

# Introduction to the Theory of Computation

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### Solutions to the Practice Final Exam

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**Problem 1 (10 pts).** The regular expression  $R^*S^*$  can be converted to an NFA,  $N$ , accepting the language denoted by  $R^*S^*$ , using a well-known algorithm. Then,  $N$  can be converted to DFA,  $D$ , using the subset construction. Since  $L(D) = \Sigma^*$  is equivalent to  $\overline{L(D)} = \emptyset$ , we can construct the DFA,  $\overline{D}$ , accepting  $\overline{L(D)}$ , by swapping final and rejecting states, and then, we just have to test whether  $\overline{D}$  accepts the empty language. However,  $\overline{D}$  accepts nothing iff there is no path from the start state to any final state, and this can be easily decided (graph reachability).

**Problem 2 (20 pts).** The language  $L_1 = \{ww^R \mid w \in R\}$  may be regular, for example, if  $R$  is finite or if  $\Sigma$  consists of a single letter. We claim that if  $R = \{a, b\}^*$ , then  $L_1$  is not regular. We proceed by contradiction using Myhill-Nerode. If  $L_1$  were regular, there would be some right-invariant equivalence relation,  $\cong$ , of finite index, say  $n$ , so that  $L_1$  is the union of classes of  $\cong$ . Since there are infinitely many strings of the form

$$ab, a^2b, \dots, a^nb,$$

we must have

$$a^ib \cong a^jb$$

for some  $i < j$ . But then, by right-invariance, we get

$$a^ibba^i \cong a^jba^i,$$

with  $i < j$ . Since  $a^ibba^i \in L_1$ , we should also have  $a^jba^i \in L_1$ , which is absurd.

Let  $D = (Q, \Sigma, \delta_R, q_0, F)$  be a DFA accepting  $R$ . We construct an NFA,  $N$ , accepting  $L_2 = \{w \in \Sigma^* \mid ww^R \in R\}$ , as follows:

$$N = ((Q \times Q) \cup \{s\}, \Sigma, \delta, s, \{(p, p) \mid p \in Q\}),$$

where  $\delta$  is given by:

$$(q_0, p) \in \delta(s, \epsilon) \quad \text{for all } p \in F,$$

and

$$(q_1, q_2) \in \delta((p_1, p_2), a) \quad \text{iff} \quad \delta_R(p_1, a) = q_1 \quad \text{and} \quad \delta_R(q_2, a) = p_2,$$

for every  $a \in \Sigma$ .

**Problem 3 (15 pts).** (i)

$$\begin{aligned} S &\longrightarrow XcY \\ X &\longrightarrow aXb \\ X &\longrightarrow ab \\ Y &\longrightarrow aaYbb \\ Y &\longrightarrow aabb \end{aligned}$$

is a CFG for

$$L_3 = \{a^m b^n c a^{2m} b^{2n} \mid m, n \geq 1\}.$$

(ii) A *deterministic* PDA accepting  $L_3$  by empty stack is easily constructed.

**Problem 4 (20 pts).** We prove that

$$L_4 = \{a^m b^n c a^{2m} b^{2n} \mid m, n \geq 1\}$$

is not context-free by contradiction, using Ogden's lemma. If  $K > 1$  is the constant of Ogden's lemma, we pick  $w = a^K b^K c a^{2K} b^{2K}$ , with the middle  $b$ 's and  $a$ 's distinguished. Then, the cases are pretty much as in the class notes on context-free languages, page 59-60.

Proving that

$$L_5 = \{a^n \mid n \text{ is not a prime}\}$$

is not context-free is a little more tricky. We use the fact proved as a homework that a context-free language over a one-letter alphabet is actually regular. Were  $L_5$  context-free, then it would be regular. But then, its complement,  $L_6$ , would also be regular. However, we can easily show using the pumping lemma for the regular languages that

$$L_6 = \{a^n \mid n \text{ is a prime}\}$$

is not regular. Thus, in the end,  $L_5$  is not context-free.

**Problem 5 (20 pts).** Pick any  $a \in \Sigma$ , let  $G$  be a linear context-free grammar and let  $L = L(G)$ . We want to prove that  $L/a = \{w \in \Sigma^* \mid wa \in L\}$  is still linear context-free. One should be aware that  $\epsilon$ -rules, which *are allowed* in linear CFG's, cause a little bit of trouble. We construct a new grammar,  $G_a$ , from  $G$ , by adding new nonterminals of the form  $[A/a]$  and new productions

$$[A/a] \longrightarrow \alpha, \quad \text{if } A \longrightarrow \alpha a \in P \quad \text{or} \quad A \longrightarrow \alpha a B \in P \quad \text{with } B \xrightarrow{+} \epsilon$$

and

$$[A/a] \longrightarrow u[B/a], \quad \text{if } A \longrightarrow uB \in P.$$

The start symbol of  $G_a$  is  $[S/a]$ . We claim that  $L(G_a) = L(G)/a = L/a$ .

