined such that

$$b_{ij} = \bar{a}_{ij}$$

for all i, j , $1 \leq i \leq m$, $1 \leq j \leq n$. The *adjoint* A^* of A is the matrix defined such that

$$A^* = \overline{(A^\top)} = (\bar{A})^\top.$$

Proposition 11.9. *Let E be any Hermitian space of finite dimension n , and let $f: E \rightarrow E$ be any linear map. The following properties hold:*

(1) *The linear map $f: E \rightarrow E$ is an isometry iff*

$$f \circ f^* = f^* \circ f = \text{id}.$$

(2) *For every orthonormal basis (e_1, \dots, e_n) of E , if the matrix of f is A , then the matrix of f^* is the adjoint A^* of A , and f is an isometry iff A satisfies the identities*

$$A A^* = A^* A = I_n,$$

where I_n denotes the identity matrix of order n , iff the columns of A form an orthonormal basis of E , iff the rows of A form an orthonormal basis of E .

Proposition 9.11 also motivates the following definition.

Definition 11.6. A complex $n \times n$ matrix is a *unitary matrix* iff

$$A A^* = A^* A = I_n.$$

Remarks: The conditions $A A^* = I_n$, $A^* A = I_n$, and $A^{-1} = A^*$, are equivalent.

Given any two orthonormal bases (u_1, \dots, u_n) and (v_1, \dots, v_n) , if P is the change of basis matrix from (u_1, \dots, u_n) to (v_1, \dots, v_n) , it is easy to show that the matrix P is unitary.

The proof of Proposition 11.8 (3) also shows that if f is an isometry, then the image of an orthonormal basis (u_1, \dots, u_n) is an orthonormal basis.

If f is unitary and A is its matrix with respect to any orthonormal basis, we have $|\det(A)| = 1$.

Definition 11.7. Given a Hermitian space E of dimension n , the set of isometries $f: E \rightarrow E$ forms a subgroup of $\mathbf{GL}(E, \mathbb{C})$ denoted as $\mathbf{U}(E)$, or $\mathbf{U}(n)$ when $E = \mathbb{C}^n$, called the *unitary group (of E)*. For every isometry, f , we have $|\det(f)| = 1$, where $\det(f)$ denotes the determinant of f . The isometries such that $\det(f) = 1$ are called *rotations, or proper isometries, or proper unitary transformations*, and they form a subgroup of the special linear group $\mathbf{SL}(E, \mathbb{C})$ (and of $\mathbf{U}(E)$), denoted as $\mathbf{SU}(E)$, or $\mathbf{SU}(n)$ when $E = \mathbb{C}^n$, called the *special unitary group (of E)*. The isometries such that $\det(f) \neq 1$ are called *improper isometries, or improper unitary transformations, or flip transformations*.

The Gram–Schmidt orthonormalization procedure immediately yields the QR -decomposition for matrices.

Proposition 11.10. *Given any $n \times n$ complex matrix A , if A is invertible then there is a unitary matrix Q and an upper triangular matrix R with positive diagonal entries such that $A = QR$.*

The proof is absolutely the same as in the real case!

