Fundamentals of Linear Algebra and Optimization Lagrangian Duality

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Primal Minimization Problem

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It turns out that under certain conditions the original Problem (P), called *primal problem*, can be solved in two stages with the help another Problem (D), called the *dual problem*.

Dual Problem

The Dual Problem (D) is a maximization problem involving a function G, called the Lagrangian dual, and it is obtained by minimizing the Lagrangian $L(v,\mu)$ of Problem (P) over the variable $v \in \mathbb{R}^n$, holding μ fixed, where $L \colon \Omega \times \mathbb{R}^m_+ \to \mathbb{R}$ is given by

$$L(\mathbf{v},\mu) = J(\mathbf{v}) + \sum_{i=1}^{m} \mu_i \varphi_i(\mathbf{v}),$$

with $\mu \in \mathbb{R}_+^m$.

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- (2) Solve the *maximization problem* of finding the maximum of the function $\mu \mapsto \mathcal{G}(\mu)$ over all $\mu \in \mathbb{R}_+^m$. This is basically an unconstrained problem, except for the fact that $\mu \in \mathbb{R}_+^m$.

If Steps (1) and (2) are successful, under some suitable conditions on the function J and the constraints φ_i (for example, if they are convex), for any solution $\lambda \in \mathbb{R}^m_+$ obtained in Step (2), the vector u_λ obtained in Step (1) is an optimal solution of Problem (P).

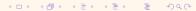
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In this presentation we do not discuss saddle points since this would take too much time.



Primal Minimization Problem

We now return to our main Minimization Problem (P):

minimize
$$J(v)$$

subject to $\varphi_i(v) \leq 0$, $i = 1, ..., m$,

where $J: \Omega \to \mathbb{R}$ and the constraints $\varphi_i: \Omega \to \mathbb{R}$ are some functions defined on some open subset Ω of some finite-dimensional Euclidean vector space V (more generally, a real Hilbert space V).

Lagrangian of the Minimization Problem

Definition. The *Lagrangian* of the Minimization Problem (P) defined above is the function $L \colon \Omega \times \mathbb{R}^m_+ \to \mathbb{R}$ given by

$$L(\mathbf{v},\mu) = J(\mathbf{v}) + \sum_{i=1}^{m} \mu_i \varphi_i(\mathbf{v}),$$

with $\mu = (\mu_1, \dots, \mu_m)$. The numbers μ_i are called *generalized Lagrange multipliers*.

Dual Maximization Problem

We are naturally led to introduce the function $G: \mathbb{R}^m_+ \to \mathbb{R}$ given by

$$G(\mu) = \inf_{\mathbf{v} \in \Omega} L(\mathbf{v}, \mu) \quad \mu \in \mathbb{R}_+^m,$$

and then λ will be a solution of the problem

find
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 such that $G(\lambda) = \sup_{\mu \in \mathbb{R}_+^m} G(\mu),$

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which is equivalent to the *Maximization Problem* (D):

maximize
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subject to $\mu \in \mathbb{R}_+^m$.

Lagrangian Duality

Definition. Given the Minimization Problem (P)

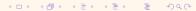
minimize
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subject to $\varphi_i(v) \leq 0$, $i = 1, ..., m$,

where $J: \Omega \to \mathbb{R}$ and the constraints $\varphi_i: \Omega \to \mathbb{R}$ are some functions defined on some open subset Ω of some finite-dimensional Euclidean vector space V (more generally, a real Hilbert space V), the function $G: \mathbb{R}^m_+ \to \mathbb{R}$ given by

$$G(\mu) = \inf_{\mathbf{v} \in \Omega} L(\mathbf{v}, \mu) \quad \mu \in \mathbb{R}_+^m,$$

is called the Lagrange dual function (or simply dual function).



Lagrange Dual Problem

Problem (D)

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Problem (P) is often called the *primal problem*, and (D) is the *dual problem*. The variable μ is called the *dual variable*. The variable $\mu \in \mathbb{R}_+^m$ is said to be *dual feasible* if $G(\mu)$ is defined (not $-\infty$). If $\lambda \in \mathbb{R}_+^m$ is a maximum of G, then we call it a *dual optimal* or an *optimal Lagrange multiplier*.

Dual as a Convex Optimization Problem

Since

$$L(\mathbf{v},\mu) = J(\mathbf{v}) + \sum_{i=1}^{m} \mu_i \varphi_i(\mathbf{v}),$$

the function $G(\mu) = \inf_{v \in \Omega} L(v, \mu)$ is the pointwise infimum of some affine functions of μ , so it is *concave*, even if the φ_i are not convex.

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One of the main advantages of the dual problem over the primal problem is that it is a *convex optimization problem*, since we wish to maximize a concave objective function G (thus minimize -G, a convex function), and the constraints $\mu \geq 0$ are convex. In a number of practical situations, the dual function G can indeed be computed.