*Claim 1:*  $L/a \subseteq L(G_a)$ . For this, we prove by induction on the length of derivations in  $G$  that if  $S \xRightarrow{+} \alpha a$ , then  $[S/a] \xRightarrow{+} \alpha$  in  $G_a$  and if  $S \xRightarrow{+} \alpha B$ , then  $[S/a] \xRightarrow{+} \alpha[B/a]$  in  $G_a$ , with  $\alpha \in V^*$ . Note that for linear context-free grammars, all derivations are leftmost derivations (and rightmost derivations).

If  $S \xrightarrow{1} \alpha a$ , then by construction, the production  $[S/a] \rightarrow \alpha$  is in  $G_a$ ; similarly, if  $S \xrightarrow{1} \alpha B$ , then by construction, the production  $[S/a] \rightarrow \alpha[B/a]$  is in  $G_a$ . Thus, the base step holds.

Now, assume the induction hypothesis holds for  $n$  and consider a derivation  $S \xRightarrow{n+1} \alpha a$  or  $S \xRightarrow{n+1} \alpha B$  of length  $n + 1$ , with  $n \geq 1$ . There are four cases:

*Case 1.* The derivation is of the form  $S \xRightarrow{n} wA\beta a \Rightarrow w\gamma\beta a$ , where  $w \in \Sigma^*$  and  $\beta, \gamma \in V^*$

By the induction hypothesis, there is a derivation  $[S/a] \xRightarrow{+} wA\beta$  in  $G_a$ . Thus, we get the derivation

$$[S/a] \xRightarrow{+} wA\beta \Rightarrow w\gamma\beta \quad \text{in } G_a.$$

*Case 2.* The derivation is of the form  $S \xRightarrow{n} wA \Rightarrow wuBva$ , where  $u, v, w \in \Sigma^*$ .

By the induction hypothesis, there is a derivation  $[S/a] \xRightarrow{+} w[A/a]$  in  $G_a$  and by construction, there is a production  $[A/a] \rightarrow uBv$  in  $G_a$ . Thus, we get the derivation

$$[S/a] \xRightarrow{+} w[A/a] \Rightarrow wuBv \quad \text{in } G_a.$$

*Case 3.* The derivation is of the form  $S \xRightarrow{n} waA \Rightarrow wa$ , where  $w \in \Sigma^*$ . This is the case when the last step is an  $\epsilon$ -rule,  $A \rightarrow \epsilon$ .

In this case, there is a first derivation step during which  $a$  appears and the above derivation must be of the form

$$S \xRightarrow{+} w_1A_1 \Rightarrow w_1w_2aB \xRightarrow{*} w_1w_2aA \Rightarrow w_1w_2a,$$

where  $w = w_1w_2$ . Thus,  $B \xRightarrow{+} \epsilon$ . By the induction hypothesis, there is a derivation  $[S/a] \xRightarrow{+} w_1[A_1/a]$  in  $G_a$  and by construction, there is a production  $[A_1/a] \rightarrow w_2$  in  $G_a$ . Thus, we get the derivation

$$[S/a] \xRightarrow{+} w_1[A_1/a] \Rightarrow w_1w_2 \quad \text{in } G_a.$$

*Case 4.* The derivation is of the form  $S \xRightarrow{n} wA \Rightarrow wuB$ , where  $u, w \in \Sigma^*$ .

By the induction hypothesis, there is a derivation  $[S/a] \xRightarrow{+} w[A/a]$  in  $G_a$  and by construction, there is a production  $[A/a] \rightarrow u[B/a]$  in  $G_a$ . Thus, we get the derivation

$$[S/a] \xRightarrow{+} w[A/a] \Rightarrow wu[B/a] \quad \text{in } G_a.$$

*Claim 2:*  $L(G_a) \subseteq L/a$ . For this, we prove by induction on the length of derivations in  $G_a$  that if  $[S/a] \xRightarrow{+} \alpha$ , then  $S \xRightarrow{+} \alpha a$ , and if  $[S/a] \xRightarrow{+} \alpha[B/a]$ , then  $S \xRightarrow{+} \alpha B$ , with  $\alpha \in V^*$ .

