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THE $\mathfrak{p}$-FUNCTION IN λ-K-CONVERSION

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In the theory of conversion it is important to have a formally defined function which assigns to any positive integer n the least integer not less than n which has a given property. The definition of such a formula is somewhat involved:[1] I propose to give the corresponding formula in λ-K-conversion,[2] which will (naturally) be much simpler. I shall in fact find a formula $\mathfrak{p}$ such that if T be a formula for which $T(n)$ is convertible[3] to a formula representing a natural number, whenever n represents a natural number, then $\mathfrak{p}(T, r)$ is convertible to the formula q representing the least natural number q, not less than r, for which $T(q)$ conv 0.[2] The method depends on finding a formula Θ with the property that Θ conv $\lambda u.u(\Theta(u))$, and consequently if $M \to \Theta(V)$ then M conv $V(M)$. A formula with this property is,

$$\Theta \to \{\lambda vu.u(v(v, u))\}(\lambda vu.u(v(v, u))).$$

The formula $\mathfrak{p}$ will have the required property if $\mathfrak{p}(T, r)$ conv r when $T(r)$ conv 0, and $\mathfrak{p}(T, r)$ conv $\mathfrak{p}(T, S(r))$ otherwise. These conditions will be satisfied if $\mathfrak{p}(T, r)$ conv $T(r, \lambda x.\mathfrak{p}(T, S(r)), r)$, i.e. if $\mathfrak{p}$ conv $\{\lambda ptr.t(r, \lambda x.p(t, S(r)), r)\}(\mathfrak{p})$. We therefore put,

$$\mathfrak{p} \to \Theta(\lambda ptr.t(r, \lambda x.p(t, S(r)), r)).$$

This enables us to define also a formula,

$$\mathcal{P} \to \lambda tn.n(\lambda v.\mathfrak{p}(t, S(v)), 0),$$

such that $\mathcal{P}(T, n)$ is convertible to the formula representing the nth positive integer q for which $T(q)$ conv 0.

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[1] Such a function was first defined by S. C. Kleene, *A theory of positive integers in formal logic*, *American journal of mathematics*, vol. 57 (1934), see p. 231.

[2] For the definition of λ-K-conversion see S. C. Kleene, *λ-definability and recursiveness*, *Duke mathematical journal*, vol. 2 (1936), pp. 340–353, footnote 12. In λ-K-conversion we are able to define the formula $0 \to \lambda fx.x$. The same paper of Kleene contains the definition of a formula L with a property similar to the essential property of Θ (p. 346).

[3] "Convertible" and "conv" refer to λ-K-conversion throughout this note.