Dual as a Partial Function

To be perfectly rigorous, we should mention that the dual function G is actually a partial function, because it takes the value $-\infty$ when the map $v \mapsto L(v, \mu)$ is unbounded below.

Example. Consider the Linear Program (P)

minimize $c^{\top}v$ subject to $Av \leq b$, $v \geq 0$,

where A is an $m \times n$ matrix, $v \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.

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The constraints $v \ge 0$ are rewritten as $-v_i \le 0$, so we introduce Lagrange multipliers $\mu \in \mathbb{R}^m_+$ and $\nu \in \mathbb{R}^n_+$, and we have the Lagrangian

$$L(\mathbf{v}, \mu, \nu) = \mathbf{c}^{\top} \mathbf{v} + \mu^{\top} (A\mathbf{v} - \mathbf{b}) - \nu^{\top} \mathbf{v}$$
$$= -\mathbf{b}^{\top} \mu + (\mathbf{c} + \mathbf{A}^{\top} \mu - \nu)^{\top} \mathbf{v}.$$

The linear function $\mathbf{v} \mapsto (\mathbf{c} + \mathbf{A}^{\top} \mu - \nu)^{\top} \mathbf{v}$ is unbounded below unless $\mathbf{c} + \mathbf{A}^{\top} \mu - \nu = 0$, so the dual function $G(\mu, \nu) = \inf_{\mathbf{v} \in \mathbb{R}^n} L(\mathbf{v}, \mu, \nu)$ is given for all $\mu \geq 0$ and $\nu \geq 0$ by

$$\textit{G}(\mu,\nu) = \begin{cases} -\emph{b}^\top \mu & \text{if } \emph{A}^\top \mu - \nu + \emph{c} = 0, \\ -\infty & \text{otherwise}. \end{cases}$$

The linear function $\mathbf{v}\mapsto (\mathbf{c}+\mathbf{A}^{\top}\mu-\nu)^{\top}\mathbf{v}$ is unbounded below unless $\mathbf{c}+\mathbf{A}^{\top}\mu-\nu=0$, so the dual function $G(\mu,\nu)=\inf_{\mathbf{v}\in\mathbb{R}^n}L(\mathbf{v},\mu,\nu)$ is given for all $\mu\geq 0$ and $\nu\geq 0$ by

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The domain of G is a proper subset of $\mathbb{R}^m_+ \times \mathbb{R}^n_+$.

Observe that the value $G(\mu, \nu)$ of the function G, when it is defined, is independent of the second argument ν .



Another way to obtain $G(\mu, \nu)$ is to observe that the function $v \mapsto (c + A^{\top}\mu - \nu)^{\top}v - b^{\top}\mu$ is affine, thus convex, and since \mathbb{R}^n is convex and open, by an earlier theorem, this function (for μ, ν fixed) has a minimum at ν iff

$$\nabla L(-,\mu,\nu)_{\mathbf{v}} = \mathbf{c} + \mathbf{A}^{\mathsf{T}}\mu - \nu = 0,$$

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But if $c + A^{\top}\mu - \nu = 0$, the function $v \mapsto (c + A^{\top}\mu - \nu)^{\top}v - b^{\top}\mu$ is the constant function with value $-b^{\top}\mu$, so indeed $G(\mu, \nu)$ is defined as above.



Since $G(\mu, \nu)$ is independent of ν , we introduce the function \widehat{G} of the single argument μ given by

$$\widehat{G}(\mu) = -b^{\top}\mu,$$

which is defined for all $\mu \in \mathbb{R}_+^m$.

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Of course, $\sup_{\mu\in\mathbb{R}^m_+}\widehat{G}(\mu)$ and $\sup_{(\mu,\nu)\in\mathbb{R}^m_+\times\mathbb{R}^n_+}G(\mu,\nu)$ are generally different, but note that $\widehat{G}(\mu)=G(\mu,\nu)$ iff there is some $\nu\in\mathbb{R}^n_+$ such that $A^\top\mu-\nu+c=0$ iff $A^\top\mu+c\geq0$.

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maximize
$$-b^{\top}\mu$$

subject to $A^{\top}\mu \ge -c$, $\mu \ge 0$.



Hidden Constraints Within the Dual

In summary, the dual function G of a Primary Problem (P) often contains hidden inequality constraints that define its domain, and sometimes it is possible to make these domain constraints $\psi_1(\mu) \leq 0, \ldots, \psi_p(\mu) \leq 0$ explicit, to define a new function \widehat{G} that depends only on q < m of the variables μ_i and is defined for all values $\mu_i \geq 0$ of these variables, and to replace the Maximization Problem (D), find $\sup_{\mu \in \mathbb{R}_+^m} G(\mu)$, by the constrained Problem (D_1)

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Problem (D_1) is different from the Dual Program (D), but it is *equivalent* to (D) as a maximization problem.