If  $[S/a] \xRightarrow{1} w$ , then by construction, either the production  $S \rightarrow wa$  is in  $G$  or the production  $S \rightarrow waB$  is in  $G$  with  $B \xRightarrow{+} \epsilon$ . In the first case,  $S \xRightarrow{1} wa$  and in the second case,

$$S \xRightarrow{+} waB \xRightarrow{+} wa.$$

Thus, the base step holds.

Now, assume the induction hypothesis holds for  $n$  and consider a derivation  $[S/a] \xRightarrow{n+1} \alpha$  or  $S \xRightarrow{n+1} \alpha[B/a]$  of length  $n + 1$ , with  $n \geq 1$ .

Due to the form of the productions involving the new nonterminals  $[A/a]$ , an easy induction shows that for any derivation in  $G_a$  of the form  $[S/a] \xRightarrow{+} \alpha[B/a]\beta$ , we must have  $\beta = \epsilon$ . There are three cases.

*Case 1.* The derivation is of the form  $[S/a] \xRightarrow{n} wA\beta \xRightarrow{+} w\gamma\beta$ , where  $w \in \Sigma^*$  and  $\beta, \gamma \in V^*$ .

By the induction hypothesis, there is a derivation  $S \xRightarrow{+} wA\beta a$  in  $G$ . Thus, we get the derivation

$$S \xRightarrow{+} wA\beta a \xRightarrow{+} w\gamma\beta a \quad \text{in } G.$$

*Case 2.* The derivation is of the form  $[S/a] \xRightarrow{n} w[A/a] \xRightarrow{+} w\alpha$ , where  $w \in \Sigma^*$  and  $\alpha \in V^*$ .

By the induction hypothesis, there is a derivation  $S \xRightarrow{+} wA$  in  $G$  and by construction, either there is a production  $A \rightarrow \alpha a$  in  $G$  or a production  $A \rightarrow \alpha aB$  in  $G$  with  $B \xRightarrow{+} \epsilon$ . Thus, we get the derivation

$$S \xRightarrow{+} wA \xRightarrow{+} w\alpha a \quad \text{in } G$$

or a derivation

$$S \xRightarrow{+} wA \xRightarrow{+} w\alpha aB \xRightarrow{+} w\alpha a \quad \text{in } G.$$

*Case 3.* The derivation is of the form  $[S/a] \xRightarrow{n} w[A/a] \xRightarrow{+} w\alpha[B/a]$ , where  $w \in \Sigma^*$  and  $\alpha \in V^*$ .

By the induction hypothesis, there is a derivation  $S \xRightarrow{+} wA$  in  $G$  and by construction, there is a production  $A \rightarrow \alpha B$  in  $G$ . Thus, we get the derivation

$$S \xRightarrow{+} wA \xRightarrow{+} w\alpha B \quad \text{in } G.$$

Putting Claim 1 and Claim 2 together, we get  $L(G_a) = L/a$ .

**Problem 6 (25 pts).** First, assume that  $M$  halts only for finitely inputs. Is so, the language  $L_M$  is finite, thus context-free. Now, assume that  $M$  accepts infinitely many inputs and that

$L_M$  is context-free. Then,  $M$  accepts arbitrarily long strings and so, if  $K$  is the constant of the pumping lemma, there is some proper halting ID

$$w = w_1 \# w_2^R \# w_3 \# \dots$$

with  $|w_1|, |w_2| > K$ . Let  $w_2$  be marked. Then, in any decomposition  $uvxyz$  of  $w$ , as  $x$  must contain some marked occurrence and  $vxy$  contains at most  $K$  marked occurrences, it is impossible that at the same time  $v$  contains symbols in  $w_1$  and that  $y$  contain symbols in  $w_3$ , as  $vxy$  would contain too many marked occurrences. But then, when pumping up  $v$  and  $y$  (recall,  $uv^i xy^i z \in L_M$ , for all  $i \geq 0$ ), either  $w_1$  remains intact or  $w_3$  remains intact while  $w_2$  gets strictly bigger. As the TM is deterministic, this is impossible: there can't be a valid move from  $w_1$  to the new  $w_2$  or from the new  $w_2$  to  $w_3$ . Therefore,  $L_M$  can't be context-free and if  $L_M$  is context-free, then it is finite.

(ii) Given any TM,  $M$ , first modify  $M$  so that every computation makes at least two moves on every input. This can be done by adding two extra states. We know that  $L = \overline{L_M}$  is context-free and from (i),  $M$  halts on finitely many inputs iff  $L_M = \overline{L}$  is context-free. However, it is undecidable whether a TM accepts a finite language (by Rice's theorem, as it is undecidable whether a partial function has a finite domain). Therefore, deciding for any CFG,  $G$ , whether  $\overline{L(G)}$  is context-free must also be undecidable.